

SURFACES OF REVOLUTION WITH POINTWISE 1-TYPE GAUSS MAP

BANG-YEN CHEN, MIEKYUNG CHOI¹, AND YOUNG HO KIM²

ABSTRACT. In this article, we introduce the notion of pointwise 1-type Gauss map of the first and second kinds and study surfaces of revolution with such Gauss map. Our main results state that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right cones are the only rational surfaces of revolution with pointwise 1-type Gauss map of the second kind.

1. Introduction

The notion of finite type submanifolds in Euclidean or pseudo-Euclidean space, introduced by the first-named author during the late 1970's, has become a useful tool for investigating and characterizing many important submanifolds (cf. [3, 4]). In [5], the notion of finite type was extended to differential maps, in particular, to Gauss map of submanifolds.

If a submanifold M of a Euclidean or pseudo-Euclidean space has 1-type Gauss map G , then G satisfies $\Delta G = \lambda(G + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C , where Δ is the Laplace operator corresponding to the induced metric on M (cf [1, 2, 8]). However, the Laplacian of the Gauss map of several important surfaces such as helicoid, catenoid and right cones take a somewhat different form; namely,

$$(1.1) \quad \Delta G = f(G + C)$$

Received October 30, 2003.

2000 Mathematics Subject Classification: 53B25, 53C40.

Key words and phrases: surfaces of revolution, circular cylinder, right cone, pointwise 1-type Gauss map.

¹ This work is supported by the Brain Korea 21.

² This work is supported by Research Fund provided by Korea Research Foundation and Support for Faculty Research Abroad.

for some non-constant function f and some constant vector C . For this reason, a submanifold is said to have pointwise 1-type Gauss map if its Gauss map satisfies (1.1) for some smooth function f on M and vector C . A pointwise 1-type Gauss map is called *proper* if the function f defined by (1.1) is non-constant. A submanifold with pointwise 1-type Gauss map is said to be *of the first kind* if the vector C in (1.1) is the zero vector. Otherwise, the pointwise 1-type Gauss map is said to be *of the second kind*.

In [8, 9], the second and third named authors together with D. W. Yoon studied ruled surfaces with pointwise 1-type Gauss map.

In this article we study surfaces of revolution with pointwise 1-type Gauss map. In particular, we prove that surfaces of revolution with pointwise 1-type Gauss map of the first kind coincide with surfaces of revolution with constant mean curvature; and the right cones are the only rational surfaces of revolution with pointwise 1-type Gauss map of the second kind.

2. Preliminaries

Let M be a surface of the Euclidean 3-space \mathbb{E}^3 (surfaces are assumed to be connected unless mentioned otherwise). The map $G : M \rightarrow S^2 \subset \mathbb{E}^3$ which sends each point of M to the unit normal vector to M at the point is called the *Gauss map* of the surface M , where S^2 is the unit sphere in \mathbb{E}^3 centered at the origin. For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the metric on M , we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. \mathcal{G}) the inverse matrix (resp. the determinant) of the matrix (\tilde{g}_{ij}) . The Laplacian Δ on M is, in turn, given by

$$(2.1) \quad \Delta = -\frac{1}{\sqrt{\mathcal{G}}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{\mathcal{G}} \tilde{g}^{ij} \frac{\partial}{\partial x^j} \right).$$

Assume M is a surface of revolution in \mathbb{E}^3 parametrized by

$$(2.2) \quad x(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v))$$

for some functions φ and ψ , where $a < v < b$ and $0 < u < 2\pi$.

We need the following lemma for later use.

LEMMA 2.1. *Let M be a surface of revolution in \mathbb{E}^3 with pointwise 1-type Gauss map. Then either the Gauss map is harmonic, that is, $\Delta G = 0$ or the function f defined by (1.1) depends only on v and the vector C in (1.1) is parallel to the axis of the surface of revolution.*

Proof. We may assume that the profile curve of (2.2) is of unit speed. A direct computation shows that the Gauss map G of M is given by

$$(2.3) \quad G = (\psi'(v)\cos u, \psi'(v)\sin u, -\varphi'(v))$$

and the Laplacian ΔG of the Gauss map G satisfies

$$(2.4) \quad \Delta G = \left(\left(\frac{\psi'}{\varphi^2} - \frac{\varphi'\psi''}{\varphi} - \psi''' \right) \cos u, \left(\frac{\psi'}{\varphi^2} - \frac{\varphi'\psi''}{\varphi} - \psi''' \right) \sin u, \frac{\varphi'\varphi''}{\varphi} + \varphi''' \right).$$

If M has pointwise 1-type Gauss map, then (1.1) holds for some function f and some vector C . When the Gauss map is not harmonic, (1.1), (2.3) and (2.4) imply that the first two components of C must be zero and

$$(2.5) \quad \frac{\psi'}{\varphi^2} - \frac{\varphi'\psi''}{\varphi} - \psi''' = f\psi'(v), \quad \frac{\varphi'\varphi''}{\varphi} + \varphi''' = f(-\varphi'(v) + c),$$

where $C = (0, 0, c)$. Since $\psi'(v)$ and $\varphi'(v)$ are not both zero, the function f is independent of u . \square

Here, we provide examples of surfaces of revolution with proper pointwise 1-type Gauss map of the first kind and the second kind, respectively.

EXAMPLE 2.1. Consider a catenoid parameterized by

$$x(u, v) = (\sqrt{1+v^2} \cos u, \sqrt{1+v^2} \sin u, \sinh^{-1} v).$$

Then its Gauss map G is given by

$$G = \frac{1}{\sqrt{1+v^2}}(\cos u, \sin u, -v).$$

Hence, the Laplacian ΔG of the Gauss map G satisfies

$$\Delta G = \frac{2}{(1+v^2)^2}G.$$

This implies that the catenoid is a surface of revolution with pointwise 1-type Gauss map of the first kind.

EXAMPLE 2.2. Consider the right cone C_a which is parametrized by

$$x(u, v) = (v \cos u, v \sin u, av), \quad a \geq 0.$$

Then the Gauss map G is given by

$$G = \frac{1}{\sqrt{1+a^2}}(a \cos u, a \sin u, -1).$$

Hence, the Laplacian ΔG of the Gauss map G satisfies

$$\Delta G = \frac{1}{v^2} \left(G + \left(0, 0, \frac{1}{\sqrt{1+a^2}} \right) \right),$$

which implies that the right cone has pointwise 1-type Gauss map of the second kind.

3. Surfaces of revolution with pointwise 1-type Gauss map of the first kind

The purpose of this section is to prove the following.

THEOREM 3.1. *A surface of revolution in \mathbb{E}^3 has constant mean curvature if and only if it has pointwise 1-type Gauss map of the first kind.*

Proof. Assume that M is a surface of revolution parametrized by

$$x(u, v) = (\varphi(v) \cos u, \varphi(v) \sin u, \psi(v))$$

for some positive function φ . We may assume that the profile curve is of unit speed; thus

$$(3.1) \quad \varphi'(v)^2 + \psi'(v)^2 = 1.$$

We may put

$$(3.2) \quad \varphi'(v) = \cos \theta(v), \quad \psi'(v) = \sin \theta(v)$$

for some function $\theta = \theta(v)$. The mean curvature H and the Gaussian curvature K of M are given by

$$(3.3) \quad H = \frac{-\psi' + \varphi(\psi'\varphi'' - \psi''\varphi')}{2\varphi}$$

and

$$(3.4) \quad K = -\frac{\varphi''}{\varphi}.$$

Suppose that M has pointwise 1-type Gauss map of the first kind. Then condition (1.1) implies

$$(3.5) \quad \Delta G - \langle \Delta G, G \rangle G = 0,$$

where $\langle \cdot, \cdot \rangle$ is the natural inner product on \mathbb{E}^3 . A straightforward computation, with help of (2.3), (2.4) and (3.2), yields

$$(3.6) \quad -\frac{\sin \theta \cos \theta}{\varphi^2} + \frac{\theta' \cos \theta}{\varphi} + \theta'' = 0,$$

which implies that $(\frac{\sin \theta}{\varphi} + \theta')$ is a constant. On the other hand, (3.2) and (3.3) yield

$$(3.7) \quad H = -\frac{1}{2} \left(\frac{\sin \theta}{\varphi} + \theta' \right).$$

Thus, the mean curvature is constant. The converse is straightforward. \square

REMARK. Surfaces of revolution with constant mean curvature are also known as *surfaces of Delaunay* (cf. [10, p.115]).

4. Surfaces of revolution with pointwise 1-type Gauss map of the second kind

Let M be a surface of revolution in \mathbb{E}^3 parametrized by

$$(4.1) \quad x(\theta, t) = (t \cos \theta, t \sin \theta, g(t))$$

for some smooth function $g(t)$. The first named author and S. Ishikawa introduced in [6] the notion of surfaces of revolution of polynomial and rational kinds: A surface of revolution M is said to be of *polynomial kind* if $g(t)$ is a polynomial and it is of *rational kind* if $g(t)$ is a rational function. A surface of revolution of rational kind is simply called a *rational surface of revolution*.

The Gauss map G of M parametrized by (4.1) is given by

$$(4.2) \quad G = \left(\frac{g'}{\sqrt{1+g'^2}} \cos \theta, \frac{g'}{\sqrt{1+g'^2}} \sin \theta, \frac{-1}{\sqrt{1+g'^2}} \right)$$

and the Laplacian Δ of M is given by

$$(4.3) \quad \Delta = -\frac{1}{1+g'^2} \frac{\partial^2}{\partial t^2} - \left(\frac{1}{t(1+g'^2)} - \frac{g'g''}{(1+g'^2)^2} \right) \frac{\partial}{\partial t} - \frac{1}{t^2} \frac{\partial^2}{\partial \theta^2}.$$

Assume M has pointwise 1-type Gauss map of the second kind. Then, by definition, the vector C in (1.1) is a nonzero vector.

Applying (1.1), (4.2), (4.3) and Lemma 2.1, we find

$$(4.4) \quad \frac{1}{t} g''(1+g'^2) + g'''(1+g'^2) - 4g'g''^2 - \frac{1}{t^2} g'(1+g'^2)^3 = -f g'(1+g'^2)^3$$

and

$$(4.5) \quad \begin{aligned} & \frac{1}{t} g' g''(1+g'^2) + g' g'''(1+g'^2) - 3g'^2 g''^2 + g''^2 \\ & = f(1+g'^2)^3 (1 - c\sqrt{1+g'^2}), \end{aligned}$$

where $C = (0, 0, c), c \neq 0$.

Equations (4.4) and (4.5) imply

$$(4.6) \quad \begin{aligned} &g''(1 + g'^2)^2t + g'''(1 + g'^2)^2t^2 - 3g'g''^2(1 + g'^2)t^2 - g'(1 + g'^2)^3 \\ &= c\sqrt{1 + g'^2}\{g''(1 + g'^2)t + g'''(1 + g'^2)t^2 - 4g'g''^2t^2 - g'(1 + g'^2)^3\}. \end{aligned}$$

Let us rewrite equation (4.6) as

$$(4.7) \quad P(t) = c\sqrt{1 + g'^2(t)}Q(t),$$

where

$$(4.8) \quad P(t) = (g'' + g'''t)(1 + g'^2)^2t - 3g'g''^2(1 + g'^2)t^2 - g'(1 + g'^2)^3$$

and

$$(4.9) \quad Q(t) = g''(1 + g'^2)t + g'''(1 + g'^2)t^2 - 4g'g''^2t^2 - g'(1 + g'^2)^3.$$

Suppose that M is a surface of revolution of polynomial kind, that is, $g(t)$ is a polynomial in t . Denote by $\deg g(t)$ the degree of $g(t)$.

If $\deg g(t) \geq 2$, then $\deg P(t) = \deg Q(t) \geq 7$, which is a contradiction. So, the only possibility for (4.7) holding is that $\deg g(t) = 1$. In this case, $g'(t) = a$ for some constant $a \neq 0$. Thus, by applying (4.6), we obtain $c = \frac{1}{\sqrt{1+a^2}}$. Therefore, the parametrization of M reduces to

$$(4.10) \quad x(\theta, t) = (t \cos \theta, t \sin \theta, at), \quad a \neq 0, \quad a \in \mathbb{R}.$$

Consequently, we obtain

THEOREM 4.1. *A surface of revolution of polynomial kind has point-wise 1-type Gauss map of the second kind if and only if it is a right cone.*

Next, we consider surfaces of revolution of rational kind. In this case, the function $g(t)$ of (4.1) and $g'(t)$ are both rational functions in t . If $g'(t)$ is not a constant, we may put $g'(t) = r(t)/q(t)$, where $r(t)$ and $q(t)$ are relative prime polynomials, that is, $r(t)$ and $q(t)$ do not have a common factor of degree ≥ 1 .

Assume $\deg r(t) = n$ and $\deg q(t) = m$. From (4.7) we know that $\sqrt{1 + g'^2(t)}$ is also a rational function. Hence, if $g'(t)$ is non-constant, then there exists a polynomial $p(t)$ satisfying $q^2(t) + r^2(t) = p^2(t)$, where $q(t), r(t)$ and $p(t)$ are relatively prime. We put

$$P_1(t) = g''(t)(1 + g'^2(t))^2t, \quad P_2(t) = g'''(t)(1 + g'^2(t))^2t^2,$$

$$\begin{aligned} P_3(t) &= g'(t)g''^2(t)(1+g'^2(t))t^2, & P_4(t) &= g'(t)(1+g'^2(t))^3, \\ Q_1(t) &= g''(t)(1+g'^2(t))t, & Q_2(t) &= g'''(t)(1+g'^2(t))t^2, \\ Q_3(t) &= g'(t)g''^2(t)t^2, & Q_4(t) &= P_4(t). \end{aligned}$$

Then $P_1, \dots, P_4, Q_1, \dots, Q_4$ are rational functions, too.

CASE 1: $m = 0$, that is, $g(t)$ is a polynomial. Then it follows from Theorem 4.1 that the surface of revolution is nothing but the right cone.

CASE 2: $m \geq 1$. Then, for each $i = 1, \dots, 4$, we see that $q^7(t)P_i(t)$ is a polynomial. Similarly, we see that for each $i = 1, 2, 3$, $q^6(t)Q_i(t)$ is a polynomial. But, we have

$$q^6(t)Q_4(t) = \frac{r(t)p^6(t)}{q(t)}.$$

Since (4.7) means

$$P(t) = c \frac{p(t)}{q(t)} Q(t),$$

it follows that $q^6(t)Q_4(t)$ is a polynomial. This is a contradiction because $p(t), q(t), r(t)$ are relative prime.

Consequently, we have the following.

THEOREM 4.2. *There do not exist rational surfaces of revolution, except the polynomial kind, with pointwise 1-type Gauss map of the second kind.*

Finally, we give the following.

THEOREM 4.3. *A rational surface of revolution has pointwise 1-type Gauss map if and only if it is an open part of a plane, a circular cylinder, or a right cone.*

Proof. Let M be a surface of revolution parametrized by (2.2). If the function φ is constant, then the surface is a circular cylinder. When φ is non-constant, the surface of revolution can be expressed by

$$(4.11) \quad x(\theta, t) = (t \cos \theta, t \sin \theta, g(t)).$$

In this case, the surface of revolution has constant mean curvature if and only if $g = g(t)$ is a solution of the following differential equation:

$$(4.12) \quad g'' + \frac{g'}{t}(1+g'^2) + 2\alpha(1+g'^2)^{3/2} = 0$$

for some constant α . If we make the following change of variable: $g' = \sinh y$, then (4.12) becomes

$$(4.13) \quad y' + \frac{1}{t} \sinh y \cosh y + 2\alpha \cosh^2 y = 0.$$

After we make another change of variable: $y = \tanh^{-1} w$, we get

$$y' = \frac{w'}{1-w^2}, \quad \sinh y = \frac{w}{\sqrt{1-w^2}}, \quad \cosh y = \frac{1}{\sqrt{1-w^2}}.$$

Thus, (4.13) becomes

$$(4.14) \quad tw'(t) + w + 2\alpha t = 0.$$

Solving (4.14) yields $w(t) = (a - \alpha t^2)/t$ for some constant a . Hence

$$(4.15) \quad g'(t) = \sinh \left(\tanh^{-1} \left(\frac{a - \alpha t^2}{t} \right) \right) = \frac{a - \alpha t^2}{\sqrt{t^2 - (a - \alpha t^2)^2}},$$

where a is a constant. Therefore $g(t)$ is given by

$$(4.16) \quad g(t) = \int^t \frac{a - \alpha t^2}{\sqrt{t^2 - (a - \alpha t^2)^2}} dt.$$

If $a = \alpha = 0$, g is a constant. In this case, the surface is an open part of a plane. If $\alpha = 0$ and $a \neq 0$, then (4.16) gives $g(t) = a \cosh^{-1}(t/a) + c_1$ for some constant c_1 . In this case, the surface is a catenoid which is not of rational kind. If $a = 0$ and $\alpha \neq 0$, then (4.16) gives $g(t) = \sqrt{\alpha^{-2} - t^2} + c_2$. In this case, the surface is a sphere which is also not of rational kind. If $a, \alpha \neq 0$, then (4.16) implies that $g(t)$ can be expressed in terms of elliptic functions and $g(t)$ is not a rational function of t .

If M is a rational surface of revolution which has pointwise 1-type Gauss map of the second kind, then M is an open part of a right cone according to Theorems 4.1 and 4.2.

The converse is easy to verify. □

ACKNOWLEDGEMENT. This work was done while the second and third named authors visited Michigan State University in 2000. They would like to express their deep gratitude to Professors D. E. Blair and B. -Y. Chen for their warm hospitality during their staying.

References

- [1] C. Baikoussis and D. E. Blair, *On the Gauss map of ruled surfaces*, Glasg. Math. J. **34** (1992), 355–359.
- [2] C. Baikoussis, B. Y. Chen, and L. Verstraelen, *Ruled surfaces and tubes with finite type Gauss map*, Tokyo J. Math. **16** (1993), 341–348.

- [3] B. Y. Chen, *Finite type submanifolds and generalizations*, University of Rome, 1985.
- [4] ———, *Total mean curvature and submanifolds of finite type*, World Scientific Publ. New Jersey, 1984.
- [5] B. Y. Chen and S. Ishikawa, *On classification of some surfaces of revolution of finite type*, Tsukuba J. Math. **17** (1993), 287–298.
- [6] B. Y. Chen and P. Piccinni, *Submanifolds with finite type Gauss map*, Bull. Austral. Math. Soc. **35** (1987), 161–186.
- [7] M. Choi and Y. H. Kim, *Characterization of the helicoid as ruled surfaces with pointwise 1-type Gauss map*, Bull. Korean Math. Soc. **38** (2001), 753–761.
- [8] Y. H. Kim and D. W. Yoon, *Ruled surfaces with finite type Gauss map in Minkowski spaces*, Soochow J. Math. **26** (2000), 85–96.
- [9] ———, *Ruled surfaces with pointwise 1-type Gauss map*, J. Geom. Phys. **34** (2000), 191–205.
- [10] J. Oprea, *Differential geometry and its applications*, Prentice Hall, New Jersey, 1997.

Bang-Yen Chen
Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA
E-mail: bychen@math.msu.edu

Miekyung Choi and Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail: miekyung@math.msu.edu & yhkim@knu.ac.kr