

DIFFERENTIABILITY OF QUASI-HOMOGENEOUS CONVEX AFFINE DOMAINS

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ABSTRACT. In this article we show that every quasi-homogeneous convex affine domain whose boundary is everywhere differentiable except possibly at a finite number of points is either homogeneous or covers a compact affine manifold. Actually we show that such a domain must be a non-elliptic strictly convex cone if it is not homogeneous.

1. Introduction

An affinely flat manifold M is a manifold which is locally modelled on the affine space with its natural affine geometry, i.e., M admits a cover of coordinate charts into the affine space \mathbb{R}^n whose coordinate transitions are affine transformations. By an analytic continuation of coordinate maps from its universal covering \tilde{M} , we obtain a developing map from \tilde{M} into \mathbb{R}^n and this map is rigid in the sense that it is determined only by a local data. Therefore the deck transformation action on \tilde{M} induces the holonomy action via the developing map by the rigidity. (see [3, 8, 9], etc for more details.) More generally, An (X, G) -manifold is a manifold which is locally modelled on X with the geometry determined by the Lie group G acting on X analytically. For example, projectively flat manifold is a special case of (X, G) -manifold with $X = \mathbb{RP}^n$ and $G = \mathrm{PGL}(n+1, \mathbb{R})$ and so is an affinely flat manifold with $X = \mathbb{R}^n$, the standard Euclidean space and $G = \mathrm{Aff}(n)$, the group of affine transformations on \mathbb{R}^n . An affinely flat manifold also can be viewed as a projectively flat manifolds whose holonomy preserves the points at infinity, \mathbb{RP}_∞^{n-1} , by identifying \mathbb{R}^n with the affine space given by $x_{n+1} = 1$ in \mathbb{R}^{n+1} so that \mathbb{RP}^n becomes a compactification of \mathbb{R}^n .

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From the definition of (X, G) -manifold, we get obviously that the developing image Ω of a closed (X, G) -manifold is a *quasi-homogeneous domain* in X , i.e., Ω is an open subset of X which has a compact subset K of Ω and a subgroup H of $\text{Aut}(\Omega) < G$ such that $HK = \Omega$ (in this case, we say sometimes that H acts on Ω *syndetically*). So the quasi-homogeneous domain theory is important to understand (X, G) -manifolds, particularly when the developing map is a diffeomorphism onto the developing image Ω . For examples, the class of convex affine (projective, resp.) manifolds is a subclass of affinely (projectively, resp.) flat manifolds whose developing maps are diffeomorphisms onto convex domains. So the study of compact convex affine manifolds is equivalent to that of divisible convex affine domains and their automorphism groups. Here a *divisible affine domain* is a domain of \mathbb{R}^n whose automorphism group contains a cocompact discrete subgroup acting properly. Note that the set of divisible domains is a subclass of the set of quasi-homogeneous domains as the set of homogeneous domains is so.

In this viewpoint we investigated quasi-homogeneous affine (respectively, projective) domains in [4, 5, 6]. More precisely, we studied two questions : (i) which quasi-homogeneous convex affine domain can cover a compact affine manifold? (in other words, which quasi-homogeneous convex affine domain is divisible?) (ii) how many quasi-homogeneous domains are there other than homogeneous domains? (that is, we want to find out all shapes of quasi-homogeneous domains.)

From the results of those previous papers, we could see that the differentiability of the boundary is closely related to the quasi-homogeneity and homogeneity of a domain and there seems to be just a little quasi-homogeneous domains which are not homogeneous. Actually it seems that every irreducible (see [6] for a definition) quasi-homogeneous convex affine domain is either homogeneous or divisible.

In this article, we get an answer of these two questions for convex affine domains whose boundaries are everywhere differentiable except possibly at a finite number of points : by finding out all their shapes, we will show that such a domain is homogeneous if is not a strictly convex quasi-homogeneous cone. Note that any strictly convex quasi-homogeneous cone which is not an elliptic cone is non-homogeneous even if its boundary is everywhere continuously differentiable except cone point. And furthermore we will show that it is true in this case that every quasi-homogeneous domain is either homogeneous or divisible.

2. Quasi-homogeneous convex projective domains

To study affine domains, sometimes it is useful to look at them in the projective space $\mathbb{R}P^n = (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ because of the compactness of $\mathbb{R}P^n$ and $PM(n + 1, \mathbb{R})$. Remind that a domain in \mathbb{R}^n can be viewed as a domain in $\mathbb{R}P^n$ whose automorphism group preserves the set of points at infinity, $\mathbb{R}P_\infty^{n-1}$, by identifying \mathbb{R}^n with the affine space given by $x_{n+1} = 1$ in \mathbb{R}^{n+1} . Since $PM(n + 1, \mathbb{R})$, which is the projectivization of the group of all $(n + 1)$ by $(n + 1)$ matrices, is a compactification of $PGL(n + 1, \mathbb{R})$, any infinite sequence of non-singular projective transformations contains a convergent subsequence. Note that the limit projective transformation may be singular in general. For a singular projective transformation g we will denote the projectivization of the kernel and the range of g by $K(g)$ and $R(g)$. Then g maps $\mathbb{R}P^n \setminus K(g)$ onto $R(g)$ and the images of any closed set in $\mathbb{R}P^n \setminus K(g)$ under the convergent sequence g_i , converges uniformly to the images under the limit transformation g of g_i (See [1]).

An open subset Ω of $\mathbb{R}P^n$ is called *convex* if there exists an affine space $H \subset \mathbb{R}P^n$ such that Ω is a convex affine subset of H . A convex domain Ω in $\mathbb{R}P^n$ is called *properly convex* if there is no non-constant projective map of \mathbb{R} into Ω and *strictly convex* if $\partial\Omega$ has no line segment. From this definition we see that any strictly convex domain is a properly convex domain.

The following definitions are originally introduced by Benzécri in [1].

DEFINITION 1. Let Ω be a properly convex projective domain of $\mathbb{R}P^n$.

- (i) A *face* of Ω is an equivalence class with respect to the equivalence relation given as follows:
 - (a) $x \sim y$ if $x \neq y$ and $\overline{\Omega}$ has an open line segment l containing both x and y .
 - (b) $x \sim y$ if $x = y$.
- (ii) The *support* of a face F , which will be denoted by $\langle F \rangle$, is the projective subspace generated by F .
- (iii) Zero dimensional faces are called *extreme points*. Note that p is an extreme points if and only if there is no open line segment which lies in $\partial\Omega$ entirely and contains p . (When we consider Ω as a convex affine domain by choosing an affine space containing $\overline{\Omega}$, we can say that $p \in \partial\Omega$ is an extreme point if it cannot be expressed as the convex combination of any two points in $\partial\Omega$.)
- (iv) Ω is called a *convex sum* of its faces Ω_1 and Ω_2 , which will be denoted by $\Omega = \Omega_1 \dot{+} \Omega_2$, if $\langle \Omega_1 \rangle \cap \langle \Omega_2 \rangle = \emptyset$ and Ω is the union

of all open line segments joining points in Ω_1 to points in Ω_2 . Note that if the dimensions of Ω , Ω_1 and Ω_2 are n , k_1 and k_2 , respectively, then $n = k_1 + k_2 + 1$. In this case, we say sometimes that $\bar{\Omega}$ is a convex sum of its closed faces $\bar{\Omega}_1$ and $\bar{\Omega}_2$ and denote it by $\bar{\Omega} = \bar{\Omega}_1 + \bar{\Omega}_2$, where a closed face is the closure of a face in its support.

From the definition, we see that a face is a convex subset of $\bar{\Omega}$ which is open in its support and $\bar{\Omega}$ is a disjoint union of faces.

Now we state some useful lemma and theorems.

LEMMA 2. *Let Ω be a quasi-homogeneous properly convex domain in \mathbb{RP}^n and G a subgroup of $\text{Aut}(\Omega)$ acting on Ω syndetically. Then for each point $p \in \partial\Omega$, there exists a sequence $\{g_i\} \subset G$ and $x \in \Omega$ such that $g_i(x)$ converges to p . Furthermore for any accumulation point g of $\{g_i\}$ in $\text{PM}(n + 1, \mathbb{R})$, $R(g)$ is the support of the face containing p and $K(g) \cap \Omega = \emptyset$ and $K(g) \cap \bar{\Omega} \neq \emptyset$.*

Proof. See Lemma 3.2 of [4]. □

THEOREM 3. *Let Ω be a properly convex quasi-homogeneous projective domain in \mathbb{RP}^n . Then the following are satisfied.*

- (i) *Suppose $\Omega = \Omega_1 + \Omega_2$. Then Ω is homogeneous (respectively, quasi-homogeneous) if and only if Ω_i is homogeneous (respectively, quasi-homogeneous) for $i = 1, 2$.*
- (ii) *Suppose the boundary $\partial\Omega$ of Ω has a line segment, that is, Ω is not strictly convex. Then Ω has a triangular section Δ . (Here a section of Ω is a nonempty intersection with a projective subspace of \mathbb{RP}^n .)*

Proof. See [1] or Theorem 2.8 and Corollary 2.9 of [4]. □

THEOREM 4. *Let Ω be a properly convex quasi-homogeneous projective domain. Then*

- (i) *Ω is a simplex if it is a polyhedron.*
- (ii) *Ω is an ellipsoid if its boundary is twice differentiable.*
- (iii) *Ω is homogeneous if its boundary is everywhere twice differentiable except possibly at a finite number of points.*

Proof. See Proposition 5, Theorem 6 and Theorem 7 of [5]. □

3. Quasi-homogeneous convex affine domains

From the definition, it is obvious that every quasi-homogeneous affine domain is a quasi-homogeneous projective domain. But some affine domains in \mathbb{R}^n are not quasi-homogeneous as affine domains even though they are quasi-homogeneous as projective domains. For a simple example, a triangle is not a quasi-homogeneous affine domain but a quasi-homogeneous projective domain. In fact any bounded affine domain cannot be a quasi-homogeneous affine domain (for a proof see [1]). Note that a quadrant (that is, $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$) is a quasi-homogeneous affine domain which is projectively equivalent to a triangle even though they are not affinely equivalent. More complicated examples are strictly convex quasi-homogeneous projective domains whose boundaries are not twice differentiable (for details, see [3, 7, 11] as mentioned in the introduction). For this reason, we have been using the terminologies ‘quasi-homogeneous affine domain’ and ‘quasi-homogeneous projective domain’ to avoid ambiguity and from now on we will denote the group of all affine transformations preserving Ω by $\text{Aut}_{\text{aff}}(\Omega)$. We usually denote the boundary of a domain Ω by $\partial\Omega$, but sometimes we will also use the notation $\partial_a\Omega$ for the boundary of Ω as a subset of \mathbb{R}^n and $\partial_p\Omega$ for the boundary of Ω as a subset of \mathbb{RP}^n when it is necessary to avoid ambiguity. We call them an affine boundary and a projective boundary of an affine domain Ω , respectively. Note that $\partial_a\Omega$ is a subset of $\partial_p\Omega$ and in fact $\partial_a\Omega = \mathbb{R}^n \cap \partial_p\Omega$. We will use $\partial_\infty\Omega$ for the infinite boundary in \mathbb{RP}^n , that is, $\partial_\infty\Omega = \bar{\Omega} \cap \mathbb{RP}_\infty^{n-1} = \partial_p\Omega \cap \mathbb{RP}_\infty^{n-1} = \partial_p\Omega - \partial_a\Omega$, where $\bar{\Omega}$ is the closure of Ω in \mathbb{RP}^n . (In this article we will not denote by $\bar{\Omega}$ the closure of Ω in \mathbb{R}^n even if Ω is in \mathbb{R}^n .)

EXAMPLE 5. Let $\Omega = \{(x, y) \mid x > 0, y > 0\} \subset \mathbb{R}^2$. Then

$$\Omega = \{[x, y, 1] \mid x > 0, y > 0\} \subset \mathbb{RP}^2$$

when it is considered as a projective domain and the following are satisfied.

$$\partial_a\Omega = \{(x, 0) \mid x \geq 0\} \cup \{(0, y) \mid y \geq 0\} \subset \mathbb{R}^2 \subset \mathbb{RP}^2$$

$$\partial_\infty\Omega = \{[x, y, 0] \mid x \geq 0, y \geq 0\} \subset \mathbb{RP}_\infty^1 = \mathbb{RP}^2 \setminus \mathbb{R}^2$$

$$\partial_p\Omega = \partial_a\Omega \cup \partial_\infty\Omega \subset \mathbb{RP}^2$$

$$= \{[x, 0, 1] \mid x \geq 0\} \cup \{[0, y, 1] \mid y \geq 0\} \cup \{[x, y, 0] \mid x \geq 0, y \geq 0\}.$$

A point of $\partial_\infty\Omega \subset \mathbb{RP}^n$ can be regarded as a direction parallel to a half line which is entirely contained in Ω . From these observation we can

see that $\partial_\infty\Omega$ is related to the asymptotic cone of Ω , which is originally introduced by Vey in [10], defined as follows.

DEFINITION 6. Let Ω be a convex domain in \mathbb{R}^n . The *asymptotic cone* of Ω is defined as follows:

$$\text{AC}(\Omega) = \{u \in \mathbb{R}^n \mid x + tu \in \Omega, \text{ for all } x \in \Omega, t \geq 0\}.$$

By the convexity of Ω , for any $x_0 \in \Omega$,

$$\text{AC}(\Omega) = \text{AC}_{x_0}(\Omega) := \{u \in \mathbb{R}^n \mid x_0 + tu \in \Omega, \text{ for all } t \geq 0\}.$$

Note that $\text{AC}(\Omega)$ is a properly convex closed cone in \mathbb{R}^n if Ω is properly convex. We will denote the interior of $\text{AC}(\Omega)$ relative to its affine hull by $\text{AC}^\circ(\Omega)$.

REMARK 7. We see the following facts immediately.

- (i) The asymptotic cone of Ω is the maximal closed cone which can be contained in Ω .
- (ii) $\overline{\text{AC}(\Omega)}$ is a convex sum of the origin o and $\partial_\infty\Omega$, that is, $\overline{\text{AC}(\Omega)} = \{o\} \dot{+} \partial_\infty\Omega$, when $\text{AC}(\Omega)$ is considered as a projective domain.

It's essential in studying quasi-homogeneous affine domains to investigate the asymptotic cones. The following theorem and proposition was proved in [4].

THEOREM 8. Let Ω be a properly convex quasi-homogeneous affine domain in \mathbb{R}^n . Then Ω admits a parallel foliation by cosets of $\text{AC}^\circ(\Omega)$ of Ω .

Theorem 8 implies that for each $x \in \Omega$ there exists a point $s(x)$ in its boundary such that $\Omega_x = \text{AC}^\circ(\Omega) + s(x)$ is a k -dimensional section of Ω containing x , where k is the dimension of $\text{AC}(\Omega)$.

DEFINITION 9.

- (i) We call the foliation of a properly convex quasi-homogeneous domain Ω by cosets of $\text{AC}^\circ(\Omega)$ the *asymptotic foliation* of Ω .
- (ii) We call $p \in \partial\Omega$ an *asymptotic cone point* of a properly convex affine domain Ω if p is a cone point of Ω_x for some $x \in \Omega$, that is, $p = s(x)$ for some $x \in \Omega$.
- (iii) A convex cone C is called a *strictly convex cone* if every line segment in $\partial_a\Omega$ lies in a ray, where a ray means a half line in \overline{C} which starts from the cone point.

Note that any strictly convex cone is a convex sum of a point set and a strictly convex projective domain in \mathbb{RP}_∞^{n-1} , when it is considered as a projective domain. Or equivalently, we can say that every $(n - 1)$ -dimensional section of a strictly convex cone is a strictly convex domain if it does not contain the cone point.

EXAMPLE 10. Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 \mid y > x^2, z > 0\}$. Then Ω is homogeneous and $AC^\circ(\Omega) = \{(0, y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\}$. By Theorem 8, the leaves of its asymptotic foliation are all translations of this cone $AC^\circ(\Omega)$. Actually for each $p = (x_0, y_0, z_0) \in \Omega$ the leaf containing p is

$$\begin{aligned} \Omega_p &= \{(x_0, x_0^2 + y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\} \\ &= (x_0, x_0^2, 0) + \{(0, y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\} \\ &= s(p) + \{(0, y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\}. \end{aligned}$$

That is, any intersection of Ω and a plane parallel to yz -plane is $\{(x_0, x_0^2 + y, z) \in \mathbb{R}^3 \mid y > 0, z > 0\}$ for some $x_0 \in \mathbb{R}$.

PROPOSITION 11. *Let Ω be a quasi-homogeneous properly convex affine domain in \mathbb{R}^n . Let E be the set of all extreme points of $\Omega \subset \mathbb{RP}^n$ and S the set of all asymptotic cone points $s(x)$. Then*

$$E = S \cup E_\infty,$$

where E_∞ denote the set of all extreme points of

$$\overline{\Omega} \cap \mathbb{RP}_\infty^{n-1} = \overline{AC(\Omega)} \cap \mathbb{RP}_\infty^{n-1}.$$

Let $\text{Lin}(\Omega)$ be the image of $\text{Aut}_{\text{aff}}(\Omega)$ by the canonical homomorphism from $\text{Aff}(n, \mathbb{R})$ to $\text{GL}(n, \mathbb{R})$. Then $\text{Lin}(\Omega)$ preserves $AC(\Omega)$ since $AC(\Omega)$ is the maximal closed cone which can be contained in Ω . So the restriction map of $\text{Lin}(\Omega)$ to the affine hull AV of $AC(\Omega)$ becomes a subgroup G of $\text{Aut}_{\text{aff}}(AC(\Omega)) < \text{GL}(AV)$. Obviously G acts on $AC^\circ(\Omega)$ transitively if Ω is homogeneous. More generally, we can show the following.

PROPOSITION 12. *Let Ω be a properly convex affine domain. Then $AC^\circ(\Omega)$ is (quasi)-homogeneous if Ω is (quasi)-homogeneous.*

Proof. It suffices to show that the linear part $G = \text{Lin}(\Omega)$ of $\text{Aut}_{\text{aff}}(\Omega)$ acts on $AC^\circ(\Omega)$ syndetically.

Let Ω be a properly convex quasi-homogeneous affine domain and K be a compact subset of Ω such that $\text{Aut}_{\text{aff}}(\Omega)K = \Omega$. Choose a point $x_0 \in K$. Then we may assume that $s(x_0)$ is the origin. For each

$y \in K$, $K_y = K \cap (s(y) + \text{AC}^\circ(\Omega))$ is mapped to a compact subset of $\text{AC}^\circ(\Omega) = s(x_0) + \text{AC}^\circ(\Omega)$ under the translation by $-s(y)$. Let

$$\tilde{K} = \cup_{y \in K} (K_y - s(y)).$$

Then \tilde{K} is a compact subset of $\text{AC}^\circ(\Omega)$ since K is a compact subset of Ω .

Now we show that $G\tilde{K} = \text{AC}^\circ(\Omega)$. For any $z \in \text{AC}^\circ(\Omega) = s(x_0) + \text{AC}^\circ(\Omega)$, there exist $g = (A, a) \in \text{Aut}_{\text{aff}}(\Omega)$ such that $g(z) = y \in K$. Since $s(z) = s(x_0) = o$ and $g(s(z)) = s(g(z)) = s(y)$, a must be $s(y)$. So $A(z) = g(z) - s(y)$ must be a point in $K_y - s(y)$ and thus in \tilde{K} . This complete the proof because $A \in G$. \square

The above proposition together with Remark 7 (ii) and Theorem 3 (i) implies that the relative interior of $\partial_\infty \Omega$ is a (quasi)-homogeneous projective domain if Ω is (quasi)-homogeneous.

LEMMA 13. *Let Ω be a quasi-homogeneous properly convex affine domain in \mathbb{R}^n . $\text{Aut}_{\text{aff}}(\Omega)$ fixes a point $\xi \in \mathbb{R}^n$. Then Ω is a cone with cone point ξ .*

Proof. Suppose there is an extreme point $p \in \partial_a \Omega$ such that $p \neq \xi$. (Note that it follows from Proposition 4 of Vey [10] that ξ must be a boundary point of Ω . But this is not necessary to proceed with our argument.) Then there exists a sequence $\{g_i\} \subset \text{Aut}_{\text{aff}}(\Omega)$ and $x \in \Omega$ such that $g_i(x)$ converges to p by Lemma 2. We may assume that there is a singular projective transformation $g \in \text{PM}(n + 1, \mathbb{R})$ such that $\{g_i\}$ converges to g . Then since p is an extreme point of Ω , $R(g) = \{p\}$ by the second statement of Lemma 2 and $K(g) \cap \mathbb{R}^n = \emptyset$ by Lemma 3.5 of [4]. So $g_n(x)$ converges to p for all x in \mathbb{R}^n as n goes to ∞ . But $g_n(\xi)$ cannot converges to p because $g_n(\xi) = \xi$. This contradiction implies that any point $p \in \partial_a \Omega$ is not an asymptotic point of Ω by Proposition 11 if it is not ξ . But by Theorem 8 there must be at least one asymptotic cone point of Ω and so ξ is a unique asymptotic cone point. Therefore we conclude that Ω is a cone with a cone point ξ by Theorem 8 again. \square

4. Differentiability and homogeneity of domains

In this section, we study relationship between the homogeneity, quasi-homogeneity and the differentiability of convex affine domains.

In [4] we studied the differentiability of quasi-homogeneous strictly convex projective domains : any quasi-homogeneous strictly convex projective domain Ω in \mathbb{RP}^n has a continuously differentiable boundary and

it must be an ellipsoid if $\partial\Omega$ is twice differentiable. Furthermore such a quasi-homogeneous strictly convex domain Ω fails to be twice differentiable on a dense subset and has a discrete automorphism group $\text{Aut}(\Omega)$ and so cannot be homogeneous if it is not an ellipsoid. (This was also proved independently by Y. Benoist in [2].) It is well known that there exist infinitely many such non-homogeneous strictly convex quasi-homogeneous projective domains (see [3, 7, 11]).

In [5] we studied that under which condition on the differentiability of their boundaries quasi-homogeneous projective domains must be homogeneous. More precisely, as stated in Theorem 4, we showed that every convex quasi-homogeneous projective domain whose boundary is everywhere twice differentiable except possibly at a finite number of points is homogeneous, which is a generalization of the 2-dimensional result of Vinberg and Kats [11].

Affine domains, as a special subclass of projective domains, seem to be quite different from projective domains. For example, there is no strictly convex quasi-homogeneous affine domain which is not homogeneous : we already showed in [4] that an n - paraboloid

$$\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + x_2^2 + \dots + x_{n-1}^2\}$$

is the only strictly convex quasi-homogeneous affine domain in \mathbb{R}^n up to affine equivalence. (Observe $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^p + x_2^p + \dots + x_{n-1}^p\}$ is strictly convex if p is any positive even integer, but it is quasi-homogeneous only when $p = 2$.)

A subset Q of \mathbb{R}^n is called a *polyhedral set* if Q is the intersection of a finite number of closed half spaces or $Q = \mathbb{R}^n$. A polyhedral set is called a *simplex cone* in \mathbb{R}^n if it is a properly convex cone bounded by n hyperplanes. That is, a simplex cone is a polyhedral set affinely isomorphic to a domain $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_i > 0, i = 1, 2, \dots, n\}$. As an immediate consequence of Theorem 4 (i) we have the following proposition.

PROPOSITION 14. *The only quasi-homogeneous polyhedral sets except \mathbb{R}^n are the products of affine subspaces of \mathbb{R}^n and simplex cones.*

Proof. Let P be a quasi-homogeneous polyhedral set which is not \mathbb{R}^n . Then by convexity of P , there is a nonnegative integer k and a $(n - k)$ -dimensional quasi-homogeneous polyhedral set P' such that $P = \mathbb{R}^k \times P'$. P' is a polyhedron as a projective domain and so it must be a simplex by (i) of Theorem 4. Since every quasi-homogeneous affine domain cannot be bounded, $\mathbb{R}\mathbb{P}_\infty^{n-k-1}$ must intersect the boundary of P' . But $\partial_a P'$ cannot have any bounded face with non-zero dimension by Lemma 3.6

of [4]. So the interior of $\mathbb{RP}_\infty^{n-k-1} \cap \overline{P'}$ relative to $\mathbb{RP}_\infty^{n-k-1}$ must be a maximal face of the simplex P' , which means that P' is a simplex cone when it is viewed as an affine domain again. \square

As concerns (ii) of Theorem 4, the particularity of affine domains distinguishing from other projective domains leads us to some stronger result.

THEOREM 15. *Let Ω be a convex quasi-homogeneous affine domain in \mathbb{R}^n whose boundary is differentiable. Then Ω is homogeneous. Furthermore, Ω is affinely equivalent to $\mathbb{R}^k \times \{(x_1, x_2, \dots, x_{n-k}) \in \mathbb{R}^{n-k} \mid x_{n-k} > x_1^2 + x_2^2 + \dots + x_{n-k-1}^2\}$ for some $k \in \{0, 1, 2, \dots, n-2\}$, if it is neither \mathbb{R}^n nor an affine half space.*

Proof. Note that affine spaces, affine half spaces and convex affine domains bounded by paraboloids are all homogeneous. So it suffices to show the second statement. Assume that Ω is neither \mathbb{R}^n nor an affine half space. If Ω is not properly convex, then $\Omega = \mathbb{R}^k \times \Omega'$ for some $0 < k < n-1$ and $(n-k)$ -dimensional properly convex domain Ω' by convexity. Obviously, Ω' is a quasi-homogeneous affine domain with dimension ≥ 2 . We first show that Ω' is strictly convex. Suppose not, that is, suppose that $\partial\Omega$ has a line segment. Then by Theorem 3, Ω has a triangular section Δ . Since Δ must intersect Ω by definition of a section, it cannot lie in infinite boundary and thus one of its vertices must be in \mathbb{R}^n , which contradicts the hypothesis that $\partial_a\Omega$ is differentiable. This shows that Ω' must be strictly convex. So Ω' is affinely equivalent to $\{(x_1, x_2, \dots, x_{n-k}) \in \mathbb{R}^{n-k} \mid x_{n-k} > x_1^2 + x_2^2 + \dots + x_{n-k-1}^2\}$ by Theorem 5.9 of [4]. \square

REMARK 16. We cannot deduce immediately from (ii) of Theorem 4 that every convex quasi-homogeneous affine domain with twice differentiable boundary is affinely isomorphic to a paraboloid, because we cannot say that its projective boundary is also twice differentiable everywhere. There are many convex affine domains such that their affine boundaries are twice differentiable and their projective boundaries are not twice differentiable at infinite points. But we can prove that there is no such a convex affine domain if domains are restricted to those which are quasi-homogeneous.

This theorem implies that any properly convex quasi-homogeneous affine domain whose boundary is everywhere differentiable is affinely equivalent to $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + x_2^2 + \dots + x_{n-1}^2\}$. Observe that the differentiability condition for being homogeneous is a little

weaken in affine case. Then what can we say for properly convex quasi-homogeneous affine domains whose boundaries are not differentiable everywhere? In contrast to (iii) of Theorem 4, we cannot conclude that a properly convex quasi-homogeneous affine domain whose boundary is everywhere differentiable except possibly at a finite number of points is also homogeneous. Instead we get

THEOREM 17. *Let Ω be a properly convex quasi-homogeneous affine domain such that the boundary of Ω is everywhere differentiable except possibly at a finite number of points. Then Ω is either \mathbb{R}^+ or $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ or $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + x_2^2 + \dots + x_{n-1}^2\}$ or a strictly convex cone whose boundary is everywhere continuously differentiable except cone point.*

Proof. Since \mathbb{R}^+ , $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ and $\{(x, y) \in \mathbb{R}^2 \mid y > x^2\}$ are the only quasi-homogeneous properly convex affine domains in \mathbb{R} or \mathbb{R}^2 up to affine equivalence, we may assume that $\Omega \subset \mathbb{R}^n$ for $n \geq 3$.

From Theorem 15, we see that if $\partial\Omega$ is differentiable everywhere it is a paraboloid, that is, $\Omega = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + x_2^2 + \dots + x_{n-1}^2\}$. Now we consider the case that $\partial\Omega$ is not differentiable everywhere and assume that $S = \{p_1, \dots, p_k\}$ is the set of all points at which the boundary of Ω is not differentiable. Since $\text{Aut}_{\text{aff}}(\Omega)$ preserves a finite subset S of \mathbb{R}^n , it must have a fixed point $p = \frac{p_1 + p_2 + \dots + p_k}{k} \in \mathbb{R}^n$ and this implies that Ω is a properly convex cone with cone point p by Lemma 13.

Since if $\partial_\infty\Omega$ has a singular point ζ then $\partial\Omega$ is not differentiable at all points of the ray ending ζ , $\partial_\infty\Omega$ must be the closure of a properly convex quasi-homogeneous projective $(n - 1)$ -dimensional domain with differentiable boundary by the hypothesis. Therefore by Theorem 5.10 of [4], the interior of $\partial_\infty\Omega$ relative to \mathbb{RP}_∞^{n-1} must be a strictly convex projective domain with C^1 boundary and thus Ω is a strictly convex cone whose boundary is C^1 everywhere except the cone point p . \square

REMARK 18. From Theorem 8, we can see that every asymptotic cone point of a quasi-homogeneous properly convex affine domain Ω is a singular point if $\dim \text{AC}(\Omega) > 1$. So if we use the results in [4] together with this observation, we can divide quasi-homogeneous properly convex affine domains in \mathbb{R}^n into the following three classes, which leads us to an another proof of Theorem 15 and 17.

- (i) $\partial_a\Omega$ is differentiable : This case occurs only when $\dim \text{AC}(\Omega) = 1$ and it must be strictly convex by Proposition 5.5 of [4] and thus it

is $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + x_2^2 + \dots + x_{n-1}^2\}$ by Theorem 5.9 of [4].

- (ii) $\partial_a \Omega$ has a unique singular point : This case occurs only when $\dim \text{AC}(\Omega) = n$ and $\partial_\infty \Omega$ is differentiable.
- (iii) $\partial_a \Omega$ has infinitely many singular points : This case occurs either when $2 \leq \dim \text{AC}(\Omega) \leq n - 1$ or when $\dim \text{AC}(\Omega) = n$ and $\partial_\infty \Omega$ has a singular point.

Now we get the following corollary for convex quasi-homogeneous affine domains which may contain complete lines.

COROLLARY 19. *Let Ω be a convex quasi-homogeneous affine domain such that the boundary of Ω is everywhere differentiable except possibly at a finite number of points. Then Ω is affinely equivalent to one of the following:*

- (i) \mathbb{R}^+ ,
- (ii) $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$,
- (iii) $\{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + x_2^2 + \dots + x_{n-1}^2\}$,
- (iv) $\mathbb{R}^k \times \mathbb{R}^+$,
- (v) $\mathbb{R}^k \times \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_n > x_1^2 + x_2^2 + \dots + x_{n-1}^2\}$,
- (vi) an affine space,
- (vii) a strictly convex quasi-homogeneous cone.

Proof. Let Ω be an n -dimensional convex quasi-homogeneous affine domain in \mathbb{R}^n . If Ω is properly convex, then it is one of (i), (ii), (iii) and (vii) by Theorem 17.

Otherwise, either $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^k \times \Omega'$ for some $0 < k < n$ and $(n - k)$ -dimensional properly convex domain Ω' by convexity. Obviously, Ω' is a quasi-homogeneous affine domain whose boundary is everywhere differentiable except possibly at a finite number of points and thus Ω' must be affinely equivalent to \mathbb{R}^+ or $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$, or a paraboloid, or a strictly convex quasi-homogeneous cone by Theorem 17 again. But Ω' can be neither $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ nor a strictly convex quasi-homogeneous cone because the products \mathbb{R}^k with them have infinitely many singular points, which contradicts the hypothesis. This completes the proof. \square

In [4], we showed that every strictly convex quasi-homogeneous projective domain is a divisible projective domain. (Note that a paraboloid is a quasi-homogeneous affine domain, but not a divisible affine domain.) Actually it seems to be true that every irreducible (see [6] for a definition) quasi-homogeneous convex projective domain is either homogeneous or

divisible, which is proved under the condition that its automorphism group is irreducible by Y. Benoist [2]. The following corollary is related to this question.

COROLLARY 20. *Let Ω be a convex quasi-homogeneous affine domain such that the boundary of Ω is everywhere differentiable except possibly at a finite number of points. Then Ω is either homogeneous or divisible.*

Proof. Since domains of Corollary 19 except the case (vii) are all homogeneous it suffices to show that any strictly convex quasi-homogeneous cone is either homogeneous or divisible. This follows from Proposition 5.15 of [4] that every strictly convex quasi-homogeneous projective domain which is not an ellipsoid has a discrete projective automorphism group. This implies that a strictly convex quasi-homogeneous projective domain is divisible and so is a strictly convex quasi-homogeneous cone. \square

This Corollary shows that every quasi-homogeneous convex affine domain whose boundary is everywhere differentiable except possibly at a finite number of points is either homogenous or covers a compact affinely flat manifold.

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