J. Korean Math. Soc. **42** (2005), No. 3, pp. 471–484

SPACES OF CONFORMAL VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

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ABSTRACT. We study Riemannian or pseudo-Riemannian manifolds which carry the space of closed conformal vector fields of at least 2-dimension. Subject to the condition that at each point the set of closed conformal vector fields spans a non-degenerate subspace of the tangent space at the point, we prove a global and a local classification theorems for such manifolds.

1. Introduction

Conformal mappings and conformal vector fields are important in general relativity, as is well known since the early 1920's [6, 18]. In 1925, Brinkmann studied conformal mappings between Riemannian or pseudo-Riemannian Einstein spaces [1]. Later conformal vector fields, or infinitesimal conformal mappings on Einstein spaces were reduced to the case of gradient vector fields, leading to a very fruitful theory of conformal gradient vector fields in general. Brinkmann's work has attracted renewed interest, especially in the context of general relativity [2, 3, 4, 5, 9, 13, 16], and the following local theorems have been shown:

PROPOSITION 1.1. [3] *Let (M^4, g) be a 4-dimensional Ricci-flat Lorentz manifold. If M^4 admits a nonhomothetic conformal vector field, then M^4 is a plane gravitational wave.*

PROPOSITION 1.2. [4] *Let (M^4, g) be a 4-dimensional Einstein but not Ricci flat Lorentz manifold. If M^4 admits a nonisometric conformal vector field, then M^4 has constant sectional curvature.*

Received October 31, 2003.
2000 Mathematics Subject Classification: 53C50, 53A30, 53C25.
Key words and phrases: pseudo-Riemannian manifold, warped product, conformal vector field.
This work was supported by Korea Research Foundation Grant.(KRF-2001-041-D00026).

PROPOSITION 1.3. (Kerckhove [9])*Let* (M^n, g) *be an n-dimensional Einstein but not Ricci flat pseudo-Riemannian manifold with* $Ric = (n-1)kg, k \neq 0$, *which carries a conformal vector field. Here we denote by Ric the Ricci tensor of* (M^n, g). *If each subspace* $\Delta(p)$ *spanned by the set of conformal gradient vector fields at* $p \in M^n$ *is a non-degenerate subspace of* T_pM *whose dimension m is independent of the choice of the point p, then* (M^n, g) *is locally isometric to a warped product* $B^m(k) \times_f F$. *The base B is an m-dimensional space of constant sectional curvature k; the fibre* (F, g_F) *is an Einstein manifold with* $Ric_F = (n-m-1)\alpha g_F$ *for some constant* α.

For an arbitrary pseudo-Riemannian manifold (M^n, g) we denote by $A(M^n, g)$ and φ the space of functions f on M^n whose hessian tensor H^f satisfies $H^f = fg$ and the symmetric bilinear form on the space $A(M^n, g)$ defined by $\varphi(f,h) =< \nabla f, \nabla h > -fh$, respectively. Then for an arbitrary complete connected pseudo-Riemannian manifold (M^n, g) with $A(M^n, g) \neq \{0\}$, in [7] Kerbrat shows the following global theorems:

PROPOSITION 1.4. (Kerbrat [7]) *If* $\dim A(M^n, g) = m > 0$, *and the bilinear form* φ *is positive definite, then* (M^n, g) *is isometric to a warped product* $H^m \times_f F$, *where the base* H^m *is the hyperbolic space with constant curvature* -1, *and the fibre F is a complete pseudo-Riemannian manifold satisfying* $A(F, g_F) = \{0\}$.

PROPOSITION 1.5. (Kerbrat [7]) *If the metric g is indefinite and* $\varphi(f,f) < 0$ *for some* $f \in A(M^n, g)$, *then* (M^n, g) *is isometric to a space form or to a covering of a space form.*

The case in which φ is positive semi-definite and degenerate was not treated by Kerbrat (See [9], p.825).

In this paper, we study pseudo-Riemannian manifolds which carry the space of closed conformal vector fields of at least 2-dimension. In Section 3 we improve the global theorems of Kerbrat (Proposition 1.4 and Proposition 1.5) and the local theorem of Kerckhove (Proposition 1.3) under the condition that each subspace $\Delta(p)$ is nondegenerate, which is a necessary condition for (M^n, g) to admit a warped product structure in the sense of Kerbrat or of Kerckhove (Theorem 3.1 and Theorem 3.2). Furthermore, we give a necessary and sufficient condition on the fiber space F for any closed conformal vector fields on the warped product space $M^n = B^m(k) \times_f F$ to be lifted from the base space (Theorem 3.4).

2. Preliminaries and closed conformal vector fields on space forms

We consider an n-dimensional connected pseudo-Riemannian manifold (M^n, g) carrying a closed conformal vector fields V. Hence there is a smooth function ϕ on M^n such that

$$\nabla_X V = \phi X \tag{2.1}$$

for all vector fields X. Here ∇ denotes the Levi-Civita connection on M^n. Then for every point $p \in M^n$ one can find a neighborhood U and a function f such that $V = \nabla f$, where ∇f denotes the gradient of f. It follows that the Hessian $\nabla^2 f$ satisfies

$$\nabla^2 f = \phi g. \tag{2.2}$$

Therefore, $\Delta f = \mathrm{div} V = n\phi$.

From equation (2.1) we immediately obtain the following Ricci identity for the Riemannian curvature tensor:

$$R(X,Y)V = X(\phi)Y - Y(\phi)X, \tag{2.3}$$

and by contraction we get

$$Ric(X,V) = (1-n)X(\phi). \tag{2.4}$$

We denote by $CC(M^n, g)$ the vector space of closed conformal vector fields. First of all, we state some useful lemmas for later use.

LEMMA 2.1. *Let V be a non-trivial closed conformal vector field.*

(1) *If $\gamma : [0, \ell) \to M^n$ is a geodesic with $V(\gamma(0)) = a\gamma'(0)$ for some $a \in R$, then we have*

$$V(\gamma(t)) = (a + \int_0^t \phi(\gamma(s))ds)\gamma'(t). \tag{2.5}$$

(2) *If $V(p) = 0$, then $divV(p) = n\phi(p) \neq 0$, in particular, all zeros of V are isolated.*

Proof. See Propositions 2.1 and 2.3 in [15]. □

LEMMA 2.2. *Let (M^n, g) be an n-dimensional connected pseudo-Riemannian manifold. Then the following hold:*

(1) $\dim CC(M^n, g) \leq n+1$.

(2) *If* $\dim CC(M^n, g) \geq 2$, *there exists a constant $k \in R$ such that for all $V \in CC(M^n, g)$*

$$\nabla\phi = -kV,$$

where $n\phi$ is the divergence of V.

Proof. See Proposition 2.3 in [15] and Proposition 4 in [7]. □

In [15], W. Kühnel and H. B. Rademacher observed that if the dimension of the space of closed conformal vector fields is maximal, that is, $\dim CC(M^n, g) = n+1$, then the manifold has constant sectional curvature [15, Remark 2.4].

Now we prove the following :

PROPOSITION 2.3. *Let (M^n, g) be an n-dimensional pseudo-Riemannian manifold. If $\dim CC(M^n, g) \geq \max\{2, n-1\}$, then M^n has constant sectional curvature.*

Proof. Since $\dim CC(M^n, g) \geq 2$, Lemma 2.2 together with (2.3) shows that there exists a constant $k \in R$ which satisfies

$$R(X,Y)V = k\{< V,Y > X - < V,X > Y\}, \quad X, Y \in TM \tag{2.6}$$

for all $V \in CC(M^n, g)$. Choose $V_1, \cdots, V_{n-1}$ in $CC(M^n, g)$ in a way that they are linearly independent. Let U be the set of all points p at which $V_1(p), \cdots, V_{n-1}(p)$ are linearly independent. Then (2.1) and Lemma 2.1 show that U is an open dense subset of M^n.

For each fixed $p \in U$, choose $V_n(p)$ so that $\{V_1(p), \cdots, V_{n-1}(p), V_n(p)\}$ forms a basis for T_pM. It suffices to show that (2.6) holds for V_n instead of V on U. Then the open dense set U has constant sectional curvature k. By continuity, M^n has constant sectional curvature k.

For $V = V_i$, $\quad 1 \leq i \leq n-1$, we have from (2.6)

$$\begin{aligned} < R(X,Y)V_n, V > &= - < R(X,Y)V, V_n > \\ &= < k\{< V_n, Y > X - < V_n, X > Y\}, V > . \end{aligned} \tag{2.7}$$

For $V = V_n$, (2.7) is trivial, and hence (2.6) holds for $V = V_n$ due to nondegeneracy of the metric. This completes the proof. □

In [10], using the work of Kühnel, W.([12]), the authors characterize the Riemannian space forms in terms of the dimension of the space of conformal gradient vector fields.

The model spaces $B^n(k)$ of constant sectional curvature $k = \epsilon a^2$ with $\epsilon = \pm 1, \quad a > 0$ and index ν are the hyperquadrics in pseudo-Euclidean space :

$$S^n_\nu(a^2) = \{x \in R^{n+1}_\nu | < x, x >= 1/a^2\},$$

$$H^n_\nu(-a^2) = \{x \in R^{n+1}_{\nu+1} | < x, x >= -1/a^2\}.$$

For a fixed vector T in R^{n+1}_ν or $R^{n+1}_{\nu+1}$, let σ_T be the height function in the direction of T defined by $\sigma_T(x) =< T, x >$. Then one can easily show that on $B^n(k)$,

$$\nabla \sigma_T(x) = T - k\sigma_T(x)x, \tag{2.8}$$

$$\nabla_X \nabla \sigma_T = -k\sigma_T X \tag{2.9}$$

for all vector fields X of $B^n(k)$([9]). (2.9) implies that for any constant vector T in R^{n+1}_ν or $R^{n+1}_{\nu+1}$, $\nabla\sigma_T$ is a closed conformal vector field on the hyperquadric $B^n(k), k = \epsilon a^2$. Furthermore, by counting dimensions (Lemma 2.2) we see that $\nabla\sigma_T$ represents every element of $CC(B^n(k))$.

For the flat space form R^n_ν with index ν, the vector field V defined by $V(x) = bx + c, b \in R, c \in R^n_\nu$ is a closed conformal vector field. Obviously, by counting dimensions, we have

$$CC(R^n_\nu, g) = \{bx + c | b \in R, \ \ c \in R^n_\nu\}.$$

For the space of conformal vector fields of pseudo-Riemannian space forms, the authors et al. gave a complete description about it([11]).

Now we introduce a function space $A_k(M^n, g)(k \neq 0)$ and a symmetric bilinear form Φ_k on the space as follows:

$$A_k(M^n, g) = \{f \in C^\infty(M) | \nabla_X \nabla f = -kfX, \quad X \in TM\}, \tag{2.10}$$

$$\Phi_k(f, h) =< \nabla f, \nabla h > +kfh, \quad f, h \in A_k(M^n, g). \tag{2.11}$$

In [7], Kerbrat defined a function space $A(M^n, g)$ by

$$A(M^n, g) = \{f \in C^\infty(M) | \nabla_X \nabla f = fX, \quad X \in TM\}$$

and a symmetric bilinear map φ on the space by

$$\varphi(f,h) =< \nabla f, \nabla h > -fh, \quad f,h \in A(M^n,g).$$

By the scale change $g \to -kg$, we see that

$$A_k(M^n,g) = A(M^n,-kg), \quad \Phi_k(f,f) = -k\varphi(f,f). \tag{2.12}$$

For the non-flat space form $B^n(k), k = \epsilon a^2$, (2.8) shows that

$$\Phi_k(\sigma_T,\sigma_S) =< \nabla\sigma_T, \nabla\sigma_S > +k\sigma_T\sigma_S =< T,S > . \tag{2.13}$$

This implies that the symmetric bilinear form Φ_k is just the usual scalar product on the ambient pseudo-Euclidean space.

3. Closed conformal vector fields

In this section we consider the vector space $CC(M^n,g)$ of closed conformal vector fields on a pseudo-Riemannian manifold (M^n,g) with indefinite metric g. For $p \in M^n$, let $\Delta(p)$ be the span of the set of closed conformal vector fields at p, that is,

$$\Delta(p) = \{V(p) \in T_p(M) | V \in CC(M^n,g)\}.$$

Suppose that $CC(M^n,g)$ is of dimension $m \geq 2$. Then (2.1) and Lemma 2.2 imply that there exists a constant $k \in R$ such that for all V in $CC(M^n,g)$ with $\phi = (1/n)\text{div}V$

$$\nabla\phi = -kV, \tag{3.1}$$

so that we have

$$\nabla_X\nabla\phi = -k\phi X, \quad X \in TM. \tag{3.2}$$

Hence, if k is nonzero, then the space $CC(M^n,g)$ may be identified with the space $A_k(M^n,g)$ and we have $\Delta(p) = \{\nabla f(p) | f \in A_k(M^n,g)\}$.

First of all, we establish a global classification theorem.

THEOREM 3.1. *Let (M^n, g) be an n-dimensional connected and complete pseudo-Riem- annian manifold with indefinite metric g. Suppose that there exists $k = \epsilon a^2$ with $\epsilon = \pm 1, a > 0$ such that (M^n, g) satisfies*

(a) $\dim A_k(M^n, g) = m \geq 1$,

(b) *each subspace $\Delta(p)$ is nondegenerate.*

Then one of the following holds:

(1) *(M^n, g) is isometric to $S^n_\nu(a^2)$, $H^n_\nu(-a^2)$ or a covering space of $S^n_{n-1}(a^2)$ or $H^n_1(-a^2)$.*

(2) *(M^n, g) is isometric to a warped product space $S^m_m(a^2) \times_{\sigma_T} F^{n-m}$ $(\epsilon = 1)$ or $H^m(-a^2) \times_{\sigma_T} F^{n-m} (\epsilon = -1)$, where the fiber (F, g_F) is an $(n-m)$-dimensional connected and complete pseudo-Riemannian manifold with $A_k(F, g_F) = \{0\}$, and T is a vector in $R^{m+1}_m(\epsilon = 1)$ or $R^{m+1}_1(\epsilon = -1)$ with $< T, T >= k$.*

Proof. First, we show that if (M^n, g) is not isometric to a space form in case (1), then the bilinear form Φ_k is definite on $A_k(M^n, g)$. Suppose that $\Phi_k(f, f)$ is trivial for some nontrivial function $f \in A_k(M^n, g)$. If there exists $h \in A_k(M^n, g)$ such that $\Phi_k(f, h) \neq 0$, then it is obvious that

$$\Phi_k(tf + h, tf + h) = 2t\Phi_k(f, h) + \Phi_k(h, h)$$

for all $t \in R$. This implies that there exists a function $l \in A_k(M^n, g)$ such that $\epsilon\Phi_k(l, l) > 0$. Hence, by (2.12) $(M^n, -kg)$ carries a function $l \in A(M^n, -kg)$ which satisfies $\varphi(l, l) < 0$. By Proposition 1.5, we see that $(M^n, -kg)$ is isometric to a space form listed in (1) with curvature -1, so that (M^n, g) is a space form in (1), which is a contradiction. This shows that for the function $f \in A_k(M^n, g)$ we have $\Phi_k(f, h) = 0$ for all $h \in A_k(M^n, g)$. For any point $m \notin f^{-1}(0)$, (2.11) with the condition $\Phi_k(f, f) = 0$ shows that $\nabla f(m)$ is not zero. Since the metric g is indefinite, we can always choose a null vector v in $T_m M$ such that $< v, \nabla f(m) >\neq 0$. Let γ be the null geodesic with initial velocity vector v. Then (2.10) implies that

$$(f \circ \gamma)''(t) = -k(f \circ \gamma)(t) < \gamma'(t), \gamma'(t) >= 0,$$

so that we have

$$f(\gamma(t)) = f(m) + < v, \nabla f(m) > t.$$

Hence we see that $f^{-1}(0)$ is not empty. Fix a point $p \in f^{-1}(0)$. Then $\nabla f(p)$ is a nonzero vector (Lemma 2.1) with the property that for all $h \in A_k(M^n, g)$

$$< \nabla f(p), \nabla h(p) >=< \nabla f(p), \nabla h(p) > +kf(p)h(p) = \Phi_k(f, h) = 0,$$

which means that $\Delta(p)$ is degenerate. This contradiction shows the definiteness of Φ_k.

If $\epsilon\Phi_k(f,f) > 0$ for some function $f \in A_k(M^n,g)$, then as above, Proposition 1.5 with (2.12) shows that (M^n,g) is isometric to a space form in (1). Hence we may assume that $\epsilon\Phi_k$ is negative definite on $A_k(M^n,g)$, that is, φ is positive definite on $A(M^n,-kg)$. Then Proposition 1.4 shows that $(M^n,-kg)$ is isometric to a warped product space $H^m(-1)\times_f F$ with metric $g_{H^m(-1)}+f^2\tilde{g}_F$, where f is given by $\sqrt{\rho}$ ([7]) and $(F,\tilde{g}_F)$ is a connected complete pseudo-Riemannian manifold with $A(F,\tilde{g}_F)=\{0\}$. Hence (M^n,g) is isometric to $B^m(k)\times_f(F,g_F)$, where the base $B^m(k)$ is a space form $H^m(-a^2)(\epsilon=-1)$ or $S^m_m(a^2)(\epsilon=1)$ and the metric g_F is given $\frac{-1}{k}\tilde{g}_F$. It is straightforward to show that $f=\sqrt{\rho}$ belongs to $A(H^m(-1))$ with $\varphi(f,f)=-1$. Therefore $f\in A_k(B^m(k))$ with $\Phi_k(f,f)=k$ due to (2.12). Hence f is a height function σ_T for some vector T in R^{m+1}_1 or R^{m+1}_m with $<T,T>=k$. Since $A(F,\tilde{g}_F)=\{0\}$, we also have $A_k(F,g_F)=\{0\}$. □

In the case (2) of Theorem 3.1, if the base space form $B^m(k)$ is neither $S^m_m(a^2)$ nor $H^m(-a^2)$, then for any constant vector T in the ambient pseudo-Euclidean space the function σ_T vanishes on a hypersurface in $B^m(k)$ preventing the warped product construction from extending over all of $B^m(k)$. By contrast, if $B^m(k)$ is either $S^m_m(a^2)$ or $H^m(-a^2)$ and T satisfies $<T,T>=k$, then the function σ_T is nowhere zero since T is nowhere tangent to $B^m(k)$.

Now we prove a local classification theorem(cf. [8, 9]), which is a generalization of Kerckhove's results(Proposition 1.3).

THEOREM 3.2. *Let (M^n,g) be an n-dimensional connected pseudo-Riemannian manifold. Suppose that there exists a nonzero constant $k\in R$ such that*

(a) $\dim A_k(M^n,g)=m\geq 1$,

(b) each subspace $\Delta(p)$ is nondegenerate.

Then, for a fixed $p_o\in M^n$ the following hold:

(1) If $\dim\Delta(p_o)<m$, then (M^n,g) is locally isometric to a space $B^n(k)$ of constant sectional curvature k.

(2) If $\dim\Delta(p_o)=m$, then (M^n,g) is locally isometric to a warped product space $B^m(k)\times_{\sigma_T}F$, where the base $B^m(k)$ is a space of constant sectional curvature k and the fiber (F^{n-m},g_F) is a pseudo-Riemannian manifold. Furthermore, F satisfies the following :

(i) *In case $<T,T>\neq 0$, we have $A_\alpha(F,g_F)=\{0\}$, where $\alpha=<T,T>$.*

(ii) *In case* $< T, T >= 0$, *F carries no nontrivial homothetic gradient vector fields.*

In either case, we have $A_k(M^n, g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), < S, T >= 0\}$, *where* $\tilde{\sigma}_S$ *denotes the lifting of* σ_S.

Proof. (1) If $\dim\Delta(p_o) < m$, there exists a nontrivial function $f \in A_k(M^n, g)$ which satisfies $\nabla f(p_o) = 0$. Lemma 2.1 shows that $f(p_o) \neq 0$. Since $\Phi_k(f, f) = kf(p_o)^2$, by the scale change $g \to -kg$ we have $\varphi(f, f) = -f(p_o)^2 < 0$. Hence (1) follows from Proposition 1.5 with the scale change.

(2) If $\dim\Delta(p_o) = m$, then $\dim\Delta(p) = m$ in a neighborhood of p_o in M^n. Hence, as in the proof of Proposition 1.3 [9], it can be shown that there exists a neighborhood U of p_o which is isometric to a warped product space $B^m(k) \times_f F$ for some positive function f on $B^m(k)$. The base space is the integral submanifold of Δ through p_o and has constant sectional curvature k. Note that the fibre $p \times F$ is totally umblic. Hence the second fundamental form h of the fibre satisfies $h(V, W) =< V, W > H$. For any $\sigma \in A_k(M^n, g)$, we have the following:

$$\begin{aligned} < V, W >< H, \nabla\sigma > &= < h(V, W), \nabla\sigma > \\ &= V < W, \nabla\sigma > - < W, \nabla_V \nabla\sigma > \\ &= -k\sigma < V, W > . \end{aligned}$$

Thus we obtain $< H, \nabla\sigma >= -k\sigma$. Since the mean curvature vector field H is given by $-\nabla f / f$, we have

$$< \nabla f, \nabla\sigma > + kf\sigma = 0 \tag{3.3}$$

for all $\sigma \in A_k(M^n, g)$. By taking the covariant derivative of (3.3) with respect to any vector field X on $B^m(k)$, we find

$$< \nabla_X \nabla f, \nabla\sigma > + < kfX, \nabla\sigma >= 0. \tag{3.4}$$

Since $\nabla\sigma(\sigma \in A_k(M^n, g))$ spans the tangent spaces of $B^m(k)$, (3.4) shows that f belongs to $A_k(B^m(k))$. Hence f is a height function σ_T for some vector T in R^{m+1}. It is easy to show that if S is a vector in R^{m+1} with $< S, T >= 0$, then the lifting $\tilde{\sigma}_S$ of the height function σ_S belongs to $A_k(M^n, g)$. By counting dimensions, we see that $A_k(M^n, g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), < S, T >= 0\}$.

Suppose that the constant vector T satisfies $< T, T >= \alpha \neq 0$ and h belongs to $A_\alpha(F, g_F)$. Then we have

$$\nabla(\sigma_T h) = h\nabla\sigma_T + \frac{1}{\sigma_T}\nabla^* h,$$

where $\nabla^* h$ denotes the gradient vector of h on F. Hence the condition $\Phi_k(\sigma_T, \sigma_T) =< T, T >= \alpha$ on $B^m(k)$ shows that the function $\sigma_T h$ lies in the function space $A_k(M^n, g)$. Since the leaves $B^m(k) \times q,\ q \in F$ are the integral submanifolds of the distribution Δ, we see that $\nabla(\sigma_T h)$ must be tangent to the leaves. This shows that $A_\alpha(F, g_F) = \{0\}$.

Finally, we suppose that T is a null vector in R^{m+1} and $\nabla^* h$ is a homothetic gradient vector field on F with $\nabla^*_V \nabla^* h = cV, c \in R,\ V \in TF$, where we denote by ∇^* the Levi-Civita connection on F. Then it is not difficult to show that for a null vector $\bar{T}$ in R^{m+1} with $< T, \bar{T} >= -1$, the function l defined by $l = \sigma_T h + c\sigma_{\bar{T}}$ belongs to $A_k(M^n, g)$. Hence as in the proof of case (1), $\nabla l = h\nabla\sigma_T + \frac{1}{\sigma_T}\nabla^* h + c\nabla\sigma_{\bar{T}}$ must be tangent to the leaves. This completes the proof of our theorem. □

Note that if (F, g_F) has constant sectional curvature $\alpha =< T, T >$, then so does $M^n = B^m(k) \times_{\sigma_T} F$. Hence we see that not all closed conformal vector fields on the warped product need to be lifted from the base. Thus it is worthwhile to find a condition on the fibre F which guarantee that any closed conformal vector fields on the warped product space $M^n = B^m(k) \times_{\sigma_T} F$ to be lifted from the base space.

To find the condition, we state a useful lemma which can be easily shown. Recall that $\mathfrak{L}_V g$ denotes the Lie derivative of g with respect to V.

LEMMA 3.3. [20] *Let (M^n, g) be a totally umbilic submanifold of a pseudo-Riemannian space $(\bar{M}, \bar{g})$. If V is a conformal vector field on $\bar{M}$ with $\mathfrak{L}_V g = 2\sigma g$, then the tangential part V^T of V on M^n is a conformal vector field on M^n with*

$$\mathfrak{L}_{V^T} g = 2\{\sigma + \bar{g}(V, H)\}g,$$

where H denotes the mean curvature vector field of M^n in $\bar{M}$.

In [11], the authors et al. proved a converse of Lemma 3.3 for hypersurfaces of a pseudo-Riemannian space form.

Now we prove that the necessary condition on F in Theorem 3.2 is sufficient for any closed conformal vector fields on the warped product space $M^n = B^m(k) \times_{\sigma_T} F$ to be lifted from the base space as follows.

THEOREM 3.4. *Let (M^n, g) be a warped product space $B^m(k) \times_{\sigma_T} F$, where T is a vector in the ambient pseudo-Euclidean space R^{m+1} of $B^m(k)$. Suppose that the fibre (F, g_F) satisfies the following :*

(1) *In case* $< T,T > \neq 0$, *we have* $A_\alpha(F, g_F) = \{0\}$, *where* $\alpha =< T,T >$.

(2) *In case* $< T,T >= 0$, *F carries no nontrivial homothetic gradient vector fields.*

Then (M^n, g) *satisfies the following :*

$$A_k(M^n, g) = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), < S,T >= 0\}.$$

In particular, each subspace $\Delta(p)$ *is nondegenerate and of dimension* m.

Proof. First note that for a vector S in R^{m+1} the lifting $\tilde{\sigma}_S$ of a height function σ_S belongs to $A_k(M^n, g)$ if and only if $< S,T >= 0$ [9, p.824]. For such a nontrivial vector S, (2.8) and $\Phi_k(\sigma_S, \sigma_T) = < S,T >= 0$ imply that $\nabla\sigma_S(p) \neq 0$ for any $p \in B^m(k)$. This means that for each $p \in B^m(k)$, $\{\nabla\sigma_S(p) | < S,T >= 0\}$ spans $T_pB^m(k)$.

For any $f \in A_k(M^n, g)$ let f_p denote the restriction of f to the fibre $p \times F$ and ∇f_p the gradient of f_p on $p \times F$. Then ∇f_p is the vertical part of ∇f. Since each fibre $p \times F$ is a totally umbilic submanifold of M^n with mean curvature vector field $\frac{-1}{\sigma_T(p)}\nabla\sigma_T(p)$ and $\mathfrak{L}_{\nabla f} g = -2kfg$ on M^n, we obtain from Lemma 3.3

$$\mathfrak{L}_{\nabla f_p} g|_{p\times F} = \frac{-2}{\sigma_T(p)}\{< \nabla f, \nabla\sigma_T(p) > + k f_p \sigma_T(p)\} g|_{p\times F}. \tag{3.5}$$

Since ∇f_p is closed, (3.5) implies for all $V \in TF$

$$\nabla^*_V \nabla f_p = \frac{-1}{\sigma_T(p)}\{< \nabla f, \nabla\sigma_T(p) > + k f_p \sigma_T(p)\} V, \tag{3.6}$$

where ∇^* denotes the Levi-Civita connection on F.

Suppose that $< T,T >= 0$. Then σ_T belongs to $A_k(M^n, g)$. Hence (3.6) shows that

$$\nabla^*_V \nabla f_p = \frac{-1}{\sigma_T(p)}\Phi_k(\sigma_T, f)V, \quad V \in TF. \tag{3.7}$$

Since $g|_{p\times F} = \sigma_T(p)^2 g_F$, it follows from (3.7) that on F

$$\nabla^*_V \nabla f_p = -\sigma_T(p)\Phi_k(\sigma_T, f)V, \quad V \in TF.$$

Hence the hypothesis shows that for each $p \in B^m(k)$, f_p is constant. This implies that f is a function on the base $B^m(k)$. Thus f is a height function σ_S for some vector S in R^{m+1} with $< S,T >= 0$.

Now, we suppose that $< T, T > \neq 0$. Then the subspace $W = \{\tilde{\sigma}_S | \sigma_S \in A_k(B^m(k)), < S, T >= 0\}$ of $A_k(M^n, g)$ is nondegenerate with respect to Φ_k because it is nothing but the orthogonal complement of T in the ambient pseudo-Euclidean space. Hence it suffices to show that the orthogonal complement of W with respect to Φ_k is trivial. For $f \in A_k(M^n, g)$ and a fixed point $p \in B^m(k)$, by differentiating both sides of (3.6) with respect to an arbitrary vector on F, we see that on $p \times F$

$$< \nabla f, \nabla \sigma_T > + k\sigma_T(p) f_p = \frac{< T, T >}{\sigma_T(p)} f_p + c \tag{3.8}$$

for some constant c. Suppose that f lies in the orthogonal complement of W. Since $\nabla \sigma_S(p)$ with $< S, T >= 0$ generates the tangent space of $B^m(k)$ at p, we can choose $S_1, \cdots, S_m$ in the orthogonal complement of T in R^{m+1} such that $< \nabla \sigma_{S_i}(p), \nabla \sigma_{S_j}(p) >= \epsilon_i \delta_{ij}$. Using $\Phi_k(f, \sigma_{S_i}) = \Phi_k(\sigma_T, \sigma_{S_i}) = 0$, on $p \times F$ we obtain

$$< \nabla f, \nabla \sigma_T >= k^2 \sigma_T(p) f_p \sum \epsilon_i \sigma_{S_i}(p)^2,$$

$$< T, T >= k^2 \sigma_T(p)^2 \sum \epsilon_i \sigma_{S_i}(p)^2.$$

Hence (3.8) shows that the constant c vanishes. Since $g|_{p \times F} = \sigma_T(p)^2 g_F$, it follows from (3.6) and (3.8) that f_p belongs to $A_\alpha(F)$, where $\alpha =< T, T >$. Thus the hypothesis on $A_\alpha(F)$ shows that every element f in the orthogonal complement of W is trivial. Thus our theorem is proved. □

ACKNOWLEDGEMENT. The authors would like to express their deep thanks to the referee for valuable suggestions.

References

[1] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Math. Ann. **94** (1925), 119–145.

[2] K. L. Duggaland Sharma and R. Sharma, *Symmetries of Spacetimes and Riemannian Manifolds*, Kluwer Academic Publishers, Dordrecht, 1999.

[3] D. Eardley, J. Isenberg, J. Marsden, and V. Moncrief, *Homothetic and conformal Symmetries of solutions to Einstein's equations*, Comm. Math. Phys. **106** (1986), 137–158.

[4] D. Garfinkle and Q. Tian, *Spacetimes with cosmological constant and a conformal Killing field have constant curvature*, Classical Quantum Gravity **4** (1987), 137–139.

[5] W. D. Halford, *Brinkmann's theorem in general relativity*, Gen. Relativity Gravitation **14** (1982), 1193–1195.

[6] G. S. Hall, *Symmetries and geometry in general relativity*, Differential Geom. Appl. 1 (1991), 35-45.

[7] Y. Kerbrat, *Transformations conformes des varietes pseudo-riemanniannes*, J. Differential Geom. **11** (1976), 547–571.

[8] M. G. Kerckhove, *Conformal transformations of pseudo-Riemannian Einstein manifolds*, Thesis, Brown University, 1988.

[9] ———, *The structure of Einstein spaces admitting conformal motions*, Classical Quantum Gravity **8** (1991), 819–825.

[10] D. -S. Kim and Y. H. Kim, *A characterization of space forms*, Bull. Korean Math. Soc. **35** (1998), no. 4, 757–767.

[11] D. -S. Kim, Y. H. Kim, S. -B. Kim, and S. -H. Park, *Conformal vector fields and totally umbilic hypersurfaces of a pseudo-Riemannian space form*, Bull. Korean Math. Soc. **39** (2002), no. 4, 671–680.

[12] W. Kühnel, *Conformal transformations between Einstein spaces*, In: Conformal Geometry, R. S. Kulkarni and U. Pinkal, Aspects Math. **E12** (1988), 105–146.

[13] W. Kühnel and H. B. Rademacher, *Twistor spinors with zeros*, Internat. J. Math. **5** (1994), 877–895.

[14] ———, *Conformal vector fields on pseudo-Riemannian spaces*, Differential Geom. Appl. **7** (1997), 237–250.

[15] ———, *Essential conformal fields in pseudo-Riemannian geometry*, J. Math. Pures Appl. **74** (1995), no. 9, 453–481.

[16] B. T. McInnes, *Brinkmann's theorem in general relativity and non-Riemannian field theories*, Gen. Relativity Gravitation **12** (1980), 767–773.

[17] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York, 1983.

[18] P. Penrose and W. Rindler, *Spinors and space time, Vol. 1, 2*, Cambridge Monogr. Math. Phys. 1986.

[19] H. B. Rademacher, *Generalized Killing Spinors with imaginary Killing function and conformal Killing fields*, In: Global differential geometry and global analysis(Berlin, 1990), Lecture Notes in Math. **1481** (1991), Springer, Berlin, 192–198.

[20] R. Sharma and K. L. Duggal, *A characterization of affine conformal vector field*, C. R. Math. Acad. Sci. Soc. R. Can. **7** (1985), 201–205.

[21] K. Yano, *The theory of Lie derivatives and its applications*, North-Holland, Amsterdam, 1957.

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