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FRACTIONAL INTEGRAL ALONG HOMOGENEOUS CURVES IN THE HEISENBERG GROUP

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ABSTRACT. We obtain the type set for the fractional integral operator along the curve $(t, t^2, \alpha t^3)$ on the three dimensional Heisenberg group when $\alpha \neq \pm 1/6$. The proof is based on the Fourier inversion formula and the angular Littlewood-Paley decompositions in the Heisenberg group in [5].

1. Introduction

Let us consider the fractional integral along a curve with $0 < \sigma \leq 1$:

$$\mathcal{A}_d^\sigma f(x) = \int_0^\infty f(x - (t, \cdots, t^d)) t^{\sigma - 1} dt.$$

The type set of $\mathcal{A}_d^\sigma$ is the set of points $(1/p, 1/q)$ such that $\mathcal{A}_d^\sigma$ maps $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$. Let $ABCD$ be the closed trapezoid with vertices

$$A = (0,0),\ B = (1,1),\ C = (2/3, 1/2), \text{and } D = (1/2, 1/3).$$

For each $\sigma > 0$ and $d \in \mathbb{Z}^+$, let l_d^σ be the line on $\mathbb{R}^2$-plane given by

$$l_d^\sigma = \{(u,v) : u - v = \frac{\sigma}{1 + 2 + \cdots + d}\}.$$

In [6] and [8], it is known that the type set of $\mathcal{A}_3^\sigma$ is given by

$$ABCD \cap l_3^\sigma.$$

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The purpose of this paper is to investigate the type set for the operator

$$(1.1)\qquad \mathcal{S}^{\sigma}_{\alpha}f(x) = \int_0^{\infty} f(x\cdot(t,t^2,\alpha t^3)^{-1})t^{\sigma-1}dt,$$

where $\alpha\in\mathbb{R}$ and $\cdot$ is the group multiplication on the 3 dimensional Heisenberg group $\mathbb{H}^1$ defined by

$$(x_1,x_2,x_3)\cdot(y_1,y_2,y_3) = (x_1+y_1, x_2+y_2, x_3+y_3+\frac{1}{2}(x_1y_2-x_2y_1)).$$

For a finite interval I on $\mathbb{R}$, we define an average operator:

$$(1.2)\qquad \mathcal{T}_{\alpha}f = \int_I f(x\cdot(t,t^2,\alpha t^3)^{-1})dt.$$

In the theorems 5.2 and 5.3 of [7], S. Secco obtains the typeset of $\mathcal{T}_\alpha$ for $\alpha\in\mathbb{R}$.

THEOREM 1. *Let $\mathcal{T}_\alpha$ be the operator defined in* (1.2). *If $\alpha\neq\pm1/6$, then the type set of $\mathcal{T}_\alpha$ is the closed trapezoid $ABCD$.*

For the case $\alpha=\pm1/6$, the type set is strictly contained in $ABCD$.

THEOREM 2. *Let $\mathcal{T}_\alpha$ be the operator defined in* (1.2).
(i) *If $\alpha=1/6$, then the type set of $\mathcal{T}_\alpha$ is the closed triangle ABC.*
(ii) *If $\alpha=-1/6$, then the type set of $\mathcal{T}_\alpha$ is the closed triangle ABD.*

From the result of Theorem 1 and the fact that with some $C_\sigma>0$

$$(1.3)\qquad \mathcal{T}_\alpha f(x)\le C_\sigma \mathcal{S}^\sigma_\alpha f(x),$$

we see that the type set of $\mathcal{S}^\sigma_\alpha$ is contained in $ABCD$ when $\alpha\neq\pm1/6$. From Theorem 2 and (1.3), the type set of $\mathcal{S}^\sigma_{1/6}$ is contained in ABC and $\mathcal{S}^\sigma_{-1/6}$ is contained in ABD. Let us define a dilation in $\mathbb{R}^3$ by $\delta x=(\delta x_1,\delta^2x_2,\delta^3x_3)$ with $\delta>0$. Then we have the dilation invariance under our group multiplication such as $\delta(x\cdot y)=\delta x\cdot\delta y$. This implies that $(\mathcal{S}^\sigma_\alpha f)_\delta=\delta^\sigma\mathcal{S}^\sigma_\alpha(f_\delta)$ where $g_\delta(x)=g(\delta x)$ for a function g defined on $\mathbb{H}^1$. If the operator norm $\|\mathcal{S}^\sigma_\alpha\|_{L^p\to L^q}$ is finite, then we have

$$\begin{aligned}
\delta^{-(1+2+3)/q}\,\|(\mathcal{S}^\sigma_\alpha f)\|_{L^q(\mathbb{H}^1)} &= \|(\mathcal{S}^\sigma_\alpha f)_\delta\|_{L^q(\mathbb{H}^1)}=\delta^\sigma\,\|\mathcal{S}^\sigma_\alpha f_\delta\|_{L^q(\mathbb{H}^1)}\\
(1.4)\qquad &\le \|\mathcal{S}^\sigma_\alpha\|_{L^p\to L^q}\,\delta^\sigma\,\|f_\delta\|_{L^p(\mathbb{H}^1)}\\
&= \|\mathcal{S}^\sigma_\alpha\|_{L^p\to L^q}\,\delta^{\sigma-(1+2+3)/p}\,\|f\|_{L^p(\mathbb{H}^1)}\,.
\end{aligned}$$

In order to satisfy the above inequality with both small and large δ, we need the condition $1/p - 1/q = \sigma/(1+2+3)$. Therefore from Theorems 1,2 and (1.3)-(1.4), it follows that

(i) the type set for $\mathcal{S}_\alpha^\sigma$ is contained in ABCD $\cap\, \mathrm{l}_3^\sigma$ $(\alpha \neq \pm 1/6)$.

(ii) the type set of $\mathcal{S}_{1/6}^\sigma$ is contained in ABC $\cap\, \mathrm{l}_3^\sigma$.

(iii) the type set of $\mathcal{S}_{-1/6}^\sigma$ is contained in ABD $\cap\, \mathrm{l}_3^\sigma$.

In this paper we show that $ABCD \cap l_3^\sigma$ is the type set for the operator $\mathcal{S}_\alpha^\sigma$ when $\alpha \neq \pm 1/6$. The type set of $\mathcal{S}_\alpha^\sigma$ with each $\alpha = \pm 1/6$ is not known. However we show in Section 4 that neither $ABC \cap l_3^\sigma$ nor $ABD \cap l_3^\sigma$ is the type set of each $\mathcal{S}_{1/6}^\sigma$ or $\mathcal{S}_{-1/6}^\sigma$.

MAIN THEOREM. *Let $\mathcal{S}_\alpha^\sigma$ be the operator defined in* (1.1). *If $\alpha \neq \pm 1/6$, then the type set of $\mathcal{S}_\alpha^\sigma$ is the line segment $ABCD \cap l_3^\sigma$.*

REMARK. (1) It is conjectured that the type set of the translation invariant operator $\mathcal{A}_d^\sigma$ is $ABC_dD_d \cap l_d^\sigma$ where $A = (0,0), B = (1,1), C_d = (\frac{d^2-d+2}{d(d+1)}, \frac{d-1}{d+1})$ and $D_d = (\frac{2}{d+1}, \frac{d-2}{d^2-d+2})$. This conjecture is known for $d \leq 3$. For $d \geq 4$, M. Christ has obtained the $L^p \to L^q$ bound except the endpoints C_d and D_d for the average operator along that curve in [2]. This proves the conjecture for any $d \geq 4$ except those endpoints C_d and D_d, see in [4] how we obtain the bound of $\mathcal{A}_d^\sigma$ from the bound of the average operator on the finite interval.
(2) In [7], S. Secco has obtained the typeset for more general curves satisfying some curvature and torsion properties.

The proof of $L^p(\mathbb{R}^3) \to L^q(\mathbb{R}^3)$ bounededness for $\mathcal{A}_3^\sigma$ is based on the decomposition of the frequency variables of convolution kernel into several angular sectors. Some of those sectors are estimated by using the Hardy-Littlewood-Sobolev theorem, and the others are handled by the Littlewood-Paley inequality and the interpolation argument made in [1]. In applying this standard argument on the Heisenberg group, we use the Fourier inversion formula on the Hisenberg group and the angular Littledwood-Paley decomposition developed in [5]. In Section 2 we discuss about these two basic tools. In Section 3, we show how we apply these tools for the proof of Main Theorem when $\alpha \neq \pm 1/6$. In

Section 4 we discuss about the case $\alpha = \pm 1/6$.

NOTATIONS. We write $*$ for the convolution on $\mathbb{H}^1$, and $\mathfrak{F}$ or $\mathfrak{F}^{-1}$ denotes the Fourier transform or its inverse for the given Euclidean space. Given two quantities a and b, we write $a \lesssim b$ or $b \gtrsim a$ if there is a constant $C > 0$ such that $a \leq Cb$. If $a \lesssim b$ and $b \lesssim a$, then we write $a \approx b$.

2. Background on the Heisenberg group

The Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n} \times \mathbb{R}^1$ endowed with the group multiplication:

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + \frac{1}{2}(x \cdot y' - y \cdot x'))$$

for $x, y, x', y' \in \mathbb{R}^n$ and $t, t' \in \mathbb{R}^1$. The left invariant vector fields on $\mathbb{H}^n$ are defined as for each $1 \leq j \leq n$,

$$X_j = \frac{\partial}{\partial x_j} - \frac{y_j}{2}\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} + \frac{x_j}{2}\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}.$$

Then $\{X_j, Y_j, T\}$ forms a basis of the Lie algebra $\mathfrak{h}_n$ corresponding to the Lie group $\mathbb{H}^n$. The canonical commutation relation in $\mathfrak{h}$ is given by $[X_j, Y_k] = \delta_{j,k} T$ with all the other commutators vanished. In [1], it is proved that the operator with a little less singular convolution kernel $|(x, y)|^{\sigma - 2n}\delta(t)$ maps $L^p(\mathbb{H}^n)$ to $L^q(\mathbb{H}^n)$ if and only if $1/p - 1/q = \sigma/(2n + 2)$ and $(1/p, 1/q)$ belong to ABC which is the closed triangle with vertices $(0, 0), (1, 1)$ and $(\frac{2n+1}{2n+2}, \frac{1}{2n+2})$. The essential part for the proof of this theorem is the Littlewood-Paley theory for the Laplacian $\mathcal{L} = -\sum_{j=1}^n (X_j^2 + Y_j^2)$ and T. The commutativity of these two vector fields makes it possible to use spectral calculus for the computation of dyadic decomposition of $\mathcal{L}$, T and the mixed one. However our convolution kernel is not radial, so we need to make the Littlewood-Paley decompositions for noncommutative vector fields X_j, Y_j, T (with $j = 1$ since $n = 1$ for our case) instead of $\mathcal{L}$ and T.

2.1. Group Fourier transform

For each $\lambda \in \mathbb{R}^1$, we define one parameter Shrödinger representation by a mapping R^λ from the Heisenberg group $\mathbb{H}^n$ to the group of unitary operators on $L^2(\mathbb{R}^n)$ such that for $h \in L^2(\mathbb{R}^n)$ and $(p,q,t) \in \mathbb{H}^n$,

$$[R^\lambda(p,q,t)h](x) = e^{2\pi i\lambda[q\cdot x+p\cdot q/2+t]}h(x+p).$$

Let $\mathbf{B}(L^2(\mathbb{R}^n))$ be the space of bounded operators on $L^2(\mathbb{R}^n)$. The group Fourier transform of $f \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$ is defined as an operator-valued function from $\mathbb{R}^1$ to $\mathbf{B}(L^2(\mathbb{R}^n))$ such that $\lambda \in \mathbb{R}^1 \mapsto \widehat{f}(\lambda) \in \mathbf{B}(L^2(\mathbb{R}^n))$, given by

$$[\widehat{f}(\lambda)h](x) = \int_{\mathbb{H}^n} [R^\lambda(-p,-q,-t)h](x) f(p,q,t)\,dpdqdt. \tag{2.1}$$

From (2.1), $\widehat{f}(\lambda)$ is an integral operator on $L^2(\mathbb{R}^n)$ given by

$$[\widehat{f}(\lambda)h](x) = \int_{\mathbb{R}^n} \mathfrak{F}^{2,3} f(x-y, \lambda(x+y)/2, \lambda)h(y)\,dy, \tag{2.2}$$

where $\mathfrak{F}^{2,3}$ is the Euclidean Fourier transform with respect to the second and third component of f. From the definition of (2.1), we can show

$$\widehat{k * f}(\lambda) = \widehat{k}(\lambda) \cdot \widehat{f}(\lambda), \tag{2.3}$$

where the multiplication on the right is the composition of operators. From (2.1) and (2.3), we can prove the Fourier inversion formula such as

$$f(p,q,t) = \int_{\mathbb{R}^1} tr\big(\widehat{f}(\lambda) \cdot R^\lambda(p,q,t)\big)|\lambda|^n d\lambda. \tag{2.4}$$

Here $tr(T)$ denotes the trace of the operator T. If $Tf(x)$ is given by $\int L(x,y)f(y)dy$, then $tr(T) = \int L(x,x)dx$. For further study of the group Fourier transform, see [3] and [9].

2.2. Dyadic decomposition

Choose an even function $\psi \in C_0^\infty(-1,1)$ such that $\psi \equiv 1$ on $[-1/2,1/2]$. Put $\chi(\xi) = \psi(\xi/2) - \psi(\xi)$ and define

$$L_j^2(y) = 2^{2j}[\mathfrak{F}^{-1}\chi](2^{2j}y_2)\delta(y_1,y_3),$$
$$L_j^3(y) = 2^{3j}[\mathfrak{F}^{-1}\chi](2^{3j}y_3)\delta(y_1,y_2)$$

where δ is the Dirac measure at 0 in $\mathbb{R}^2$. In [5], it is shown that L_j^2 and L_j^3 are convolution kernels of the dyadic decompositions for Y_1 and T. They satisfy the Littlewool-Paley type inequalities:

LEMMA 1. *For $1 < p < \infty$, and $\nu = 2, 3$,*

$$\left\| \left(\sum_j |L_j^\nu * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{H}^1)} \lesssim \| f \|_{L^p(\mathbb{H}^1)} .$$

2.3. Angular decomposition

Put $\Omega(\xi, \eta) = \chi(\xi/\eta)$ and define three measures for each $j \in \mathbb{Z}$:

$$A_k^1(y_1, y_2, y_3) = 2^{5k}[\mathfrak{F}^{-1}\Omega](2^{2k}y_2, 2^{3k}y_3)\delta(y_1),$$
$$A_k^2(y_1, y_2, y_3) = 2^{3k}[\mathfrak{F}^{-1}\Omega](2^{k}y_1, 2^{2k}y_2)\delta(y_3 - y_1y_2/2),$$
$$A_k^3(y_1, y_2, y_3) = 2^{4k}[\mathfrak{F}^{-1}\Omega](2^{k}y_1, 2^{3k}y_3)\delta(y_2).$$

In [5], it is also proved that A_k^ν is a convolution kernel of the angular decomposition for each pair of X_1, Y_1, T. We can see that $\mathfrak{F}(A_k^1)(\xi_1, \xi_2, \xi_3)$ is supported on $|\xi_2/\xi_3| \approx 2^{-k}$. They also satisfy the following Littlewood-Paley type inequalities:

LEMMA 2. *For $1 < p < \infty$ and for $\nu = 1, 2, 3$,*

$$\left\| \left(\sum_{j=-\infty}^{\infty} |A_j^\nu * f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{H}^1)} \lesssim \| f \|_{L^p(\mathbb{H}^1)} .$$

Let $\epsilon_\alpha = 1 - 3(\alpha + 1/2)$ and $\epsilon_\alpha^* = 1 - 3(-\alpha + 1/2)$. Then we choose $n_\alpha, m_\alpha \in \mathbb{Z}$ such that $2^{n_\alpha - 1} < \min\{|\epsilon_\alpha|, |\epsilon_\alpha^*|\} \leq 2^{n_\alpha}$ and $2^{m_\alpha - 1} < |\epsilon_\alpha| + 1 \leq 2^{m_\alpha}$. Note that we can choose n_α when $\alpha \neq \pm 1/6$. Let us define three subsets of $\mathbb{Z}$ so that

$$H(1, j) = \{k \in \mathbb{Z} : 2^{n_\alpha - m_\alpha - j - 5} \leq 2^{-k} \leq 2^{m_\alpha - j + 5}\}$$
$$H(2, j) = \{k \in \mathbb{Z} : 2^{-k} > 2^{m_\alpha - j + 5}\}$$
$$H(3, j) = \{k \in \mathbb{Z} : 2^{-k} < 2^{n_\alpha - m_\alpha - j - 5}\}.$$

For fixed $\alpha \neq \pm 1/6$, let us define three measures by using A_k^1's for each $\nu = 1, 2, 3$:

$$E_j^\nu(y_1, y_2, y_3) = \sum_{k \in H(\nu, j)} A_k^1(y_1, y_2, y_3).$$

We have the following vector valued inequalities for the operator $\{E_j^\nu * f_j\}$ on $L^p(l^2)$ in [5]:

LEMMA 3. *We have for each* $\nu = 1,2,3$, *with* $1 < p < \infty$

$$\| \Big(\sum_{j=-\infty}^{\infty} |E_j^\nu * f_j|^2 \Big)^{\frac{1}{2}} \|_{L^p(\mathbb{H}^1)} \lesssim \| \Big(\sum_{j=-\infty}^{\infty} |f_j|^2 \Big)^{\frac{1}{2}} \|_{L^p(\mathbb{H}^1)} .$$

All measures defined in this section are dilation invariant in the sense that $U_j(y_1,y_2,y_3) = 2^{6j}U_0(2^j y_1, 2^{2j}y_2, 2^{3j}y_3)$ for $U_j = L_j^\nu,\ A_j^\nu,\ E_j^\nu$, or the convolution of these meaures.

3. Proof of main theorem with $\alpha \neq \pm 1/6$

Let $\omega(t)$ be a smooth function on $\mathbb{R}^+$ and such that $\sum_{j\in\mathbb{Z}} \omega(2^j t) = 1$ for $t > 0$. For each fixed σ, put $\varphi_j(t) = \omega(2^j t)t^{\sigma-1}$. We define the convolution kernel K_j^α of the operator S_j^α by

$$< K_j^\alpha, f >= \int f(t,t^2,\alpha t^3)\varphi_j(t)dt.$$

Then $S_\alpha^\sigma f = \sum_j S_j^\alpha f = \sum_j K_j^\alpha * f$, and each convolution kernel K_j^α is scaled so that:

$$K_j^\alpha(x_1,x_2,x_3) = 2^{(6-\sigma)j}K_0^\alpha(2^j x_1, 2^{2j}x_2, 2^{3j}x_3).$$

By using (2.2), the integral kernel of the operator $\widehat{K_j^\alpha}(\lambda)\mathfrak{F}^{-1}$ can be written as,

$$\int e^{-2\pi i(\lambda(x-y)^2(x+y)/2+\lambda\alpha(x-y)^3-\xi y)}\varphi_j(x-y)dy. \tag{3.1}$$

The size of the derivative of phase function is determined by $\xi, \lambda y, \lambda$ which can be considered as the three frequency variables $\mathbb{H}^1$. These three frequency variables are controlled by the convolution with L_j^ν's or A_j^ν's. For example we can observe that by using the computation formula (2.2) and (2.4), each of $\widehat{L_j^\nu}(\lambda)$ determines the range of the frequency variable λy ($\nu = 2$) or λ ($\nu = 3$) in (3.1). Moreover $\widehat{A_j^\nu}(\lambda)$ controls the ratio $\lambda y/\lambda$ ($\nu = 1$), $\xi/\lambda y$ ($\nu = 2$) or ξ/λ ($\nu = 3$) in (3.1).

For each $\nu = 2, 3$, we put $[L_j^\nu]_0 = \sum_{l=-\infty}^{10} L_{j+l}^\nu$. Fix σ, and split

$$\begin{aligned} K_j^\alpha &= K_j^\alpha * [L_j^2]_0 * [L_j^3]_0 + K_j^\alpha * (\delta - [L_j^2]_0 * [L_j^3]_0) \\ &= K_j^{loc} + K_j^{glo}. \end{aligned}$$

With $\nu = 0, 1, 2, 3$, we set $\mathcal{R}^\nu = \sum_j R_j^\nu$ such that

$$\begin{aligned} R_j^0 f &= K_j^{loc} * f, \\ R_j^\nu f &= K_j^{glo} * E_j^\nu * f. \end{aligned}$$

Note that the group Fourier transform of E_j^ν splits the range of $(\xi, \lambda y, \lambda)$ in (3.1) into three sectors

$$\begin{aligned} \frac{2^{-5}\min\{|\epsilon_\alpha|, |\epsilon_\alpha^*|\}}{1+|\epsilon_\alpha|}|\lambda 2^{-j}| \le |\lambda y| \le 2^5(1+|\epsilon_\alpha|)|\lambda 2^{-j}| &\quad \text{when } \nu = 1, \\ |\lambda y| \ge 2^5(1+|\epsilon_\alpha|)|\lambda 2^{-j}| &\quad \text{when } \nu = 2, \\ |\lambda y| \le \frac{2^{-5}\min\{|\epsilon_\alpha|, |\epsilon_\alpha^*|\}}{1+|\epsilon_\alpha|}|\lambda 2^{-j}| &\quad \text{when } \nu = 3. \end{aligned}$$

For the proof of Main Theorem, we show that there exists a constant $C > 0$ such that for each fixed σ, $\alpha \neq \pm 1/6$ and $\nu = 0, 1, 2, 3$,

$$(3.2) \quad \| \mathcal{R}^\nu \|_{L^q(\mathbb{H}^1)} \le C(1 + \| \mathcal{S}_\alpha^\sigma \|_{L^p \to L^q}^{1-\theta}) \| f \|_{L^p(\mathbb{H}^1)} \quad \text{for } \theta \in (0,1).$$

3.1. Proof of (3.2) for $\nu = 0$

Let $\phi_2(\xi) = \psi(\xi/2^{20})$ and $\phi_3(\xi) = \psi(\xi/2^{30})$. Then by the straightforward computation, $K_0^{loc}(x_1, x_2, x_3)$ is written as

$$(3.3) \quad \varphi_0(x_1)(\mathfrak{F}^{-1}\phi_2)(x_2 - x_1^2)(\mathfrak{F}^{-1}\phi_3)(x_3 - x_1^3 - x_1x_2/2).$$

Let $\rho(x_1, x_2, x_3) = (|x_1|^6 + |x_2|^3 + |x_3|^2)^{\frac{1}{6}}$. Then from (3.3) we have for large $N > 0$,

$$(3.4) \quad |K_0^{loc}(x_1, x_2, x_3)| \le C_N(1 + \rho(x_1, x_2, x_3))^{-N}.$$

Hence,

$$\begin{aligned} \sum_j |K_j^{loc}(x_1, x_2, x_3)| &= \sum_j 2^{(6-\sigma)j} K_0^{loc}(2^j x_1, 2^{2j} x_2, 2^{3j} x_3) \\ &\lesssim \rho(x_1, x_2, x_3)^{-6+\sigma}. \end{aligned}$$

By applying the Hardy-Littlewood-Sobolev theorem with $\frac{1}{p}-\frac{1}{q}=\frac{\sigma}{6}$, we have

$$\| \mathcal{R}^0 * f \|_{L^q(\mathbb{H}^1)} \lesssim \| f \|_{L^p(\mathbb{H}^1)} .$$

3.2. Proof of (3.2) for $\nu = 2,3$

Let us decompose the support of the integral in (3.1) into two sectors $\{|\xi| \gg |\lambda y| \text{ or } |\xi| \ll |\lambda y|\}$ and $\{|\xi| \approx |\lambda y|\}$ for $\nu = 2$. For $\nu = 3$, the two sectors are $\{|\xi| \gg (|\epsilon_\alpha|+1)|\lambda| \text{ or } |\xi| \ll \min\{|\epsilon_\alpha|, |\epsilon_\alpha^*|\}|\lambda|\}$ and $\{|\xi| \approx |\lambda|\}$. For this decomposition we define two subsets of $\mathbb{Z}$:

$$\begin{aligned} M(2,j) &= \{k\in\mathbb{Z} : 2^{-j-5} \le 2^{-k} \le 2^{-j+5}\} \\ M(3,j) &= \{k\in\mathbb{Z} : 2^{n_\alpha-2j-5} \le 2^{-2k} \le 2^{m_\alpha-2j+5}\}. \end{aligned}$$

By using $M(\nu,j)$ with $\nu=2,3$, let us define measures:

$$\begin{aligned} G_j^\nu &= \sum_{k\in M(\nu,j)} A_k^\nu, \\ W_j^\nu &= \delta - G_j^\nu, \end{aligned}$$

where δ is a dirac mass at 0. Split for each $\nu = 2,3$

$$K_j^{glo} * E_j^\nu = K_j^{glo} * E_j^\nu * W_j^\nu + K_j^{glo} * E_j^\nu * G_j^\nu. \tag{3.5}$$

Let us start with the first term above. In order to apply the Hardy-Littlewood-Sobolev theorem for the kernel $\sum_j K_j^{glo} * E_j^\nu * W_j^\nu$ as (3.4), we need to show that

$$|K_0^{glo} * E_0^\nu * W_0^\nu(x_1,x_2,x_3)| \le C_N(1+\rho(x_1,x_2,x_3))^{-N}. \tag{3.6}$$

Proof of (3.6). Let $\Theta_j(u,v,\lambda) = \varphi_j(u)(1-\phi_2(2^{-2j}v)\phi_3(2^{-3j}\lambda))$. By using (2.2) and (2.3), we can compute the integral kernel $\mathcal{K}_j^\nu(x,z)$ of the group Fourier transform $\widehat{K_j^{glo} * E_j^\nu * W_j^\nu}(\lambda)$: For $\nu = 2$,

$$\begin{aligned} &\mathcal{K}_j^\nu(x,z) \\ &= \iint e^{-2\pi\lambda(x-y)^2(x+y)/2+\alpha(x-y)^3-\xi(y-z)}\Theta_j(x-y,\lambda y,\lambda) \\ &\quad \times \sum_{k\in H(\nu,j)} \chi(\frac{\lambda y}{2^{-k}\lambda}) \sum_{k\in M(\nu,j)^c} \chi(\frac{\xi}{2^{-k}\lambda y}) dy d\xi. \end{aligned} \tag{3.7}$$

For $\nu = 3$, the last sum of the above integrand is replaced by the sum $\sum_{k\in M(\nu,j)^c} \chi(\frac{\xi}{2^{-2k}\lambda})$. Let $P_{p,q,t}(z,w)$ be the integral kernel of the operator $R^\lambda(p,q,t)$. By the Fourier inversion formula in (2.4), $K_0^{glo} * E_0^\nu * W_0^\nu(p,q,t)$ is written as $\iiint \mathcal{K}_0^\nu(x,z)P_{p,q,t}(z,x)dzdx|\lambda|d\lambda$. This integral is computed as

$$(3.8) \qquad \iiiint e^{2\pi i[\Psi(p,q,t,u,\xi,v,\lambda)]}M^\nu(u,\xi,v,\lambda)dud\xi dvd\lambda,$$

where the oscillatory term $\Psi(p,q,t,u,\xi,v,\lambda)$ given by

$$\xi(p-u)+v(q-u^2)+\lambda(t-\frac{pq}{2}+qu-(\frac{1}{2}+\alpha)u^3).$$

The amplitudes M^ν's are

$$\begin{aligned} M^2(u,\xi,v,\lambda) &= \Theta_0(u)(\sum_{k\in H(2,0)}\chi(\frac{v}{2^{-k}\lambda}))\sum_{k\in M(2,0)^c}\chi(\frac{\xi}{2^{-k}v}),\\ M^3(u,\xi,v,\lambda) &= \Theta_0(u)(\sum_{k\in H(3,0)}\chi(\frac{v}{2^{-k}\lambda}))\sum_{k\in M(3,0)^c}\chi(\frac{\xi}{2^{-2k}\lambda}). \end{aligned}$$

Here we note that ξ, v, λ are three frequency variables and v and u is given by the change of variables $\lambda y = v$ and $x - y = u$ in (3.7). Let us define a differential operator D_w for each $w = u, v, \xi, \lambda$ by

$$D_w g(u,\xi,v,\lambda) = \frac{1}{2\pi i\Psi'_w(p,q,t,u,\xi,v,\lambda)}\frac{\partial g(u,\xi,v,\lambda)}{\partial w}.$$

Then $D_w e^{2\pi i\Psi(p,q,t,u,\xi,v,\lambda)} = e^{2\pi i\Psi(p,q,t,u,\xi,v,\lambda)}$. The transpose of D_w is given by

$$D_w^T g(u,\xi,v,\lambda) = \frac{1}{\partial w}(\frac{g(u,\xi,v,\lambda)}{2\pi i\Psi'_w(p,q,t,u,\xi,v,\lambda)}).$$

Let us choose a cutoff function ζ supported on the region $1-2^{-10} < t < 1+2^{-10}$ and $\zeta(t) = 1$ on $1-2^{-11} < t < 1+2^{-11}$. We define $[\theta]^2 = \lambda q/(\xi+2vu)$ and $[\theta]^3 = \lambda(q-u^2)/(\xi+\epsilon_\alpha\lambda u^2)$ as functions of u,ξ,v,λ and q. To apply integration by parts in (3.7), we define a differential operator for each $\nu = 2,3$, $r = p, q, (t-\frac{pq}{2})+qu$ and $w = \xi, v, \lambda$,

$$(3.9) \qquad \begin{aligned} \mathcal{D}^\nu g = &\ [D_u^T]^N(\zeta^c([\theta]^\nu)g)+[D_v^T]^N(\zeta([\theta]^\nu)g),\\ \mathcal{D}_w^r g = &\ [D_w^T]^N(\psi^c(r/2^{5+m_\alpha})g)+\psi(r/2^{5+m_\alpha})g. \end{aligned}$$

For each $\nu = 2, 3$, the integration by parts yields that $K_0^{glo} * E_0^\nu * W_0^\nu(p,q,t)$ is majorized by

$$\iiiint |\mathcal{D}_\lambda^{(t-\frac{pq}{2})+qu}\mathcal{D}_v^q\mathcal{D}_\xi^p\mathcal{D}^\nu M^\nu(u,\xi,v,\lambda)|dud\xi dvd\lambda \lesssim \big((1+|p|)(1+|q|)(1+|t-2pq|)\big)^{-N}. \tag{3.10}$$

This is the desired estimate for the proof of (3.6). Let us sketch the proof of (3.10). As the first step we need to observe that $\mathcal{D}^\nu$ makes the decay such as

$$|\mathcal{D}^\nu M^\nu(u,\xi,v,\lambda)| \lesssim (1+|\xi|+|v|+|\lambda|)^{-N}. \tag{3.11}$$

Proof of (3.11). Derivatives of our phase function are computed as follows:

$$\begin{aligned} \Psi_u'(p,q,t,u,\xi,v,\lambda) &= -(\xi+2uv)+\epsilon_\alpha\lambda u^2+\lambda(q-u^2),\\ \Psi_v'(p,q,t,u,\xi,v,\lambda) &= q-u^2. \end{aligned} \tag{3.12}$$

Recall that $\epsilon_\alpha = 1-3(\alpha+1/2) \neq 0$ when $\alpha \neq -1/6$. In (3.7) we need to observe that M^ν is supported on the set $B(\nu)$ where

$$\begin{aligned} B(2) &= \{u\approx 1, |v|\gg 1, |v|\gg(1+|\epsilon_\alpha|)|\lambda|\}\cap\{|\xi|\gg|v| \text{ or } |\xi|\ll|v|\},\\ B(3) &= \{u\approx 1, |\lambda|\gg 1, |v|\ll\min(|\epsilon_\alpha|,|\epsilon_\alpha^*|)|\lambda|\}\cap\\ &\quad\ \{|\xi|\gg(|\epsilon_\alpha|+1)|\lambda| \text{ or } |\xi|\ll\min(|\epsilon_\alpha|,|\epsilon_\alpha^*|)|\lambda|\}. \end{aligned}$$

On the intersection of the support of $\zeta^c([\theta]^\nu)$ and $B(\nu)$, we have rapid change of the phase function Ψ with respect to u:

$$|\Psi_u'(p,q,t,u,\xi,v,\lambda)| \gtrsim (|\xi|+|v|+|\lambda|+1). \tag{3.13}$$

On the intersection of the support of $\zeta([\theta]^\nu)$ and $B(\nu)$, we also have enough change of Ψ with respect to v:

$$|\Psi_v'(p,q,t,u,\xi,v,\lambda)| \gtrsim (|\xi|+|v|+|\lambda|+1)/|\lambda|. \tag{3.14}$$

We can also easily observe that on $B(\nu)$,

$$\begin{aligned} &|\frac{\partial}{\partial u}(\zeta^c([\theta]^\nu)M^\nu(u,\xi,v,\lambda))| \lesssim \frac{1}{u},\\ &|\frac{\partial}{\partial v}(\zeta([\theta]^\nu)M^\nu(u,\xi,v,\lambda))| \lesssim \frac{1}{|\lambda|}. \end{aligned} \tag{3.15}$$

The derivative of $\partial_u(1/\Psi'_u)$ or $\partial_v(1/\Psi'_v)$ gains the decay of $1/u$ or $1/|\lambda|$ respectively. This combined with (3.8) and (3.13)-(3.15) gives the inequality of (3.11). □

As the second step, we show that each of $\mathcal{D}^p_\xi$, $\mathcal{D}^q_v$, and $\mathcal{D}_\lambda^{(t-\frac{pq}{2})+qu}$ gives the decay of p, q and $t-pq/2$ respectively. For these estimates we need to observe that on each region of $|x| \geq 2^{4+m_\alpha}$ with each $x = p, q, (t-\frac{pq}{2})+qu$,

$$(3.16)\qquad \begin{aligned} |\Psi'_\xi(p,q,t,u,\xi,v,\lambda)| &= |p-u| \gtrsim |p|, \\ |\Psi'_v(p,q,t,u,\xi,v,\lambda)| &= |q-u^2| \gtrsim |q|, \\ |\Psi_\lambda(p,q,t,u,\xi,v,\lambda)| &\gtrsim |(\frac{t}{4}-\frac{pq}{2})+qu|. \end{aligned}$$

By using (3.11) and (3.16) with (3.9) we can apply $\mathcal{D}_\lambda^{(t-\frac{pq}{2})+qu}\mathcal{D}^q_v\mathcal{D}^p_\xi$ to obtain (3.10). Therefore the decay estimate (3.6) is proved. □

Let us turn to the second term of (3.5). Let us consider $\nu = 2$. In view of (3.7), we can observe that $x-y \approx 2^{-j}$ and $|y| \gg 2^{-j}$, which implies that $|x| \gg 2^{-j}$. Therefore from this and the Fourier inversion formula (2.4), we can write $K_j^{glo} * E_j^2 * G_j^2 = [E_j^2]_\flat * K_j^{glo} * E_j^2 * G_j^2$. Here $[E_j^2]_\flat = \sum_{k\in H_\flat(2,j)} A_k^1$ where $H_\flat(2,j) = \{k : 2^{-k} > 2^{m_\alpha - j+4}\}$. We decompose $K_j^{glo} * E_j^2 * G_j^2$ so that

$$(3.17)\qquad \begin{aligned} K_j^{glo} * E_j^2 * G_j^2 \;&=\; G_j^2 * [E_j^2]_\flat * K_j^{glo} * E_j^2 * G_j^2 \\ &\quad + W_j^2 * E_j^2 * K_j^{glo} * [E_j^2]_\flat * G_j^2. \end{aligned}$$

Let us consider the second term in (3.17). Let $1/p' = 1-1/p$ and $1/q' = 1-1/q$. Then the $L^p \to L^q$ boundedness of the convolution operator $B*f$ is equivalent to the $L^{q'} \to L^{p'}$ boundedness of the convolution operator $\tilde{B}*f$ with $\tilde{B}(x) = \overline{B(-x)}$. Let $r_j(t)$'s be the Rademacher functions. Then we have the estimate

$$\begin{aligned}
&\| \sum_j (G_j^2)\tilde{} * (E_j^2)\tilde{} * (K_j^{glo})\tilde{} * ([E_j^2]_\flat)\tilde{} * (W_j^2)\tilde{} * f \|_{L^{p'}(\mathbb{H}^1)} \\
\lesssim\ & \| \big(\sum_j |(K_j^{glo})\tilde{} * ([E_j^2]_\flat)\tilde{} * (W_j^2)\tilde{} * f|^2\big)^{1/2} \|_{L^{p'}(\mathbb{H}^1)} \\
\approx\ & \Big(\int_0^1 \| \sum_j r_j(t)(K_j^{glo})\tilde{} * ([E_j^2]_\flat)\tilde{} * (W_j^2)\tilde{} * f \|_{L^{p'}(\mathbb{H}^1)}^{p'} \, dt\Big)^{1/p'} \\
\lesssim\ & \| f \|_{L^{q'}(\mathbb{H}^1)} .
\end{aligned}$$

The first inequality follows from Lemma 3 and the dual estimates of Lemma 2. The last inequality follows from the fact that $(1/q', 1/p') \in ABCD \cap l_3^\sigma$ and

$$\sum_j |r_j(t)(K_j^{glo})\tilde{} * ([E_j^2]_\flat)\tilde{} * (W_j^2)\tilde{}(x_1, x_2, x_3)| \ \lesssim\ \rho(x_1, x_2, x_3)^{-6+\sigma},$$

which is shown in the similar way as (3.6). Now there remains the first term in (3.17). By Lemmas 1,2,3, we have the estimate

$$\begin{aligned}
&\| \sum_j G_j^2 * [E_j^2]_\flat * K_j^{glo} * E_j^2 * G_j^2 * f \|_{L^q} \\
\lesssim\ & \| \{K_j^{glo} * E_j^2 * G_j^2 * f\} \|_{L^q(l^2)} \\
\lesssim\ & \| \{S_j^\alpha\} \|_{L^p(l^2) \to L^q(l^2)} \| f \|_{L^p} \\
\lesssim\ & (1 + \| \mathcal{S}_\alpha^\sigma \|_{L^p \to L^q}^{1-\theta}) \| f \|_{L^p},
\end{aligned}$$

where $0 < \theta < 1$ such that $\frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{\infty}$. The last inequality can be shown as follows. Let us assume that $p \leq 2$, since the other range is treated by duality. By using the Hölder inequality,

$$\| \{S_j^\alpha\} \|_{L^p(l^2) \to L^q(l^2)} \leq \| \{S_j^\alpha\} \|_{L^p(l^p) \to L^q(l^p)}^\theta \| \{S_j^\alpha\} \|_{L^p(l^\infty) \to L^q(l^\infty)}^{1-\theta} .$$

We also have for $(1/p, 1/q) \in ABCD \cap l_3^\sigma$,

$$(3.18)\quad \| \{S_j^\alpha\} \|_{L^p(l^\infty) \to L^q(l^\infty)} \leq \| \mathcal{S}_\alpha^\sigma \|_{L^p \to L^q},$$

$$(3.19)\quad \| \{S_j^\alpha\} \|_{L^p(l^p) \to L^q(l^p)} \leq \sup_j \| S_j^\alpha \|_{L^p \to L^q} \leq \sup_j \| S_0^\alpha \|_{L^p \to L^q} \leq C.$$

The positivity for the kernel of S_j^α shows (3.18). The first inequality of (3.19) follows from the Minkowski inequality and the fact $q/p \geq 1$.

The second inequality follows from a scaling estimation. The last follows by the application of Theorem 1. For this, we need to use the fact that $|S_0^\alpha f(x)| \lesssim |\mathcal{T}_\alpha f(x)|$ and the condition $(1/p, 1/q) \in ABCD \cap l_3^\sigma$.
When $\nu = 3$, in view of (3.7) with the condition $|x-y| \approx 2^{-j}$ and $|y| \ll 2^{-j}$, we have the support condition $|x| \approx 2^{-j}$. So we can write $K_j^{glo} * E_j^3 * G_j^3 = E_j^1 * K_j^{glo} * E_j^3 * G_j^3$. This case is also handled in the similar way,

$$\| \sum_j E_j^1 * K_j^{glo} * E_j^3 * G_j^3 * f \|_{L^q} \lesssim (1+ \| \mathcal{S}_\alpha^\sigma \|_{L^p \to L^q}^{1-\theta}) \| f \|_{L^p} .$$

3.3. Proof of (3.2) for $\nu = 1$

By using the support condition of the integral kernel of $\widehat{K_j^{glo} * E_j^1}(\lambda)$ we have $K_j^{glo} * E_j^1 = (E_j^1 + E_j^3) * K_j^{glo} * E_j^1$. By applying the previous estimate for the case $\nu = 3$, we can show that for $(1/p, 1/q) \in ABCD \cap l_3^\sigma$ with $0 < \theta < 1$ satisfying $\frac{1}{2} = \frac{\theta}{p} + \frac{1-\theta}{\infty}$,

$$\| \sum_j E_j^1 * K_j^{glo} * E_j^1 * f \|_{L^q} \lesssim (1+ \| \mathcal{S}_\alpha^\sigma \|_{L^p \to L^q}^{1-\theta}) \| f \|_{L^p},$$

$$\| \sum_j G_j^3 * E_j^3 * K_j^{glo} * E_j^1 * f \|_{L^q} \lesssim (1+ \| \mathcal{S}_\alpha^\sigma \|_{L^p \to L^q}^{1-\theta}) \| f \|_{L^p},$$

$$\| \sum_j W_j^3 * E_j^3 * K_j^{glo} * E_j^1 * f \|_{L^q} \lesssim \| f \|_{L^p} .$$

For the last inequality we need to show the dual part estimate

$$\| \sum_j E_j^1 * [K_j^{glo}]\tilde{} * E_j^3 * W_j^3 * f \|_{L^{p'}} \lesssim \| f \|_{L^{q'}} .$$

The above estimate follows from the decay condition

$$|[K_0^{glo}]\tilde{} * E_0^3 * W_0^3(x_1, x_2, x_3)| \le C_N(1 + \rho(x_1, x_2, x_3))^{-N}. \tag{3.20}$$

We show (3.20) by the same way as (3.6). For this case the corresponding phase function $\Psi(p, q, t, u, \xi, v, \lambda)$ for the integral (3.8) is

$$\xi(p-u) + v(q-u^2) + \lambda(t - \frac{pq}{2} + qu - (\frac{1}{2} - \alpha)u^3).$$

Its derivatives are

$$
(3.21)\qquad \begin{aligned}
\Psi'_u(p,q,t,u,\xi,v,\lambda) &= -(\xi+2uv)+\epsilon^*_\alpha\lambda u^2+\lambda(q-u^2),\\
\Psi'_v(p,q,t,u,\xi,v,\lambda) &= q-u^2.
\end{aligned}
$$

If $\alpha \neq 1/6$, then $\epsilon^*_\alpha \neq 0$. So we can apply the same estimation as (3.10). Now the proof of (3.2) is finished. This implies that for some $0<\theta<1$,

$$
(3.22)\qquad \| \mathcal{S}^\sigma_\alpha \|_{L^p\to L^q} \leq C(1+ \| \mathcal{S}^\sigma_\alpha \|^{1-\theta}_{L^p\to L^q}).
$$

Let us replace $\mathcal{S}^\sigma_\alpha$ by $\mathcal{S}^\sigma_{\alpha,N} = \sum_{j=-N}^{N} S^\alpha_j$ with a finite N. Then from (3.22) we obtain that

$$
\| \mathcal{S}^\sigma_{\alpha,N} \|_{L^p\to L^q} \leq C,
$$

where C is independent of N.

4. The case $\alpha = \pm 1/6$

The decay properties (3.6) and (3.20) are crucially used for our (L^p, L^q) estimate. The decay estimates (3.6) and (3.20) are made by using the fact that at least one of Ψ'_u and Ψ'_v in each (3.12) and (3.21) does not vanish. However for the case $\alpha = 1/6$ we have $\epsilon^*_\alpha = 0$ in (3.21), which makes all the partial derivatives of Ψ with respect to u, ξ, v, and λ vanish where $|\xi + 2uv|$, $|p-u|$, $|q-u^2|$ and $|t-\alpha u^3|$ are very small. Similarly when $\alpha = -1/6$, we have $\epsilon_\alpha = 0$ in (3.12), and all the derivatives of Ψ vanish where $|\xi+2uv| \approx |p-u| \approx |q-u^2| \approx |t-\alpha u^3| \approx 0$.
We do not know what is the type set of $\mathcal{S}^\sigma_{\pm 1/6}$, but we can show that $ABD \cap l^\sigma_3$ is not the type set of $\mathcal{S}^\sigma_{-1/6}$, and the set $ABC \cap l^\sigma_3$ is not the type set of $\mathcal{S}^\sigma_{1/6}$. Let us define two sets in $\mathbb{R}^2$:

$$
\begin{aligned}
U_\sigma &= \{(1/p, 1/q) : 1/p \geq 2\sigma/3\},\\
V_\sigma &= \{(1/p, 1/q) : 1/q \leq 1 - 2\sigma/3\}.
\end{aligned}
$$

We show the following nontrivial results about the case $\alpha = \pm 1/6$:
(a) The type set of $\mathcal{S}^\sigma_{1/6}$ is contained in $ABC \cap l^\sigma_3 \cap U_\sigma$.
(b) The type set of $\mathcal{S}^\sigma_{-1/6}$ is contained in $ABD \cap l^\sigma_3 \cap V_\sigma$.

In showing (a) and (b), it suffices to show (b) since the other case $\alpha = 1/6$ follows from the dual estimate of (L^p, L^q) and the fact

$$[\mathcal{S}_\alpha^\sigma]^* f(x_1, x_2, x_3) = \mathcal{S}_{-\alpha}^\sigma \tilde{f}(-x_1, -x_2, x_3),$$

where $\tilde{f}(x_1, x_2, x_3) = f(-x_1, -x_2, x_3)$.

Proof of (b). Recall that

$$\mathcal{S}_\alpha^{-1/6} f(x) = \int f(x_1 - t, x_2 - t^2, x_3 + \frac{1}{6}t^3 - \frac{1}{2}(x_1 t^2 - x_2 t))\frac{t^\sigma}{t} dt.$$

Let $R \gg r$ and define two sets depending on r and R and an integer $1 \le k \le R/r$:

$$\begin{aligned} D(r, R) &= \{(x_1, x_2, x_3) : |x_1| < 10r, |x_2| < 10rR, |x_3| < 10r^2 R\}, \\ Q_k(r, R) &= \{(x_1, x_2, x_3) : (k-1)r < |x_1| < kr, |x_2 - x_1^2| < rR, \\ &\qquad |x_3 + \frac{1}{2}x_1(x_2 - x_1^2)| < r^2 R\}. \end{aligned}$$

Let us define $f_{D(r,R)}$ be a characteristic function supported on $D(r, R)$. Then for any $(x_1, x_2, x_3) \in Q_k(r, R)$ we can show that if $(k-1)r < t < kr$, then

$$f_{D(r,R)}(x_1 - t, x_2 - t^2, x_3 + \frac{1}{6}t^3 - \frac{1}{2}(x_1 t^2 - x_2 t)) = 1. \tag{4.1}$$

(4.1) follows from the fact that

$$|x_1 - t| < 10r, |x_2 - t^2| \le |x_2 - x_1^2| + |(x_1 - t)(x_1 + t)| < 10rR,$$

and the third component $|x_3 + \frac{1}{6}t^3 - \frac{1}{2}(x_1 t^2 - x_2 t)|$ is written as

$$|x_3 + \frac{1}{6}(t - x_1)^3 + \frac{1}{2}(x_2 - x_1^2)(t - x_1) + \frac{1}{2}x_1(x_2 - x_1^2)| < 10r^2 R.$$

Therefore for each $x \in Q_k(r, R)$,

$$\mathcal{S}_\alpha^\sigma f_{D(r,R)}(x) \ge (kr)^{\sigma - 1} r.$$

So from this and the measure estimate $m(Q_k(r,R)) = r(rR)(r^2R)$,

$$\begin{aligned}\| \mathcal{S}^{\sigma}_{\alpha} f_{D(r,R)} \|_{L^q(\mathbb{H}^1)} &\geq \Big(\sum_{1<k<R/r} \int_{Q_k(r,R)} |\mathcal{S}^{\sigma}_{\alpha} f_{D(r,R)}(x)|^q dx\Big)^{1/q} \\ &\geq \Big(\sum_{1<k<R/r} (k)^{(\sigma-1)q}\Big)^{1/q}(r)^{\sigma}(r(rR)(r^2R))^{1/q} \\ &\geq C_{\sigma,q}(R/r)^{(\sigma-1)+1/q}(r)^{\sigma}(r(rR)(r^2R))^{1/q},\end{aligned}$$

and

$$\| f_{D(r,R)} \|_{L^p(\mathbb{H}^1)} = m(D(r,R))^{1/p} = \big(r(rR)(r^2R)\big)^{1/p}.$$

Suppose that for $(1/p, 1/q) \in ABD \cap l^{\sigma}_3$, $\| \mathcal{S}^{\sigma}_{-1/6} \|_{L^p \to L^q} < \infty$. Then we can write

$$(R/r)^{(\sigma-1)+1/q}(r)^{\sigma} \leq (r(rR)(r^2R))^{1/p-1/q}.$$

Then by taking $r = R^{1-\epsilon}$ with some positive number ϵ, then we need the following relation for sufficiently large R,

$$1/q \leq 1 - 2\sigma/3. \qquad \square$$

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References

[1] M. Christ, *Endpoint bounds for singular fractional integral operators*, preprint, unpublished, 1988.

[2] ———, *Convolution, curvature, and combinatorics: a case study*, Int. Math. Res. Not. **19** (1998), 1033–1048.

[3] G. B. Folland, *Harmonic analysis in phase space*, *Annals of Mathematics Studies*, Princeton Univ. Press, **122** (1989).

[4] A. Greenleaf, S. Wainger, and A. Seeger, *A On X-ray transforms for rigid line complexes and integrals over curves in* $\mathbb{R}^4$, Proc. Amer. Math. Soc. **127** (1999), 3533–3545.

[5] J. Kim, L^p*-estimate for singular integrals and maximal operators associated with flat curves on the Heisenberg group*, Duke. Math. J. **114** (2002), 555-593.

[6] D. Obelin, *Convolution estimates for some measures on curves*, Proc. Amer. Math. Soc. **99** (1987), 56–60.

[7] S. Secco, *L^p-improving properties of measures supported on curves on the Heisenberg group*, Studia Math. **132** (1999), 179–201.

[8] ———, *Fractional integration along homogeneous curves in R^3*, Math. Scand. **85** (1999), 259–270.

[9] E. M. Stein, *Harmonic Analysis:Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, 1993.

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