Commun. Korean Math. Soc. **20** (2005), No. 2, pp. 209–219

ON TOR-TORSION THEORIES

M. GOLRIZ AND M. H. BIJANZADEH

ABSTRACT. Tor-torsion theory was defined by Jan Trlifaj in 2000. In this paper we introduce the notion of Co envelopes, CoCovers and Tor-generators as dual of envelopes, covers and generators in cotorsion(Ext-torsion) theory and deduce that each R-module has a projective and a cotorsion coprecover.

0. Introduction

Let $\mathcal{C}$ denote the class of all R-modules where $\mathcal{R}$ is a commutative Noetherian ring with identity. For a class S of R-modules, we put:

$$S^{\mathsf{T}} = \{N \in \mathcal{C} \mid \operatorname{Tor}_1^R(S, N) = 0,\ \forall s \in S\}$$

and

$$^{\mathsf{T}}S = \{N \in \mathcal{C} \mid \operatorname{Tor}_1^R(N, S) = 0,\ \forall s \in S\}.$$

Throughout we use $\mathcal{A}$, $\mathcal{B}$ to denote the classes of R-modules such that $\mathcal{A} = ^{\mathsf{T}} B$, $B = \mathcal{A}^{\mathsf{T}}$, and we call $(\mathcal{A}, B)$ a Tor-torsion theory.

EXAMPLE. $(\mathcal{C}, \mathcal{P})$ and $(\mathcal{C}, \mathcal{F})$ where $\mathcal{P}$ is the class of all projective R-modules and $\mathcal{F}$ is the class of all flat R-modules, are examples of Tor-torsion theories.

1. Properties of tor-torsion theories

PROPOSITION 1.1. *With the above notations, Both $\mathcal{A}$ and $\mathcal{B}$ are closed under extensions, direct products and direct limits.*

PROOF. We just prove that $\mathcal{A}$ is closed under extensions. The other parts are proved in [6, Theorem 8.10, 8.11].

Received June 17, 2003.
2000 Mathematics Subject Classification: 13D30.
Key words and phrases: tor-torsion theory, co(pre)cover, co(pre)envelope, tor-generator.

Let $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ be an exact sequence of R-modules such that M and L are in $\mathcal{A}$; then we have the following exact sequence.

$$\cdots \longrightarrow \mathrm{Tor}_1(M,F) \longrightarrow \mathrm{Tor}_1(N,F) \longrightarrow \mathrm{Tor}_1(L,F) \longrightarrow \cdots$$

since $\mathrm{Tor}_1(M,F) = \mathrm{Tor}_1(L,F) = 0\ \forall F \in \mathcal{B}$. So is the $\mathrm{Tor}_1(N,F)$; hence $\mathcal{A}$ is closed under extension. □

PROPOSITION 1.2. *The following conditions are equivalent:*

(i) $\mathrm{Tor}_2(A,B) = 0$, *for all A in $\mathcal{A}$ and all B in $\mathcal{B}$.*

(ii) *For each exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ with Y, Z in B(in $\mathcal{A}$) we have $X \in B(X \in A)$.*

PROOF. (i)$\longrightarrow$(ii) We have the long exact sequence

$$\cdots \mathrm{Tor}_2(A,Z) \longrightarrow \mathrm{Tor}_1(A,X) \longrightarrow \mathrm{Tor}_1(A,Y) \longrightarrow \cdots$$

since $Y \in B$; $\mathrm{Tor}_1(A,Y) = 0$. Also by hypothesis $\mathrm{Tor}_2(A,Z) = 0$. It therefore follows that $\mathrm{Tor}_1(A,X) = 0$ and so $X \in B$.

(ii)$\longrightarrow$(i) Let $0 \longrightarrow K \longrightarrow P \longrightarrow Z \longrightarrow 0$ be an exact sequence with P projective and $Z \in B$. Then we have the exact sequence

$$\cdots \longrightarrow \mathrm{Tor}_2(A,P) \longrightarrow \mathrm{Tor}_2(A,Z) \longrightarrow \mathrm{Tor}_1(A,K) \longrightarrow \mathrm{Tor}_1(A,P) \longrightarrow .$$

But $\mathrm{Tor}_2(A,P) = \mathrm{Tor}_1(A,K) = 0$; hence $\mathrm{Tor}_2(A,Z) = 0$ for all $A \in \mathcal{A}$ and all $Z \in B$. □

DEFINITION 1.3. Let $\mathcal{X}$ be a class of R-modules which is closed under extensions. Then for $X \in \mathcal{X}$, $M \in \mathcal{C}$;

(i) An R-homomorphism $\phi : X \longrightarrow M$ is called an X-copreenvelope of M if, for each $X' \in \mathcal{X}$, the following sequence is exact:

$$0 \longrightarrow X' \otimes X \longrightarrow X' \otimes M$$

(ii) An R-homomorphism $\psi : M \longrightarrow X$ is called an X-coprecover if, for each X', the following sequence is exact:

$$0 \longrightarrow X' \otimes M \longrightarrow X' \otimes X.$$

DEFINITION 1.4. The class $(\mathcal{A}, B)$ is said to have enough injectives if, for every module M, there is an exact sequence

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

with $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Also we say that $(\mathcal{A}, \mathcal{B})$ has enough projectives if, for every module M, there is an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$$

with $A \in \mathcal{A}$, $B \in \mathcal{B}$.

PROPOSITION 1.5. *Let the situation be as in 1.4 then $f : M \longrightarrow B$ of the exact sequence*

$$0 \longrightarrow M \longrightarrow B \longrightarrow A \longrightarrow 0$$

is a $\mathcal{B}$-coprecover of M.

PROOF. Let $f' : M \longrightarrow B'$, with $B' \in \mathcal{B}$, be an arbitrary R-homomorphism. Then we have the exact sequence:

$$0 = \mathrm{Tor}_1(A, B') \longrightarrow M \otimes B' \longrightarrow B \otimes B' \longrightarrow A \otimes B' \longrightarrow 0. \qquad \square$$

Note that if $0 \longrightarrow A \longrightarrow B \longrightarrow M \longrightarrow 0$ is as in the definition then it may not be a B-copre envelope of M.

PROPOSITION 1.6. *If $\phi : M \longrightarrow X$ is injective with $X \in \mathcal{X}$ and $D = \mathrm{coker}(\phi) \in^{\mathsf{T}} X$, then ϕ is an X-copercover, such a coprecover is called a special X-coprecover of M.*

PROOF. If $0 \longrightarrow M \xrightarrow{\phi} X \longrightarrow D \longrightarrow 0$ is exact, then for all $\phi' : M \longrightarrow X'$, the sequence $0 = \mathrm{Tor}_1(X', D) \longrightarrow A \otimes M \longrightarrow A \otimes X$ is exact; so by definition, $\phi : M \longrightarrow X$ is X-coprecover of M. $\square$

NOTE. (i) We can define a special X-Copre envelope but in this case it is not necessarily an X-copre envelope.

(ii) Let $M \xrightarrow{\phi} N$ be injective. M is a pure submodule of N if and only if $\mathcal{D} = \mathrm{coker}(\phi) \in^{\mathsf{T}} \mathcal{C}$ or $\mathcal{D}$ is flat.

COROLLARY 1.7. *If $D = \mathrm{coker}(\phi)$, where $\phi : M \longrightarrow N$ is injective. Then $D \in^{\mathsf{T}} X$ if and only if $0 \longrightarrow X \otimes M \longrightarrow X \otimes N \longrightarrow X \otimes D \longrightarrow 0$ is exact for all $x \in X$.*

2. Generators and tor-generators

One can define a generator and a minimal generator for $\mathcal{T}or(\mathcal{L}, M)$ as like as which is defined for $\mathcal{E}xt(\mathcal{L}, M)$ see [7].

Proposition 2.1. *Let $\mathcal{L}$ be a class of R-modules which is closed under extensions, and $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ is a minimal generator for $\mathcal{T}or(\mathcal{L}, M)$ then $K \in \mathcal{L}^{\mathsf{T}}$.*

PROOF. For any $\bar{L} \in \mathcal{L}$, consider an arbitrary extension of K by $\bar{L}$, say $0 \longrightarrow K \longrightarrow N \longrightarrow \bar{L} \longrightarrow 0$. Using a pushout diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & L & \longrightarrow & 0 \\
 & & & & \downarrow f & & \downarrow g & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & P & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bar{L} & & \bar{L} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

since L and $\bar{L}$ are in $\mathcal{L}$, $\mathcal{P}$ is in $\mathcal{L}$. Next, since $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ is a generator, there are homomoprhism h, l, making diagram commutative. So the middle column is split. Therefore $hf : K \longrightarrow K$ is an automorphism. So, $hf \otimes 1_{\bar{L}} : K \otimes \bar{L} \longrightarrow K \otimes \bar{L}$ is an automorphism. Now we can write the following exact sequence:

$$0 \longrightarrow K \xrightarrow{f} N \xrightarrow{h} K \longrightarrow 0$$

By tensoring this exact sequence with $\bar{L}$, we get the exact sequence:

$$\ldots \longrightarrow \mathrm{Tor}_1(K, \bar{L}) \longrightarrow K \otimes \bar{L} \longrightarrow N \otimes \bar{L} \longrightarrow K \otimes \bar{L} \longrightarrow 0.$$

And so $\mathrm{Tor}_1(K, \bar{L}) = 0$. Hence $K \in^{\intercal} \mathcal{L}$. □

We can establish the results concerning $\mathcal{T}or(\mathcal{L}, M)$ by using the same arguments which applied to $\mathcal{E}xt(\mathcal{L}, M)$ as wrote in [7].

Now we want to define Tor-generators for $\mathcal{T}or(L, \mathcal{M})$.

DEFINITION 2.2. Let $\mathcal{M}$ be a class of R-modules, an extension $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ with $M \in \mathcal{M}$, is called a Tor-generator for $\mathcal{T}or(L, \mathcal{M})$ if for any extension $0 \longrightarrow \bar{M} \longrightarrow \bar{N} \longrightarrow L \longrightarrow 0$ there is a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0
\end{array}$$

Furthermore, such a Tor-generator is said to be maximal if any commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \bar{L} & \longrightarrow & 0
\end{array}$$

always implies that f is an automorphism (so that g is too).

NOTE. If $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is a Tor-generator for $\mathcal{T}or(L, \mathcal{M})$ and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & \bar{L} & \longrightarrow & 0
\end{array}$$

is a commutative diagram with exact rows and $M' \in \mathcal{M}$, then $0 \longrightarrow M' \longrightarrow N' \longrightarrow L \longrightarrow 0$ is also a Tor-generator.

EXAMPLE. Any exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$ with P projective is a Tor-generator for $\mathcal{T}or(L, \mathcal{M})$. Moreover if P is projective cover of L, then it is a maximal Tor-generator.

PROPOSITION 2.3. *If $\mathcal{M}$ is closed under extensions and $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ is a maximal Tor-generator for $\mathcal{T}or(L, \mathcal{M})$, then $K \in^{\top} \mathcal{M}$.*

PROOF. For any $\bar{M} \in \mathcal{M}$, consider an arbitrary extension of $\bar{M}$ by K, say $0 \longrightarrow \bar{M} \longrightarrow N \longrightarrow K \longrightarrow 0$. Then we can write:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \bar{M} & = & \bar{M} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & G & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle l} & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & K & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

since M, $\bar{M}$ are in $\mathcal{M}$, so is G. But $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ is a Tor-generator. So there are homomorphisms h, l making the diagram commutative. Since Tor-generator is maximal $(hf), (gl)$ are automorphisms. Now $gl : K \longrightarrow K$ induces automorphism $K \otimes \bar{M} \overset{gl\otimes 1_{\bar{M}}}{\longrightarrow} K \otimes \bar{M}$. If we tensor the exact sequence $0 \longrightarrow K \longrightarrow N \longrightarrow K \longrightarrow 0$ by $\bar{M}$, then we obtain an exact sequence; so $\mathcal{T}or_1(K, \bar{M}) = 0$ for all $\bar{M} \in \mathcal{M}$. Hence $K \in^{\mathsf{T}} \mathcal{M}$. □

THEOREM 2.4. *Let $\mathcal{M}$ be closed under inverse limits. For an R-module L, if $\mathcal{T}or(L, \mathcal{M})$ has a Tor-generator, it must have a maximal Tor-generator.*

For the proof of this theorem, we need the following lemmas:

LEMMA 2.5. *Let $\mathcal{M}$ be closed under inverse limits. If $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is a Tor-generator for $\mathcal{T}or(L, \mathcal{M})$, then there is a Tor-generator $0 \longrightarrow \bar{M} \longrightarrow \bar{N} \longrightarrow L \longrightarrow 0$ and the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M' & \longrightarrow & N' & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

such that for any Tor-generator $0 \longrightarrow M^* \longrightarrow N^* \longrightarrow L \longrightarrow 0$ and any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^* & \longrightarrow & N^* & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \| & & \\ 0 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0 \\ & & \downarrow p & & \downarrow h & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

with exact rows, we have $\mathrm{Im}(h) = \mathrm{Im}(hg)$.

PROOF. We try to derive a contradiction by assuming that such a Tor-generator does not exist.

Put $M_0 = M$ and $N_0 = N$. By assumption there exsit a Tor-generator $0 \longrightarrow M_1 \longrightarrow N_1 \longrightarrow L \longrightarrow 0$ such that in the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & N_1 & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow g_{01} & & \| & & \\ 0 & \longrightarrow & M_0 & \longrightarrow & N_0 & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

g_{01} is not projective. By assumption again $0 \longrightarrow M_1 \longrightarrow N_1 \longrightarrow L \longrightarrow 0$ does not satisfy the desired property. In other words, there is a Tor-generator $0 \longrightarrow M_2 \longrightarrow N_2 \longrightarrow L \longrightarrow 0$ and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_2 & \longrightarrow & N_2 & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow g_{12} & & \| & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & N_1 & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

such that $\mathrm{Im}(g_{01}) \subsetneqq \mathrm{Im}(g_{02})$, where $g_{02} = g_{01} \circ g_{12}$. By repeting the same process, for each $n \in \mathbb{N}$, we can find a Tor-generator $0 \longrightarrow M_n \longrightarrow N_n \longrightarrow L \longrightarrow 0$ and homomorphism g_{in} for all $i < n$ such that for any $k < m < n$, $g_{kn} = g_{km} \circ g_{mn}$ and

$$\mathrm{Im}(g_{01}) \subsetneqq \mathrm{Im}(g_{02}) \subsetneqq \mathrm{Im}(g_{03}) \subseteq \cdots \subset N$$

so $card(Z') \leq card(N)$.

We wish to demonstrate that the cardinality of N must be grater than that of any ordinal number β. Consider the exact sequence

$$0 \longrightarrow \varprojlim M_n \longrightarrow \varprojlim N_n \longrightarrow L \longrightarrow 0$$

and note that $\varprojlim M_n \in \mathcal{M}$. For the first infinite ordinal w, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_w & \longrightarrow & N_w & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \varprojlim M_n & \longrightarrow & \varprojlim N_n & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

with exact rows and let $g_{nw} : N_w \longrightarrow N_n$ be the obvious maps. We have $\mathrm{Im}(g_{0n}) \subsetneqq \mathrm{Im}(g_{0w})$ for all $n \in \mathbb{N}$; otherwise there exist $n \in \mathbb{N}$ such that $\mathrm{Im}(g_{0n}) = \mathrm{Im}(g_{0w})$. Choose $x \in \mathrm{Im}(g_{0n+1})\backslash \imath(g_{0n})$ then $x = g_{0n+1}(x_{n+1})$ besides $x_{n+1} = g_{n+1w}(x_w)$ so $x = g_{0n+1}g_{n+1w}(x_w) = g_{0w}(x_w) = g_{0w}(x_w) \in \mathrm{Im}(g_{0w}) = \mathrm{Im}(g_{0n})$ which is a contradiction.

Since

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_w & \longrightarrow & N_w & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

does not satisfy the conclusion of the lemma, we can find a Tor-generator $0 \longrightarrow M_{w+1} \longrightarrow N_{w+1} \longrightarrow L \longrightarrow 0$ and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_{w+1} & \longrightarrow & N_{w+1} & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & M_w & \longrightarrow & N_w & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

such that $\operatorname{Im}(g_{0w}) \subsetneqq \operatorname{Im}(g_{0w+1})$. Proceeding in this manner, given any ordinal β, we can find Tor-generator

$$0 \longrightarrow M_\alpha \longrightarrow N_\alpha \longrightarrow L \longrightarrow 0$$

for all $\alpha \leq \beta$ with $g_{0\alpha} : N_\alpha \longrightarrow N_0$; so that for $\lambda < \mu \leq \beta$ $\operatorname{Im}(g_{0\lambda}) \subsetneqq \operatorname{Im}(g_{0\mu})$. Hence $card(N) \geq card(\beta)$. Since β is arbitrary we have the required contradiction. □

LEMMA 2.6. *If $\mathcal{M}$ is closed under inverse limits and if there exist a Tor-generator $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ for $\mathcal{T}or(L, \mathcal{M})$, then there is a Tor-generator $0 \longrightarrow \bar{M} \longrightarrow \bar{N} \longrightarrow L \longrightarrow 0$ such that for any Tor-generator $0 \longrightarrow M^* \longrightarrow N^* \longrightarrow L \longrightarrow 0$ and any commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \longrightarrow & N^* & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow g & & \| & & \\
0 & \longrightarrow & \bar{M} & \longrightarrow & \bar{N} & \longrightarrow & \bar{L} & \longrightarrow & 0
\end{array}$$

g must be projective.

PROOF. By lemma 2.5, there exist a Tor-generator $0 \longrightarrow M_1 \longrightarrow N_1 \longrightarrow L \longrightarrow 0$ such that, in any commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \longrightarrow & N^* & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow h & & \| & & \\
0 & \longrightarrow & M_1 & \longrightarrow & N_1 & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow g & & \| & & \\
0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

with exact rows and $M^*, N^* \in \mathcal{M}$, we have $\operatorname{Im}(g) = \operatorname{Im}(gh)$. So, for each $n \in \mathbb{N}$, we can find a Tor-generator $0 \longrightarrow M_n \longrightarrow N_n \longrightarrow L \longrightarrow 0$ such that for any Tor-generator $0 \longrightarrow M^* \longrightarrow N^* \longrightarrow L \longrightarrow 0$ and any commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M^* & \longrightarrow & N^* & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow h^* & & \| & & \\
0 & \longrightarrow & M_{n+1} & \longrightarrow & N_{n+1} & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow g & & \| & & \\
0 & \longrightarrow & M'_n & \longrightarrow & N'_n & \longrightarrow & L & \longrightarrow & 0
\end{array}$$

we have $\mathrm{Im}(g_{nn+1}h^*) = \mathrm{Im}(g_{nn+1})$. Now let w be the first infinite ordinal number. We have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_w & \longrightarrow & N_w & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \varprojlim M_n & \longrightarrow & \varprojlim N_n & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

where $g_{nw} : N_w \longrightarrow N_n$ are obvious maps. We claim that the above Tor-generator must have the desired property. In the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^* & \longrightarrow & N^* & \longrightarrow & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow h & & \| & & \\ 0 & \longrightarrow & M_w & \longrightarrow & N_w & \longrightarrow & L & \longrightarrow & 0 \end{array}$$

h must be projective. Otherwise, there exist $x_w \in N_w \setminus \mathrm{Im}(h)$. $g_{n+1w}(x_w) = x_{n+1}$, $x_{n+1} \in N_{n+1}$. Now $g_{nn+1}(x_{n+1}) = x_n$ and $x_n \in \mathrm{Im}(g_{nn+1}) = \mathrm{Im}(g_{nn+1} \circ h^*)$ so there is an $x^* \in N^*$ such that $g_{nn+1}h^*(x^*) = x_n$. Since the diagram

$$\begin{array}{ccc} N^* & \xrightarrow{h} & N_w \\ h^* & \searrow & \downarrow g_{n+1w} \\ & & N_{n+1} \end{array}$$

is commutative, so $h(x^*) = x_w$ and $x_w \in \mathrm{Im}(h)$ which is a contradiction. □

LEMMA 2.7. *Let $\mathcal{M}$ be closed under inverse limits. If $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ is a Tor-generator having the property stated in the previous lemma, then it is a maximal Tor-generator.*

The proof is the dual of the lemma 2.2.5 of [7], replacing generator by Tor-generator, minimal by maximal and changing the direction of arrows.

THEOREM 2.8. *Assume that $\mathcal{M}$ is closed under extensions and inverse limits. For a given R-module L, if $\mathcal{T}or(L, \mathcal{M})$ has a Tor-generator, then M admits an ${}^{\mathsf{T}}\mathcal{M}$-co(pre)cover whenever $L \in \mathcal{M}$.*

PROOF. By Theorem 2.4, we have a maximal Tor-generator $0 \longrightarrow M \longrightarrow K \longrightarrow L \longrightarrow 0$ for $\mathcal{T}or(L, \mathcal{M})$. By proposition 2.3, $K \in^{\mathsf{T}} \mathcal{M}$. Since $\mathcal{T}or(L, K') = 0$ for all $K' \in^{\mathsf{T}} \mathcal{M}$, tensoring the above exact sequence by K', gives an exact sequence. So $M \longrightarrow K$ is an ${}^{\mathsf{T}}\mathcal{M}$-coprecover. Now by maximality of tor-generator, we deduce that $M \longrightarrow K$ is a cocover. □

REMARK. A coprecover is called a cocover, if any endomorphism f of X with $\varphi = f\varphi$ always implies, f an auto morphism where φ is the same homomorphism in 1.3.

THEOREM 2.9. *Every R-module has a projective cocover.*

PROOF. If we set $\mathcal{M} = \mathcal{C}$ in Theorem 2.4, then ${}^{\mathsf{T}}\mathcal{C} = \mathcal{P}$. For any R-module L there is an exact sequence $0 \longrightarrow M \longrightarrow P \longrightarrow L \longrightarrow 0$ with $M \in \mathcal{C}$ and $P \in \mathcal{P}$. This exact sequence provides a Tor-generator for $\mathcal{T}or(L, \mathcal{M})$. So by Theorem 2.8, we are done. □

To prove the existence of cotorsion copre covers we just need the following propositions which are proved exactly in [4].

PROPOSITION 2.10. *Every R-module is a pure submodule of a pure injective R-module.*

PROPOSITION 2.11. *Every pure injective R-module is cotorsion.*

REMARK. An R-module M is cotorsion if $\mathrm{Ext}^1(F, M) = 0$ for all flat R-modules F. A submodule T of N is a pure submodule if $0 \longrightarrow A \otimes T \longrightarrow A \otimes N$ is exact for all R-module A. And M is pure injective if for every pure exact sequence $0 \longrightarrow T \longrightarrow N$ $\mathrm{Hom}(N, M) \longrightarrow \mathrm{Hom}(T, M) \longrightarrow 0$ is exact.

THEOREM 2.12. *Every R-module has a cotorsion coprecover.*

PROOF. Let M be an arbitrary R-module by proposition 2.10 there is a pure injective R-module N such that $0 \longrightarrow M \longrightarrow N$ is pure exact. Besides by proposition 2.11 N is cotorsion. Now if, $M \longrightarrow N'$ is a homomorphism with N' cotorsion, then $0 \longrightarrow M \otimes N' \longrightarrow N \otimes N'$ is exact. So $M \longrightarrow N$ is a coprecover of M. □

References

[1] K. D. Akinci and R. Alizadeh, *Special precovers in Cotorsion theoreis* Proc. Edinb. Math. Soc.(2) 2002.

[2] M. H. Bijan-Zadeh, *Torsion theories and local Cohomology over commutative Noetherian rings*, J. London Math. Soc.(2) **19** (1979), no. 2.

[3] M. P. Brodman and R. Y. Sharp, *Local cohomology*, Cammbridge University Press, 1998.

[4] E. E. Enochs and O. M. G. Jenda, *Relative Homological Algebra*, Walter de Gruyter, 2000.

[5] J. Trlifaj, *Covers, envelopes, and Cotorsion theoreis*, Lecture Notes for the Workshop, "Homological methods in modules theory", Cortona, 2000.

[6] J. Rotman, *Notes on Homological Algebras*, Reinhold, 1970.

[7] J. Xu, *Flat covers of modules*, Lecture Notes of Mathematics 1996, no. 1634.

M. Golriz
Science and Research Branch
Islamic Azad University
Iran
E-mail: Golriz@azad.ac.ir

M. H. Bijanzadeh
Faculty of Mathematical Sciences and Computer Engineering
Teacher Training University
Iran
E-mail: bijan@pnu.ac.ir