

CONVOLUTORS FOR THE SPACE OF FOURIER HYPERFUNCTIONS

KWANG WHOI KIM

ABSTRACT. We define the convolutions of Fourier hyperfunctions and show that every strongly decreasing Fourier hyperfunction is a convolutor for the space of Fourier hyperfunctions and the converse is true. Also we show that there are no differential operator with constant coefficients which have a fundamental solution in the space of strongly decreasing Fourier hyperfunctions. Lastly we show that the space of multipliers for the space of Fourier hyperfunctions consists of analytic functions extended to any strip in \mathbb{C}^n which are estimated with a special exponential function $\exp(\mu|x|)$.

0. Introduction

We introduced the following in [6].

Let $F_{(h,\nu)}$ be the space of continuously differentiable functions $\varphi(x)$ for which the norm

$$(0.1) \quad |\varphi|_{(h,\nu)} = \sup_{x \in \mathbb{R}^n, \alpha} \frac{|\partial^\alpha \varphi(x)| \exp(\nu|x|)}{h^{-|\alpha|} \alpha!}, \quad h > 0, \nu \in \mathbb{R}$$

is finite. Then the (continuous) embeddings

$$(0.2) \quad F_{(h,\nu)} \subset F_{(h',\nu')}, \quad h \geq h' > 0, \nu \geq \nu'$$

take place.

Received February 9, 2004.

2000 Mathematics Subject Classification: 46F15.

Key words and phrases: Fourier hyperfunction, convolution, convolution operator, convolutor, pseudodifferential operator, multiplier.

This work is supported by the Foundation of JeonJu University.

By virtue of (0.2), we can define the spaces \mathcal{G} , \mathcal{O} and \mathcal{M} with the aid of the operations of projective and inductive limits:

$$\begin{aligned}
 \mathcal{G} &= \bigcap_{h,\nu} F_{(h,\nu)}, \\
 (0.3) \quad \mathcal{O} &= \bigcup_{\nu} F_{(\infty,\nu)}, \quad F_{(\infty,\nu)} = \bigcap_{h>0} F_{(h,\nu)}, \\
 \mathcal{M} &= \bigcap_{h>0} F_{(h,-\infty)}, \quad F_{(h,-\infty)} = \bigcup_{\nu} F_{(h,\nu)}.
 \end{aligned}$$

Let the space $\mathcal{G}'(\mathcal{O}' \text{ resp.})$ be a space of continuous linear functionals on $\mathcal{G}(\mathcal{O} \text{ resp.})$. Since the embeddings (0.2) induce the adjoint embeddings

$$(0.4) \quad (F_{(h',\nu')})' \subset (F_{(h,\nu)})', \quad h \geq h' > 0, \nu \geq \nu'.$$

The space \mathcal{G}' regarded as a vector space coincides with the union of $(F_{(h,\nu)})'$:

$$(0.5) \quad \mathcal{G}' = \bigcup_{h,\nu} (F_{(h,\nu)})'.$$

The right-hand space can be equipped with the topology of inductive limit, and in the left-hand space we can introduce the topology of the strong conjugate space of \mathcal{G} .

Note that \mathcal{G}' is reflexive and regular inductive limit, which implies the coincidence of the two above-mentioned topologies in \mathcal{G}' . The regularity of \mathcal{G}' implies that for each bounded set $B \subset \mathcal{G}'$ there are real numbers h and μ such that $B \subset (F_{(h,\nu)})'$.

The space \mathcal{O}' regarded as a vector space can be identified with the projective limit of the conjugate spaces $(F_{(\infty,\nu)})'$. The latter, when treated as vector spaces, are identified with the inductive limits $\bigcup_h (F_{(h,\nu)})'$. Thus,

$$(0.6) \quad \mathcal{O}' = \bigcap_{\nu} \left(\bigcup_h (F_{(h,\nu)})' \right).$$

Let $H_{\langle \nu \rangle}$ ($\nu \in \mathbb{R}$) denote the space of measurable functions square integrable with weight $\exp(2\nu|x|)$. The corresponding norm is written as

$$(0.7) \quad \|f\|_{\langle \nu \rangle} = ((f, f)_{\langle \nu \rangle})^{1/2} = \|\exp(\nu|x|)f\|_{L^2}.$$

Then we see that

$$\mathcal{G} \subset F_{(h, \mu + \epsilon)} \subset H_{\langle \mu \rangle} \subset (F_{(h, -\mu + \epsilon)})' \subset \mathcal{G}', \quad \epsilon > 0.$$

Let $H_{\langle 0 \rangle} = H$. Then the mapping $f \rightarrow f \exp(\nu|x|)$ determines an isometric isomorphism of $H_{\langle \nu \rangle}$ onto H . It follows that $H_{\langle -\nu \rangle}$ and the Banach conjugate space of $H_{\langle \nu \rangle}$ are isometrically isomorphic, i.e.,

$$(0.8) \quad (H_{\langle \nu \rangle})' = H_{\langle -\nu \rangle}.$$

Consequently, $H_{\langle \nu \rangle}$ is a reflexive Banach space.

We can define in $H_{\langle \nu \rangle}$ the scalar product

$$(0.9) \quad (f, g)_{\langle \nu \rangle} = \int \exp(2\nu|x|) f(x) \overline{g(x)} dx$$

to which the norm (0.7) corresponds. In other words, $H_{\langle \nu \rangle}$ is a Hilbert space.

Since $\mathcal{G} \subset H_{\langle s \rangle} \subset \mathcal{G}'$ and the Fourier operator $\mathcal{F} : \mathcal{G}' \rightarrow \mathcal{G}'$ is one-to-one and transforms the subset $\mathcal{G} \subset \mathcal{G}'$ into itself, we denote by $H^{(s)}$ the image of $H_{\langle s \rangle}$ under the operator \mathcal{F}^{-1} :

$$H^{(s)} = \mathcal{F}^{-1} H_{\langle s \rangle}.$$

Since the composition of \mathcal{F} and \mathcal{F}^{-1} is an identity operator in \mathcal{G}' , we have

$$\mathcal{F} H^{(s)} = H_{\langle s \rangle}.$$

We introduce the norm

$$(0.10) \quad \|f\|^{(s)} = \|\mathcal{F}f\|_{\langle s \rangle}$$

in the space $H^{(s)}$. Thus, $H^{(s)}$ consists of those functions $f \in \mathcal{G}'$, possessing Fourier transforms, which are square summable and whose norm (0.10) is finite.

We now include the zeroth space $H_{\langle \nu \rangle}$, $\nu > 0$, in the scale $\{H_{\langle \nu \rangle}^{(s)}\}$ generated with respect to pseudodifferential operators. To this end we define the symbols

$$(0.11) \quad \delta_{s,N}(\zeta) = \exp\left(s \sum_{k=1}^n (N^2 + \zeta_k^2)^{1/2}\right).$$

Note that if $s \geq 0$, then for $|Im\zeta_k| < \nu < N$,

$$(0.12) \quad C_1 \exp(s/\sqrt{2}|\zeta|) \leq |\delta_{s,N}(\zeta)| \leq C_2 \exp(s|\zeta|).$$

We put

$$(0.13) \quad H_{\langle \nu \rangle}^{(s)} = \{f \in H^{(-\infty)} \mid \delta_{s,N}(D)f \in H_{\langle \nu \rangle}, 0 \leq \nu < N\}$$

and equip this space with the norm

$$(0.14) \quad \|f\|_{\langle \nu \rangle}^{(s)} = \|\delta_{s,N}(D)f\|_{\langle \nu \rangle}, 0 \leq \nu < N.$$

We can introduce in this space a scalar product (to which the norm (0.14) corresponds):

$$(0.15) \quad (f, g)_{\langle \nu \rangle}^{(s)} = (\delta_{s,N}(D)f, \delta_{s,N}(D)g)_{\langle \nu \rangle}, 0 \leq \nu < N.$$

Since

$$(0.16) \quad F_{(h, \nu + \epsilon)} \subset H_{\langle \nu \rangle}^{(h/2)} \subset F_{(h/4, \nu)}, h, \nu, \epsilon > 0,$$

we have

$$(0.17) \quad \mathcal{G} = \bigcap_{h, \nu} H_{\langle \nu \rangle}^{(h)}.$$

For $\nu > 0$ we define $H_{\langle -\nu \rangle}^{(-s)}$, $s \in \mathbb{R}$ as the Banach conjugate space of $H_{\langle \nu \rangle}^{(s)}$ and introduce in it the norm of a conjugate space:

$$(0.18) \quad \|f\|_{\langle -\nu \rangle}^{(-s)} = \sup\{|(f, g)| \mid \|g\|_{\langle \nu \rangle}^{(s)} \leq 1\}.$$

Then we have

$$(0.19) \quad F_{(h, -\nu + \epsilon)} \subset H_{\langle -\nu \rangle}^{(h/2)} \subset F_{(h/4, -\nu)}, h, \nu > 0.$$

Since the spaces $H_{\langle \nu \rangle}^{(s)}$, $\nu \geq 0$, form a scale, the spaces $H_{\langle -\nu \rangle}^{(-s)}$ form the dual scale, and we can consider the inductive limits $H_{\langle \nu \rangle}^{(-\infty)}$, $H_{\langle -\nu \rangle}^{(-\infty)}$, and endow them with the natural topology. By virtue of the reflexivity of

$H_{\langle \nu \rangle}^{(s)}$, these limits are regular, and, according to the general properties of regular inductive limits, we have the topological isomorphisms

$$(0.20) \quad \begin{aligned} (H_{\langle \nu \rangle}^{(\infty)})' &= H_{\langle -\nu \rangle}^{(-\infty)}, \\ \mathcal{G}' &= \bigcup_{s, \nu} H_{\langle \nu \rangle}^{(s)}, \end{aligned}$$

where the left-hand and right-hand spaces are equipped with topologies of strong conjugate space and inductive limits, respectively.

(0.16) and (0.19) implies

$$(0.21) \quad \begin{aligned} \mathcal{O} &= \bigcup_{\nu} H_{\langle \nu \rangle}^{(\infty)}, \quad \mathcal{O}' = \bigcap_{\nu} H_{\langle \nu \rangle}^{(-\infty)}, \\ \mathcal{M} &= \bigcap_s H_{\langle -\infty \rangle}^{(s)}. \end{aligned}$$

In this paper, making use of the same method as in [3], we research properties of the convolutions of Fourier hyperfunctions in the spaces $H_{\langle \nu \rangle}^{(s)}$.

In Section 2, we show that the convolution of analytic functions estimated with exponential functions $\exp(\nu_i|x|)$, $i = 1, 2$, $\nu_1 > \nu_2 > 0$ is defined and associative (Proposition 2.1 and 2.3), and that every Fourier hyperfunction can be expressed as the form $f = \delta_{s,N}(D)f_0$, $f_0 \in F_{(h,\nu)}$ (Lemma 2.4). Also we show that the operator of convolution defined originally on the dense subset $\mathcal{G} \times \mathcal{O} \subset \mathcal{O}' \times \mathcal{G}'$ is continued by continuity to a mapping of $\mathcal{O}' \times \mathcal{G}'$ (Proposition 2.5). Defining the convolution of Fourier hyperfunctions as in the proof of Proposition 2.5 we show the following (Proposition 2.8):

$$\begin{aligned} \mathcal{O}' * \Phi &\subset \Phi, \quad \Phi = \mathcal{G}, \mathcal{G}', \mathcal{O}, \mathcal{O}', \\ \mathcal{G} * \mathcal{G}' &\subset \mathcal{O}, \quad \mathcal{O} * \mathcal{O}' \subset \mathcal{O}. \end{aligned}$$

In Section 3, we introduce the definition of convolution operator and show that when $\Phi = \mathcal{G}, \mathcal{G}', \mathcal{O}, \mathcal{O}'$, for each convolution operator $A : \Phi \rightarrow \Phi$ there is a strongly decreasing Fourier hyperfunction $f \in \mathcal{O}'$ such that

$$A\varphi = f * \varphi, \quad \varphi \in \Phi.$$

Also we show that there are no differential operator with constant coefficients which have a fundamental solution in the space \mathcal{O}' .

Lastly we show that the space of multipliers for the space of Fourier hyperfunctions consists of analytic functions extended to any strip in \mathbb{C}^n which are estimated with a special exponential function $\exp(\mu|x|)$.

1. Preliminaries

We introduce some theorems and propositions to need in this paper which are founded in [6].

THEOREM 1.1. $f(x) \in F_{(h,\nu)}$ if and only if $f(x)$ can be continued holomorphically to the tube domain $D_h = \{x + yi \in \mathbb{C}^n \mid |y_j| < h, j = 1, 2, \dots, n\}$ such that

$$(1.1) \quad |f(x + yi)| \leq C \exp(-\nu|x|).$$

Let $\nu > 0$. Let $F^{(\nu,s)}$ denote the Banach space of functions $\psi(\zeta)$ holomorphic in the tube domain D_ν and having a finite norm

$$(1.2) \quad |\psi|^{(\nu,s)} = \sup_{\zeta \in D_\nu} \exp(s|\zeta|)|\psi(\zeta)|.$$

PROPOSITION 1.2. The map $F_{(h,\nu)} \rightarrow F^{(h,\nu)} : f(x) \rightarrow f(x + yi)$ is a topological isomorphism and there are constants $C_1, C_2 > 0$ such that

$$(1.3) \quad C_1|f|^{(h,\nu)} \leq |f|_{(h,\nu)} \leq C_2|f|^{(h,\nu)}.$$

REMARK. From Proposition 1.2 we see that

$$(1.4) \quad \mathcal{M} = \bigcap_{\nu > 0} F^{(\nu, -\infty)}.$$

THEOREM 1.3. The Fourier-Laplace transform $\mathcal{F} : F_{(h,\nu)} \rightarrow F^{(\nu,h)} : \varphi(x) \rightarrow \hat{\varphi}(\zeta)$ is a topological isomorphism.

THEOREM 1.4. The Fourier-Laplace transform operator determines an isomorphism

$$(1.5) \quad \mathcal{FG} = \bigcap_{h,\nu > 0} F^{(\nu,h)}.$$

PROPOSITION 1.5. (i) \mathcal{M} is a commutative algebra relative to multiplication;

(ii) \mathcal{FG} is an ideal in \mathcal{M} . i.e., the operation of multiplication

$$\mathcal{M} \times \mathcal{G} \rightarrow \mathcal{G} ((a(\zeta), \psi(\zeta)) \rightarrow a(\zeta)\psi(\zeta))$$

is defined.

THEOREM 1.6. The following isomorphisms of vector spaces hold:

$$(1.6) \quad \mathcal{FO}' = \mathcal{M},$$

where \mathcal{F} is a Fourier-Laplace operator.

PROPOSITION 1.7. For $s \geq 0$ we have

$$(1.7) \quad F_{(\nu,s)} \simeq F^{(\nu,s)} \subset F_{(s)}^{<\nu>} \subset F^{(\nu,s/\sqrt{2})} \simeq F_{(\nu,s/\sqrt{2})},$$

where $F_{(s)}^{<\nu>}$ is the space of holomorphic functions in the tube domain D_ν and having a finite norm:

$$|\psi|_{(s)}^{<\nu>} = \sup_{\zeta \in D_\nu} |\delta_{s,N}(\zeta)| |f(\zeta)|.$$

PROPOSITION 1.8. ([3]) Suppose that $A_0 : \mathcal{G} \rightarrow \mathcal{G}$ be a linear operator and that for each ν there exists a number ν' and for each s there exists a number s' such that the inequality

$$(1.8) \quad \|A_0\varphi\|_{<\nu>}^{(s')} \leq C\|\varphi\|_{<\nu'>}^{(s)}, \varphi \in \mathcal{G}$$

is satisfied. Then A_0 is the restriction to \mathcal{G} of a regular operator $A : \mathcal{O}' \rightarrow \mathcal{O}'$.

3. Convolution in the space of Fourier hyperfunctions

If f and g are continuous functions and the expressions $f(x - y)g(y)$ and $g(y)f(x - y)$ regarded as functions of y are absolutely integrable for each $x \in \mathbb{R}^n$, then the operation of convolution

$$(2.1) \quad \begin{aligned} (f * g)(x) &= \int f(x - y)g(y)dy \\ &= \int f(y)g(x - y)dy \end{aligned}$$

is defined for them, and $f * g = g * f$, i.e., the operation is commutative.

PROPOSITION 2.1. Let $f \in F_{(h_1, \nu)}$ and $g \in F_{(h_2, \mu)}$, and let $\nu > |\mu|$. Then $f * g = g * f \in F_{(h, \mu)}$, $h = \min\{h_1, h_2\}$ and

$$(2.2) \quad |f * g|_{(h, \mu)} \leq \text{const} |f|_{(h_1, \nu)} |g|_{(h_2, \mu)}.$$

Proof. It follows from Theorem 1.1 and Proposition 1.2 that for $|y_j| < h$

$$\begin{aligned} & |(f * g)(x + yi)| \\ & \leq \int |f(x + yi - t)g(t)| dt \\ & \leq C |f|_{(h_1, \nu)} |g|_{(h_2, \mu)} \exp(-\mu|x|) \int \exp(\mu|x| - \nu|x - t| - \mu|t|) dt \\ & \leq C' |f|_{(h_1, \nu)} |g|_{(h_2, \mu)} \exp(-\mu|x|) \int \exp(-(\nu - |\mu|)|x - t|) dt \\ & \leq C'' |f|_{(h_1, \nu)} |g|_{(h_2, \mu)} \exp(-\mu|x|). \end{aligned}$$

This completes the proof. \square

COROLLARY 2.2. We have

$$(2.3) \quad \mathcal{G} * \Phi \subset \Phi, \quad \Phi = \mathcal{G}, \mathcal{O}, F_{(\infty, \mu)}.$$

PROPOSITION 2.3. For $h_1 > h_2 > h_3 > 0$ and $\nu_1 > \nu_2 > \nu_3$, let $f_i \in F_{(h_i, \nu_i)}$, $i = 1, 2, 3$. Then

$$(2.4) \quad f_1 * (f_2 * f_3) = (f_1 * f_2) * f_3.$$

Proof. By virtue of Proposition 2.1, all the convolutions exist and all the integrals involved in them are absolutely convergent. According to the definition of the convolution, (2.4) implies that for every $x \in \mathbb{R}^n$

$$\int f_1(x - y) \int f_2(y - z) f_3(z) dz dy = \int \int f_1(x - y - z) f_2(z) f_3(z) dz dy.$$

On the basis of Fubini's theorem, we can replace the repeated integrals by double integrals. Making change of variable $y \rightarrow y - z$ in the right-hand integral we obtain (2.4). \square

By pseudodifferential operators (*PDO*) in \mathcal{G} are meant operators having the form

$$(2.5) \quad (a(D)\varphi)(x) = (2\pi)^{-n/2} \int \exp(i \langle x, \xi \rangle) a(\xi) \hat{\varphi}(\xi) d\xi.$$

If $f \in \mathcal{G}'$ and $a(\xi) \in \mathcal{M}$, then we define by $a(D)f$ the functional

$$(2.6) \quad (a(D)f, \varphi) = (f, a(-D)\varphi), \quad \varphi \in \mathcal{G}.$$

Fix $\nu > 0$. Each function $a(\zeta) \in F^{(\nu, -\infty)}$ is a multiplier on the space $F^{(\nu', \infty)}$, $\nu' < \nu$ and for $\varphi \in F_{(\infty, \nu')}$ we can define the *PDO*

$$a(D)\varphi = (2\pi)^{-n/2} \int \exp(i \langle \xi + i\Gamma, x \rangle) a(\xi + i\Gamma) \hat{\varphi}(\xi + i\Gamma) d\xi, \quad \Gamma \in \nu' I.$$

Then Theorem 1.1 and 1.3 imply that $a(D) : F_{(\infty, \nu')} \rightarrow F_{(\infty, \nu')}$ is continuous.

LEMMA 2.4.

- (i) For any $f \in \mathcal{G}'$ there are positive numbers h, k and $\nu \in \mathbb{R}$ such that $f = \delta_{k, N}(D)g$, $g \in F_{(h, \nu)}$.
- (ii) If $f \in \mathcal{O}'$, then for every $\nu > 0$ there are positive numbers h, s such that $f = \delta_{s, N}(D)g_\nu$, $g_\nu \in F_{(h, \nu)}$.

Proof. (i) According to (0.20), for $f \in \mathcal{G}'$ there are real numbers σ, ν such that $f \in H_{<\nu>}^{(\sigma)}$.

Since for any s, μ and r the mapping $H_{<\mu>}^{(s)} \rightarrow H_{<\mu>}^{(s-r)} : f \rightarrow \delta_{r, N}(D)f$ is an isometric isomorphism, there exists a positive number k such that $\delta_{-k, N}(D)f \in H_{<\nu>}^{(\sigma+k)}$, $\sigma + k > 0$. According to (0.16) and (0.19) f can be represented in the form $f = \delta_{k, N}(D)f_0$, $f_0 \in F_{(h, \nu)}$, $h = (\sigma + k)/2$.

Taking account of (0.21), (ii) also follows from (0.16) and (0.19). \square

PROPOSITION 2.5. *The operation of convolution defined originally on the dense subset $\mathcal{G} \times \mathcal{O} \subset \mathcal{O}' \times \mathcal{G}'$ is continued by continuity to a mapping of $\mathcal{O}' \times \mathcal{G}'$ into \mathcal{G}' .*

Proof. We first of all give a constructive definition of convolution for $f \in \mathcal{G}'$ and $g \in \mathcal{O}'$. From Lemma 2.4 it follows that there are positive numbers h_1 , k and $\mu \in \mathbb{R}$ such that $f = \delta_{k,N}(D)f_0$, $f_0 \in F_{(h_1,\mu)}$.

Similarly, for $\nu > |\mu|$, there are positive numbers h_2 , s such that $g = \delta_{s,N}(D)g_0$, $g_0 \in F_{(h_2,\nu)}$.

Define by the convolution $f * g$ as follows:

$$(2.7) \quad f * g = \delta_{k+s,N}(D)(f_0 * g_0).$$

Then (2.7) depends on f and g but does not depend on the way f and g are represented.

Indeed, if $f = \delta_{k',N}(D)f'_0$, $g = \delta_{s',N}(D)g'_0$, then by (2.7)

$$f_0 * g_0 = \delta_{s'-s+k'-k,N}(D)(f'_0 * g'_0).$$

Substituting this into (2.7) we conclude that the left-hand side does not change when k , f_0 , s and g_0 are replaced by k' , f'_0 , s' and g'_0 , respectively.

If $f \in F_{(h_1,\mu)}$ and $g \in F_{(h_2,\nu)}$ in (2.7), then we can put $k = s = 0$, i.e., we arrive at the definition (2.7).

We now show that

$$\{f_j \rightarrow f \text{ in } \mathcal{G}', g_j \rightarrow g \text{ in } \mathcal{O}'\} \Rightarrow \{f_j * g_j \rightarrow f * g \text{ in } \mathcal{G}'\}.$$

It follows from the definition of convergence in \mathcal{G}' that there are h_1 , k and μ such that we have

$$f_{0j} = \delta_{-k,N}(D)f_j \rightarrow \delta_{-k,N}(D)f = f_0 \in F_{(h_1,\mu)}.$$

Similarly, given $\nu > |\mu|$, there are h_2 and $s_\nu = s$ such that

$$g_{0j} = \delta_{-s,N}(D)g_j \rightarrow \delta_{-s,N}(D)g = g_0 \in F_{(h_2,\nu)}.$$

Taking equality (2.7) into consideration we obtain

$$f_j * g_j = \delta_{k,N}(D)f_{0j} * \delta_{s,N}(D)g_{0j} = \delta_{k+s,N}(D)(f_{0j} * g_{0j}).$$

According to Proposition 2.1, the sequence $f_{0j} * g_{0j}$ converges in $F_{(h,\mu)}$, $h = \min\{h_1, h_2\}$, which implies the convergence of $f_j * g_j$ in \mathcal{G}' . \square

REMARK. Repeating literally the argument in the proposition we prove that the convolution is continued by continuity from $\mathcal{O} \times \mathcal{G}$ to $\mathcal{G}' \times \mathcal{O}'$ and

$$(2.8) \quad f * g = g * f, \quad f \in \mathcal{G}', \quad g \in \mathcal{O}'.$$

We have thus proved that

$$(2.9) \quad \mathcal{G}' * \mathcal{O}' \subset \mathcal{G}', \quad \mathcal{O}' * \mathcal{G}' \subset \mathcal{G}'.$$

The presented considerations imply

PROPOSITION 2.6. *If $f \in F_{(h,\nu)}$ and $g \in H_{<\mu>}^{(s)}$ for $h > 0$, $\nu > |\mu|$, then the convolution $f * g$ belongs to $H_{(\mu-\epsilon)}^{(s)}$, $\epsilon > 0$ and*

$$(2.10) \quad \|f * g\|_{<\mu-\epsilon>}^{(s)} \leq \text{const} \|f\|_{(h,\nu)} \|g\|_{<\mu>}^{(s)}.$$

Proof. Since

$$\begin{aligned} & \exp((\mu - \epsilon)|x|) |\delta_{s,N}(D)(f * g)| \\ & \leq \exp(-\epsilon|x|) \int \exp(\mu|x|) |f(x-t)| |\delta_{s,N}(D)g(t)| dt \\ & \leq \exp(-\epsilon|x|) \|f\|_{(h,\nu)} \exp(\mu|x| - \mu|t| - \nu|x-t|) \|g\|_{<\mu>}^{(s)}, \end{aligned}$$

it completes the proof. □

PROPOSITION 2.7. *Let $f \in H_{<\nu>}^{(s)}$ and $g \in H_{<\mu>}^{(t)}$ for $\nu > |\mu|$. Then the convolution $f * g$ belongs to $H_{(\mu-\epsilon)}^{(s+t)}$, $\epsilon > 0$ and*

$$(2.11) \quad \|f * g\|_{<\mu-\epsilon>}^{(s+t)} \leq \text{const} \|f\|_{<\nu>}^{(s)} \|g\|_{<\mu>}^{(t)}.$$

Proof. Since

$$\begin{aligned} & \exp((\mu - \epsilon)|x|) |\delta_{s+t,N}(D)(f * g)| \\ & \leq \exp(-\epsilon|x|) \|\exp(\mu|x| - \mu|t| - \nu|x-t|) \\ & \quad \times \exp(\nu|x-t|) \delta_{s,N}(D)f(x-t)\| \|g\|_{<\mu>}^{(t)} \\ & \leq \exp(-\epsilon|x|) \|\exp(\nu|x-t|) \delta_{s,N}(D)f(x-t)\| \|g\|_{<\mu>}^{(t)}, \end{aligned}$$

it completes the proof. \square

We also note that the constructed operation of convolution is associative and commutative:

$$(2.12) \quad (f * g) * \varphi = g * (f * \varphi) = f * (g * \varphi),$$

where either $f \in \mathcal{G}'$, $g \in \mathcal{O}'$, $\varphi \in \mathcal{O}'$ or $f \in \mathcal{O}'$, $g \in \mathcal{O}'$, $\varphi \in \mathcal{G}'$.

To prove (2.12) we approximate f, g , and φ with analytic functions, apply (2.7), and make use of Propositions 2.1 and 2.3.

Note that $A : \mathcal{G} \rightarrow \mathcal{G}$ ($\mathcal{G}' \rightarrow \mathcal{G}'$) is a continuous if and only if for every s, μ there exist numbers s', μ' such that

$$\begin{aligned} \|A\varphi\|_{\langle \mu \rangle}^{(s)} &\leq C \|\varphi\|_{\langle \mu' \rangle}^{(s')}, \quad \varphi \in \mathcal{G} \\ (\|A\varphi\|_{\langle \mu' \rangle}^{(s')} &\leq C \|\varphi\|_{\langle \mu \rangle}^{(s)}, \quad \varphi \in H_{\langle \mu \rangle}^{(s)}). \end{aligned}$$

A subset $B \subset \mathcal{O}$ will be called regularly bounded if, for some μ , B is contained in $H_{\langle \mu \rangle}^{(\infty)}$ and is bounded in this space, i.e., for each $s > 0$ there is a constant K_s such that

$$\|\varphi\|_{\langle \mu \rangle}^{(s)} \leq K_s$$

for every $\varphi \in B$.

A linear operator $A : \mathcal{O} \rightarrow \mathcal{O}$ is called regular if it transforms each regularly bounded set in \mathcal{O} into a regularly bounded set in \mathcal{O} .

Note that A linear operator $A : \mathcal{O} \rightarrow \mathcal{O}$ is regular if and only if for each μ there exists a number μ' and for each s there exists a number s' such that

$$\|A\varphi\|_{\langle \mu' \rangle}^{(s)} \leq C \|\varphi\|_{\langle \mu \rangle}^{(s')}$$

for every $\varphi \in H_{\langle \mu \rangle}^{(\infty)}$.

A set $B \subset \mathcal{O}'$ is said to be bounded if for each μ there exist numbers s_μ, K_μ such that

$$\|\varphi\|_{\langle \mu \rangle}^{(s_\mu)} \leq K_\mu$$

for every $\varphi \in B$.

A linear operator $A : \mathcal{O}' \rightarrow \mathcal{O}'$ is said to be regular if for each μ there exists a number μ' and for each s there exists a number s' such that

$$\|A\varphi\|_{\langle \mu \rangle}^{(s')} \leq C \|\varphi\|_{\langle \mu' \rangle}^{(s)}$$

for every $\varphi \in H_{\langle \mu' \rangle}^{(s)}$.

PROPOSITION 2.8.

(i) *The operator*

$$(2.13) \quad \text{con}_f : \Phi \rightarrow \Phi(\varphi \rightarrow f * \varphi), \quad f \in \mathcal{O}'$$

is continuous for $\Phi = \mathcal{G}$, \mathcal{G}' and is regular for $\Phi = \mathcal{O}$, \mathcal{O}' .

(ii) *The operators*

$$\text{con}_f : \mathcal{G}' \rightarrow \mathcal{O} \quad (f \in \mathcal{G}), \quad \text{con}_f : \mathcal{O}' \rightarrow \mathcal{O} \quad (f \in \mathcal{O})$$

transform bounded sets in \mathcal{G}' , \mathcal{O}' into regularly bounded sets in \mathcal{O} .

Taking account of (0.16), (0.19), (0.20) and (0.21), all the assertions of Proposition 2.8 are consequences of (2.7) and Proposition 2.7.

3. Convolutors in the spaces \mathcal{G} , \mathcal{G}' , \mathcal{O} , \mathcal{O}'

There is a well-known relationship between the convolution and the Fourier operator. Namely, if $f, g \in L_1$, then

$$(3.1) \quad (\mathcal{F}(f * g))(\xi) = (2\pi)^{n/2} \hat{f}(\xi) \hat{g}(\xi).$$

This relation also remains valid when $f, g \in \mathcal{O}'$. Indeed, by Lemma 2.4, for every $\mu > 0$ there are positive numbers t and h_2 such that $g = \delta_{t,N}(D)g_0$, $g_0 \in F_{(h_2,\mu)}$, and for $\nu > \mu$ there are positive numbers s, h_1 such that $f = \delta_{s,N}(D)f_0$, $f_0 \in F_{(h_1,\nu)}$. Then it follows from Proposition 2.1 that $f_0 * g_0 \in F_{(h,\mu)} \subset L_1$, $h = \min\{h_1, h_2\}$. If we define $f * g$ with the aid of (2.7), we obtain

$$(2\pi)^{-n/2} (\mathcal{F}(f * g))(\xi) = \delta_{s+t,N}(\xi) \hat{f}_0(\xi) \hat{g}_0(\xi) = \hat{f}(\xi) \hat{g}(\xi).$$

The operator $\text{con}_f : \Phi \rightarrow \Phi$, $f \in \mathcal{O}'$, $\Phi = \mathcal{G}$, \mathcal{O} , is a *PDO* with the symbol $(2\pi)^{n/2} \hat{f}(\xi)$, i.e.,

$$\text{con}_f = (2\pi)^{n/2} \hat{f}(D), \quad f \in \mathcal{O}'.$$

By the definition of Dirac's delta function $\delta(x)$,

$$(\hat{\delta}, \varphi) = (\delta, \hat{\varphi}) = \hat{\varphi}(0) = (2\pi)^{-n/2} \int \varphi(x) dx = (2\pi)^{-n/2} (1, \varphi)$$

i.e., $\hat{\delta} = (2\pi)^{-n/2}$, whence $\hat{\delta} \in \mathcal{M}$. Therefore from Theorem 1.6 we can see that $\delta(x) \in \mathcal{O}'$. It follows from (3.1) that

$$f * \delta = \delta * f = f$$

for $f \in \mathcal{O}'$. This relation is continued by continuity to \mathcal{G}' , i.e., $\text{con}_\delta : \mathcal{G}' \rightarrow \mathcal{G}'$ is an identity operator.

Since

$$(\delta(x-h), \hat{\varphi}(x)) = \hat{\varphi}(h) = (2\pi)^{-n/2} \int \exp(-i \langle h, \xi \rangle) \varphi(\xi) d\xi, \quad \varphi \in \mathcal{G},$$

the Fourier transform of $\delta(x-h)$ is equal to $(2\pi)^{-n/2} \exp(-i \langle h, \xi \rangle)$, and hence

$$(3.2) \quad \text{con}_{\delta(x-h)} = \exp(-i \langle h, D \rangle).$$

Note that the translation operator $\tau_h : \Phi \rightarrow \Phi$ is continuous for $\Phi = F_{(h,\nu)}, F_{(\infty,\nu)}, \mathcal{G}$ and is regular for $\Phi = \mathcal{O}$. From the definition of the Fourier operator it follows that

$$(\mathcal{F}(\tau_h \varphi))(\xi) = \exp(-i \langle h, \xi \rangle) \hat{\varphi}(\xi), \quad \varphi \in \mathcal{G}.$$

Comparing this with (3.2) we conclude that the translation operator τ_h coincides with the *PDO* (3.2) on \mathcal{G} .

Using the translation operator, the convolution (2.1) can be defined by means of the relation

$$(3.3) \quad (f * g)(x) = (f, \tau_x I g).$$

If $g \in \Phi = \mathcal{G}, \mathcal{O}$, then $\tau_x I g \in \Phi$, and the right-hand side of (3.3) makes sense for any Fourier hyperfunction $f \in \Phi'$.

On the other hand, according to Proposition 2.8 (ii), the left-hand side of (3.3) also exists for $f \in \Phi'$ (and belongs to \mathcal{O}).

PROPOSITION 3.1.

(i) Let $f \in \mathcal{O}'$ and $\Phi = \mathcal{G}, \mathcal{O}$. Then the operators

$$\text{con}_f : \Phi \rightarrow \Phi, \text{con}_{I f} : \Phi' \rightarrow \Phi'$$

are mutually adjoint relative to the canonical duality of Φ and Φ' .

(ii) Let $f \in \mathcal{G}'$. Then the operators

$$\text{con}_f : \mathcal{O}' \rightarrow \mathcal{G}', \text{con}_{I f} : \mathcal{G} \rightarrow \mathcal{O}$$

are mutually adjoint.

Proof. Both the assertions follow from

$$(3.4) \quad (f * g, \varphi) = (g, If * \varphi),$$

where either (a) $f \in \mathcal{O}'$, $g \in \mathcal{G}'$, \mathcal{O}' , $\varphi \in \mathcal{G}$, \mathcal{O} or (b) $f \in \mathcal{G}'$, $g \in \mathcal{O}'$, $\varphi \in \mathcal{G}$.

The case (ii) follows from (i) in view of (2.8). To prove (3.4) for the case (i) we note a useful formula obtained :

$$(f, \varphi) = (f * I\varphi)(0).$$

Applying (2.12) we find

$$\begin{aligned} (f * g, \varphi) &= ((f * g) * I\varphi)(0) = (g * (f * I\varphi))(0) \\ &= (g * (I(If * \varphi)))(0) = (g, If * \varphi). \end{aligned}$$

□

A continuous (regular) operator $A : \Phi \rightarrow \Phi$, $\Phi = \mathcal{G}$, \mathcal{G}' ($\Phi = \mathcal{O}$, \mathcal{O}') commutable with any *PDO* whose symbol belongs to $F^{(N, -\infty)}$ is called a convolution operator.

Note that the convolution operator A commutes the translation operator.

THEOREM 3.2. *Let $\Phi = \mathcal{G}$, \mathcal{G}' , \mathcal{O} , \mathcal{O}' . Then for each convolution operator $A : \Phi \rightarrow \Phi$ there is a strongly decreasing Fourier hyperfunction $f \in \mathcal{O}'$ such that*

$$(3.5) \quad A\varphi = \text{con}_f(\varphi) = f * \varphi, \quad \varphi \in \Phi.$$

In particular, each convolution operator on $\Phi = \mathcal{G}$, \mathcal{O} is representable in the form

$$(3.6) \quad (A\varphi)(x) = (f, \tau_x I\varphi), \quad \varphi \in \Phi, \quad f \in \mathcal{O}'.$$

We begin with proving that each convolution operator on \mathcal{G} and \mathcal{O} can be represented as (3.6), and we indicate the functional f .

PROPOSITION 3.3. *Let $\Phi = \mathcal{G}$, \mathcal{O} . For each convolution operator $A : \Phi \rightarrow \Phi$ there is a Fourier hyperfunction $f \in \Phi'$ such that A is represented in the form (3.6).*

Proof. We associate with A a family of linear functionals:

$$(3.6') \quad (f_x, \tau_x I\varphi) = (A\varphi)(x).$$

Since the translation performs one-to-one mapping of Φ into itself, the functional (3.6') is defined throughout the space Φ . The proposition will be proved if we show that

- (i) f_x is a continuous linear functional, i.e., $f_x \in \Phi'$;
- (ii) the functional f_x does not depend on x : $f_x = f \in \Phi'$;

To prove (i) we note that the topologies in \mathcal{G} and \mathcal{O} are stronger than the topology of pointwise convergence, and the operator $A : \Phi \rightarrow \Phi$ is continuous (the continuity of $A : \mathcal{O} \rightarrow \mathcal{O}$ follows from the regularity of A).

(ii) follows from the commutability of A with translations. Indeed, for $\psi \in \Phi$ we have

$$\begin{aligned} (f_x, \psi) &= (f_x, \tau_x I\tau_x I\psi) := (A\tau_x I\psi)(x) \\ &= \tau_x (AI\psi)(x) = (AI\psi)(0). \end{aligned}$$

□

We note that the proposition implies the validity of the theorem for $\Phi = \mathcal{O}$; as to $\Phi = \mathcal{G}$, in this case the theorem reduces to the following assertion:

$$(3.7) \quad \{f \in \mathcal{G}', f * \varphi \in \mathcal{G} \mid \varphi \in \mathcal{G}\} \Rightarrow \{f \in \mathcal{O}'\}.$$

Let Φ be a space of analytic functions invariant with respect to translations and reflections. Formula (3.6) makes it possible to introduce the convolution for $f \in \Phi'$ and $\varphi \in \Phi$. A Fourier hyperfunction $f \in \Phi'$ is called a convolutor if $f * \varphi \in \Phi$ for all $\varphi \in \Phi$. The set of convolutors will be denoted $\mathcal{C}(\Phi)$.

PROPOSITION 3.4.

- (i) Each convolution operator $A_0 : \Phi \rightarrow \Phi$ ($\Phi = \mathcal{G}, \mathcal{O}$) is continued by continuity to a convolution operator $A : \Psi \rightarrow \Psi$, where $\Psi = \mathcal{O}', \mathcal{G}'$, respectively.
- (ii) Let $A : \Psi \rightarrow \Psi$ ($\Psi = \mathcal{O}', \mathcal{G}'$) be a convolution operator. Then its restriction A_0 to the subspace $\Phi = \mathcal{G}, \mathcal{O}$ is a convolution operation on Φ .

Proof. (i) If $\Phi = \mathcal{G}$, the continuity of $A = \text{con}_f$, $f \in \mathcal{G}'$, implies that for every s, ν there exist numbers $s' = s'(s, \nu)$, $\nu' = \nu'(s, \nu)$ such that

$$(3.8) \quad \|f * \varphi\|_{\langle \nu \rangle}^{(s)} \leq K_{s\nu} \|\varphi\|_{\langle \nu' \rangle}^{(s')}.$$

In particular, for $s = 0$ we have

$$\|f * \varphi\|_{\langle \nu \rangle} \leq K_\nu \|\varphi\|_{\langle \lambda(\nu) \rangle}^{(\sigma(\nu))}.$$

Replacing φ by $\delta_{s,N}(D)\varphi$ we find

$$\begin{aligned} \|f * \varphi\|_{\langle \nu \rangle}^{(s)} &= \|f * \delta_{s,N}(D)\varphi\|_{\langle \nu \rangle} \\ &\leq K_\nu \|\delta_{s,N}(D)\varphi\|_{\langle \lambda(\nu) \rangle}^{(\sigma(\nu))} \\ &= K_\nu \|\varphi\|_{\langle \lambda(\nu) \rangle}^{(s+\sigma(\nu))}, \end{aligned}$$

whence, replacing s by $s - \sigma(\nu)$ we find

$$\|f * \varphi\|_{\langle \nu \rangle}^{(s-\sigma(\nu))} \leq \text{const} \|\varphi\|_{\langle \lambda(\nu) \rangle}^{(s)}, \quad \varphi \in \mathcal{G}.$$

Using Proposition 1.8 and the denseness \mathcal{G} in \mathcal{O}' we conclude that the operator con_f is continued by continuity to a regular operator on \mathcal{O}' .

If $\Phi = \mathcal{O}$. then, by Proposition 3.3, $A = \text{con}_f$, $f \in \mathcal{O}'$, and the required assertion follows from Proposition 2.5.

(ii) We note that a convolution operator $A : \Psi \rightarrow \Psi$ commutes with any *PDO* $\delta_{s,N}(D)$.

The regularity of $A : \mathcal{O}' \rightarrow \mathcal{O}'$ implies that for each ν there exist a number ν' and for each s there exist a number s' such that

$$\|A\varphi\|_{\langle \nu' \rangle}^{(s')} \leq \text{const} \|\varphi\|_{\langle \nu \rangle}^{(s)}.$$

Similarly, the continuity of $A : \mathcal{G}' \rightarrow \mathcal{G}'$ implies that for every s, ν there exist numbers s', ν' such that

$$\|A\varphi\|_{\langle \nu' \rangle}^{(s')} \leq \text{const} \|\varphi\|_{\langle \nu \rangle}^{(s)}.$$

In particular, for $s = 0$,

$$\|A\varphi\|_{\langle \nu \rangle}^{(\sigma(\nu))} \leq \text{const} \|\varphi\|_{\langle \lambda(\nu) \rangle}, \quad \|A\varphi\|_{\langle \lambda(\nu) \rangle}^{(\sigma(\nu))} \leq \text{const} \|\varphi\|_{\langle \nu \rangle}.$$

Replacing φ by $\delta_{s,N}(D)\varphi$, we find

$$\begin{aligned} \|A\varphi\|_{\langle \nu \rangle}^{(s+\sigma(\nu))} &= \|\delta_{s,N}(D)A\varphi\|_{\langle \nu \rangle}^{(\sigma(\nu))} = \|A\delta_{s,N}(D)\varphi\|_{\langle \nu \rangle}^{(\sigma(\nu))} \\ &\leq \text{const}\|\delta_{s,N}(D)\varphi\|_{\langle \lambda(\nu) \rangle} = \text{const}\|\varphi\|_{\langle \lambda(\nu) \rangle}^{(s)}, \\ \|A\varphi\|_{\langle \lambda(\nu) \rangle}^{(s+\sigma(\nu))} &= \|\delta_{s,N}(D)A\varphi\|_{\langle \lambda(\nu) \rangle}^{(\sigma(\nu))} = \|A\delta_{s,N}(D)\varphi\|_{\langle \lambda(\nu) \rangle}^{(\sigma(\nu))} \\ &\leq \text{const}\|\delta_{s,N}(D)\varphi\|_{\langle \nu \rangle} = \text{const}\|\varphi\|_{\langle \nu \rangle}^{(s)}, \end{aligned}$$

whence replacing s by $s - \sigma(\nu)$ we find

$$\begin{aligned} \|A\varphi\|_{\langle \nu \rangle}^{(s)} &\leq \text{const}\|\varphi\|_{\langle \lambda(\nu) \rangle}^{(s-\sigma(\nu))}, \\ \|A\varphi\|_{\langle \lambda(\nu) \rangle}^{(s)} &\leq \text{const}\|\varphi\|_{\langle \nu \rangle}^{(s-\sigma(\nu))}, \end{aligned}$$

meaning that the restriction of A to $\mathcal{G}(\mathcal{O})$ is continuous (regular). \square

In view of Proposition 3.4, we put

$$(3.10) \quad \mathcal{C}(\mathcal{O}') = \mathcal{C}(\mathcal{G}), \quad \mathcal{C}(\mathcal{G}') = \mathcal{C}(\mathcal{O}).$$

By Proposition 2.8,

$$\mathcal{O}' \subset \mathcal{C}(\Phi), \quad \Phi = \mathcal{G}, \quad \mathcal{O}, \quad \mathcal{O}', \quad \mathcal{G}'.$$

From the definition of $\mathcal{C}(\Phi)$, we see $\mathcal{C}(\mathcal{O}) \subset \mathcal{O}'$, and therefore

$$\mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{G}') = \mathcal{O}'.$$

Further, since the Dirac delta function $\delta(x)$ belongs to \mathcal{O}' , we have $\mathcal{C}(\mathcal{O}') \subset \mathcal{O}'$, whence

$$\mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{O}') = \mathcal{O}'.$$

Hence, Theorem 3.2 is proved, and we see that

$$(3.11) \quad \mathcal{C}(\mathcal{G}) = \mathcal{C}(\mathcal{O}') = \mathcal{C}(\mathcal{O}) = \mathcal{C}(\mathcal{G}') = \mathcal{O}'.$$

THEOREM 3.5. *Let $\Phi = \mathcal{G}, \mathcal{G}', \mathcal{O}, \mathcal{O}'$ and let $A \in \mathcal{C}(\Phi) = \mathcal{O}'$. The following two conditions are equivalent.*

(I) *For any $f \in \Phi$ the convolution equation*

$$(3.12) \quad A * u = f$$

possesses a unique solution $u \in \Phi$.

(II) *Equation (3.12) has a fundamental solution $G \in \mathcal{C}(\Phi) = \mathcal{O}'$:*

$$(3.13) \quad A * G = G * A = \delta(x).$$

Proof. (I) \Rightarrow (II). For $\Phi = \mathcal{O}'$ this assertion is a tautology since $\delta(x) \in \mathcal{O}'$. In the case $\Phi = \mathcal{G}, \mathcal{G}', \mathcal{O}$, according to the Banach inverse operator theorem, which holds for Fréchet spaces and their inductive limits, the condition (I) is equivalent to the existence of a continuous operator

$$(3.14) \quad (\text{con}_A)^{-1} : \Phi \rightarrow \Phi, \quad \Phi = \mathcal{G}, \mathcal{G}', \mathcal{O}.$$

Since the operator con_A commutes with any *PDO* whose symbol belongs to $F^{(N, -\infty)}$, the operator (3.14) possesses the same property. Consequently, $(\text{con}_A)^{-1}$ is a convolution operator on $\mathcal{G}, \mathcal{G}'$, and \mathcal{O} . By Theorem 3.2, there is an element G of \mathcal{O}' such that $(\text{con}_A)^{-1} = \text{con}_G$.

By the definition of an inverse operator,

$$(A * (G * f))(x) = ((G * A) * f)(x) = f(x), \quad f \in \mathcal{G}.$$

Putting $x = 0$ and using (2.12) we obtain

$$(A * G, If) = (G * A, If) = f(0), \quad f \in \mathcal{G},$$

i.e., (3.13) is fulfilled.

(II) \Rightarrow (I). If $G \in \mathcal{O}'$ satisfies (3.13), then $G * f$ is a solution to equation (3.13) since $A * (G * f) = (A * G) * f = \delta * f = f$. Further, if $A * u = 0$, then $0 = G * (A * u) = (G * A) * u = \delta * u = u$. i.e., Equation (3.13) possesses no more than one solution. \square

The operator con_A , $A \in \mathcal{O}'$, has a symbol $(2\pi)^{n/2} \hat{A}(\zeta) \in \mathcal{M}$, and (II) is equivalent to the following condition

$$(II') \quad \frac{1}{\hat{A}(\zeta)} \in \mathcal{M}.$$

REMARK. According to (1.4), this condition is equivalent to the property that for every $\nu > 0$ there are real numbers K_ν, s_ν such that the estimate from below

$$(3.15) \quad |\hat{A}(\zeta)| \geq K_\nu \exp(s_\nu |\zeta|), \quad \zeta \in \bar{T}_\nu$$

holds. If con_A is a differential operator, i.e., $A(x) = P(D)\delta(x)$, where $P(\zeta)$ is a polynomial, i.e., $\hat{A}(\zeta) = (2\pi)^{-n/2}P(\zeta)$, then it follows from (3.15) that $P(\zeta) \neq 0$, whence $P(\zeta) \equiv \text{const}$. Therefore there are no differential operators with constant coefficients satisfying the conditions of Theorem 3.5.

An entire function $a(\zeta)$ is called a multiplier on the space $\mathcal{G} \simeq \mathcal{FG} = \bigcap_{h, \nu > 0} F^{(\nu, h)}$ if $a\varphi \in \mathcal{G}$, for all $\varphi \in \mathcal{G}$. The multipliers form a commutative algebra which we denote $\mathfrak{M}(\mathcal{G})$. Each element $a \in \mathfrak{M}(\mathcal{G})$ generates an operator in the space \mathcal{G}' :

$$(af, \varphi) = (f, a\varphi), \quad f \in \mathcal{G}', \quad \varphi \in \mathcal{G},$$

and, by definition, we put $\mathfrak{M}(\mathcal{G}) = \mathfrak{M}(\mathcal{G}')$.

It follows from Proposition 1.5 that $\mathcal{M} \subset \mathfrak{M}(\mathcal{G})$. If $a \in \mathfrak{M}(\mathcal{G})$, then the *PDO* $a(D)$ is defined throughout \mathcal{G} , transforms onto \mathcal{FG} , and commutes with any *PDO* whose symbol belongs to $F^{(N, -\infty)}$. This operator is continuous:

$$a(D)\varphi = \mathcal{F}_{\zeta \rightarrow x}^{-1} a(\zeta) \mathcal{F}_{x \rightarrow \zeta} \varphi(x)$$

i.e., it is a convolution operator. According to Theorem 3.2, there is $f \in \mathcal{O}'$ such that

$$a(D)\varphi = f * \varphi.$$

It follows that $a(\zeta)\hat{\varphi}(\zeta) = (2\pi)^{n/2}\hat{f}(\zeta)\hat{\varphi}(\zeta)$. Since $\varphi \in \mathcal{G}$ is arbitrary, it follows from Theorem 1.6 that

$$a(\zeta) = (2\pi)^{n/2}\hat{f}(\zeta) \in \mathcal{M}.$$

Therefore we obtain

PROPOSITION 3.6. $\mathfrak{M}(\mathcal{G}) = \mathfrak{M}(\mathcal{G}') = \mathcal{M}$.

ACKNOWLEDGMENTS. The author expresses sincere thanks to Professor D. Kim at Seoul National University. He suggested to research the structure of Fourier hyperfunctions making use of the same method as in [3].

References

- [1] S.-Y. Chung, D. Kim, and S. K. Kim, *Structure of the extended Fourier hyperfunctions*, Japan. J. Math. **19** (1993), no. 2, 217–226.
- [2] I. M. Gel'fand and G. E. Shilov, *Generalized functions Vol. 2*, Acad. Press New York and London, 1968.
- [3] S. G. Gindikin and L. R. Volevich, *Distributions and Convolution Equations*, Gordon and Breach Sci. Publ., 1992.
- [4] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag Berlin New York, 1969.
- [5] A. Kaneko, *Introduction to hyperfunctions*, KTK Sci. Publ./Tokyo, 1992.
- [6] K. W. Kim, *The space of Fourier hyperfunctions as an inductive limit of Hilbert space*, Preprint.
- [7] K. W. Kim, S.-Y. Chung, and D. Kim, *Fourier hyperfunctions as the boundary values of smooth solutions of heat equations*, Publ. RIMS Kyoto Univ. **29** (1993), no. 2, 289–300.
- [8] H. Komatsu, *Introduction to the theory of generalized functions*, Iwanami Sheoten, Tokyo, 1978. (Jananeses)
- [9] S. G. Krantz and H. R. Parks, *A Primer of real analytic functions*, Birkhauser Verlag, 1992.
- [10] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean spaces*, Princeton Univ. Press, 1975.
- [11] M. E. Taylor, *Pseudodifferential operators*, Princeton Univ. Press, 1981.
- [12] F. Trèves, *Topological vector spaces, distributions and kernels*, Acad. Press New York and London, 1967.
- [13] K. Yosida, *Functional analysis*, Spriger-Verlag Berlin New York, 1980.

Department of Mathematics Education
JeonJu University
JeonJu 560–759, Korea
E-mail: kwkim@jj.ac.kr