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ON KENMOTSU MANIFOLDS

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Abstract. The purpose of this paper is to study a Kenmotsu manifold which is derived from the almost contact Riemannian manifold with some special conditions. In general, we have some relations about semi-symmetric, Ricci semi-symmetric or Weyl semi-symmetric conditions in Riemannian manifolds. In this paper, we partially classify the Kenmotsu manifold and consider the manifold admitting a transformation which keeps Riemannian curvature tensor and Ricci tensor invariant.

1. Introduction

Let (M^n, g) (where $n = 2m + 1$) be an almost contact Riemannian manifold with a contact form η, the associated vector field ξ, a (1,1)-tensor field ϕ and the associated Riemannian metric g. In 1971, K.Kenmotsu [1] studied a class of contact Riemannian manifolds satisfying some special conditions. We call it Kenmotsu manifold. Kenmotsu proved that if in a Kenmotsu manifold the condition $R(X,Y) \cdot R = 0$ holds, then the manifold is of negative curvature -1, where R is the curvature tensor of type (1,3) and $R(X,Y)$ denotes the derivation of the tensor algebra at each point of the tangent space. A Riemannian manifold satisfying the condition $R(X,Y) \cdot R = 0$ is called *semi-symmetric* [2]. In analogous manner, a Riemannian manifold is called *Ricci semi-symmetric* (respectively *Weyl semi-symmetric*) if $R(X,Y) \cdot S = 0$ (respectively $R(X,Y) \cdot C = 0$), where S is the Ricci tensor(respectively C is the Weyl conformal curvature tensor of type (1,3)) [3]. Though $R(X,Y) \cdot R = 0$ implies $R(X,Y) \cdot S = 0$, but the converse is not true, in general. So it is meaningful to undertake the study of Kenmotsu manifold satisfying the condition $R(X,Y) \cdot S = 0$. It is proved that

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if a Kenmotsu manifold is Ricci semi-symmetric, then it is an Einstein manifold. Next we prove that a Ricci recurrent manifold [4] satisfies the condition $R(X,Y) \cdot S = 0$. So we get a theorem on a Ricci recurent Kenmotsu manifold. Further it is known that every semi-symmetric manifold is Weyl semi-symmetric, but the converse is not true, in general. In section 4, we consider Weyl semi-symmetric Kenmotsu manifold.

Next we prove that a conformally recurrent manifold is Weyl semi-symmetric. It is proved that a conformally recurrent Kenmotsu manifold is locally isometric to the Hyperbolic space $H^n(1)$. In section 5, we deal with Kenmotsu manifold whose Ricci tensor is η-parallel. In section 6, we consider η-Einstein Kenmotsu manifold. In the last section, we study a Kenmotsu manifold admitting a transformation which keeps Riemannian curvature tensor and Ricci tensor invariant.

2. Kenmotsu manifold

Let $(M^n, \phi, \xi, \eta, g)$ be an n-dimensional(where $n = 2m+1$) almost contact Riemannian manifold, where ϕ is a $(1,1)$-tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known that the (ϕ, ξ, η, g)-structure satisfies the conditions [5]

$$\phi\xi = 0, \ \eta(\phi X) = 0, \ \eta(\xi) = 1, \tag{2.1}$$

$$\phi^2 X = -X + \eta(X)\xi, \ g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any vector fields X and Y on M^n.

If moreover

$$(\nabla_X \phi)Y = -g(X, \phi Y)\xi - \eta(Y)\phi X, \tag{2.4}$$

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.5}$$

where ∇ denotes the Riemannian connection of g hold, then $(M^n, \phi, \xi, \eta$ $g)$ is called a *Kenmotsu manifold.*

In this case, it is well known that [1]

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.7}$$

where S denotes the Ricci tensor. From (2.6), it easily follows that

$$R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.8}$$

$$R(X,\xi)\xi = \eta(X)\xi - X. \tag{2.9}$$

Since $S(X,Y) = g(QX,Y)$, we have

$$S(\phi X, \phi Y) = g(Q\phi X, \phi Y),$$

where Q is the Ricci operator.

Using the properties $g(X,\phi Y) = -g(\phi X, Y)$, $Q\phi = \phi Q$, (2.2) and (2.7), we get

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y). \tag{2.10}$$

Also we have [1]

$$(\nabla_X \eta)(Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2.11}$$

A Kenmotsu manifold M^n is said to be *η-Einstein* if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.12}$$

for any vector fields X and Y, where a and b are functions on M^n.

3. Ricci semi-symmetric Kenmotsu manifold

Let us consider an n-dimensional ($n = 2m+1$) Kenmotsu manifold which satisfies the condition

$$R(X,Y)\cdot S = 0. \tag{3.1}$$

From (2.6) we have

$$g(R(X,Y)\xi, V) = \eta(X)g(Y,V) - \eta(Y)g(X,V)$$

or,

$$-g(R(X,Y)V,\xi) = \eta(X)g(Y,V) - \eta(Y)g(X,V)$$

or,

$$\eta(R(X,Y)V) = \eta(Y)g(X,V) - \eta(X)g(Y,V). \tag{3.2}$$

From (3.1), we get

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{3.3}$$

Putting $U = \xi$ in (3.3) and using (2.6), (2.7) and (3.2) we get

$$\begin{aligned} &\eta(X)S(Y,V) - \eta(Y)S(X,V) \\ &\qquad - (n-1)[\eta(Y)g(X,V) - \eta(X)g(Y,V)] = 0. \end{aligned} \tag{3.4}$$

Now putting $X = \xi$ in (3.4), we get by using (2.1) and (2.7)

$$S(Y,V) = -(n-1)g(Y,V). \tag{3.5}$$

Hence we can state the following:

THEOREM 1. *A Ricci semi-symmetric Kenmotsu manifold is an Einstein manifold.*

Since $R(X,Y)\cdot R = 0$ implies $R(X,Y)\cdot S = 0$, we can state the following corollary.

COROLLARY 1. *A semi-symmetric Kenmotsu manifold is an Einstein manifold.*

The above corollary has been proved by K.Kenmotsu in another way [1].

A Riemannian manifold M^n is said to be *Ricci recurrent* [4] if the Ricci tensor S is non-zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = \alpha(X)S(Y,Z), \tag{3.6}$$

where α is non-zero 1-form.

We now define a function f on M^n by $f^2 = g(Q,Q)$, where $g(QX,Y) = S(X,Y)$ and the Riemannian metric g is extended to the inner product between the tensor fields in the standard fashion. Then we obtain

$$f(Yf) = f^2\alpha(Y).$$

So from this we have

$$Yf = f\alpha(Y) \neq 0. \tag{3.7}$$

From (3.7), we have

$$X(Yf) - Y(Xf) = \{X\alpha(Y) - Y\alpha(X)\}f. \tag{3.8}$$

Therefore we get

$$\{\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}\}f = \{X\alpha(Y) - Y\alpha(X) - \alpha[X,Y]\}f. \tag{3.9}$$

Since the left hand side of the above equation is identically zero and $f \neq 0$ on M^n by our assumption, we obtain

$$d\alpha(X,Y) = 0, \tag{3.10}$$

that is, the 1-form α is closed. Now from $(\nabla_Y S)(U,V) = \alpha(Y)S(U,V)$, we get

$$(\nabla_X\nabla_Y S)(U,V) = \{X\alpha(Y) + \alpha(X)\alpha(Y)\}S(U,V).$$

Hence from (3.10), we get

$$(R(X,Y)\cdot S)(U,V) = 2d\alpha(X,Y)S(U,V).$$

That is, our manifold satisfies $R(X,Y) \cdot S = 0$. Thus a Ricci recurrent manifold is Ricci semi-symmetric. Hence from Theorem 1, we can state the following:

THEOREM 2. *A Ricci recurrent Kenmotsu manifold is an Einstein manifold.*

4. Kenmotsu manifold satisfying certain condition on the conformal curvature tensor

In [1] it is proved that a conformally flat Kenmotsu manifold is a manifold of constant negative curvature -1.

Again it is known [6] that a manifold of constant negative curvature-1 is locally isometric with the hyperbolic space $H^n(1)$. Now we state the following theorem [6]:

THEOREM 3. *A Weyl semi-symmetric Kenmotsu manifold* M^n $(n > 3)$ *is conformally flat.*

Since the proof of this theorem is done by the same method as M.C. Chaki and M.Tarafdar proved the theorem in section 2 in [7], we shall omit it here.

Since a conformally flat Kenmotsu manifold is locally isometric with $H^n(1)$, we can restate the theorem 3 as follows: "*An n-dimensional* $(n > 3)$ *Weyl semi-symmetric Kenmotsu manifold is locally isometric with the hyperbolic space* $H^n(1)$."

A non-conformally flat Riemannian manifold M^n is called *conformally recurrent* [8] if the conformal curvature tensor C satisfies the condition $\nabla C = \alpha \otimes C$, where α is an everywhere non-zero 1-form. As in section 3, we can prove that a conformally recurrent Riemannian manifold satisfies $R(X,Y) \cdot C = 0$. Hence we can state the following:

COROLLARY 2. *A conformally recurrent Kenmotsu manifold is locally isometric with the hyperbolic space* $H^n(1)$.

5. Kenmotsu manifold with η-parallel Ricci tensor

DEFINITION. The Ricci tensor S of a Kenmotsu manifold M^n is called η-parallel, if it satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{5.1}$$

for all vector fields X, Y and Z.

The notion of Ricci η-parallelity for the Sasakian manifolds was introduced by M. Kon [9].

Now, let us consider an n-dimensional Kenmotsu manifold M^n with η-parallel Ricci tensor. We have

$$(5.2)\qquad \begin{aligned}&(\nabla_X S)(\phi Y, \phi Z)\\ &= \nabla_X S(\phi Y, \phi Z) - S(\nabla_X \phi Y, \phi Z) - S(\phi Y, \nabla_X \phi Z).\end{aligned}$$

Using (2.4), (2.7), (2.10) and $\eta(\phi X) = 0$ in (5.2), we get

$$(5.3)\qquad \begin{aligned}&(\nabla_X S)(\phi Y, \phi Z)\\ &= \nabla_X S(Y, Z) + (n-1)\{\eta(Z)\nabla_X \eta(Y) + \eta(Y)\nabla_X \eta(Z)\}\\ &\quad + \eta(Y)\{S(X, Z) + (n-1)\eta(X)\eta(Z)\}\\ &\quad + \eta(Z)\{S(Y, X) + (n-1)\eta(Y)\eta(X)\} - S(\nabla_X Y, Z)\\ &\quad - S(Y, \nabla_X Z) - (n-1)\{\eta(Z)\eta(\nabla_X Y) + \eta(Y)\eta(\nabla_X Z)\}.\end{aligned}$$

Also,

$$(5.4)\qquad (\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y),$$

and

$$(5.5)\qquad \nabla_X S(Y, Z) = (\nabla_X S)(Y, Z) + S(\nabla_X Y, Z) + S(Y, \nabla_X Z).$$

By virtue of (2.11), (5.4) and (5.5) and owing to (5.3), we get

$$(5.6)\qquad \begin{aligned}&(\nabla_X S)(\phi Y, \phi Z)\\ &= (\nabla_X S)(Y, Z) + (n-1)\{g(X, Y)\eta(Z)\\ &\quad + g(X, Z)\eta(Y)\} + \{\eta(Y)S(X, Z) + \eta(Z)S(Y, X)\}.\end{aligned}$$

Using (5.1) in (5.6), we get

$$(5.7)\qquad \begin{aligned}&(\nabla_X S)(Y, Z)\\ &= -(n-1)\{g(Y, X)\eta(Z) + g(Z, X)\eta(Y)\}\\ &\quad - \{\eta(Y)S(Z, X) + \eta(Z)S(Y, X)\}.\end{aligned}$$

Hence we can state the following:

PROPOSITION 4. *A Kenmotsu manifold $M^n(\phi, \eta, \xi, g)$ has η-parallel Ricci tensor if and only if* (5.7) *holds.*

Now let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold M^n for $i = 1, 2, ..., n$. Putting $Y = Z = e_i$ in (5.7) and then taking summation over the index i, we get

$$(5.8)\qquad dr(X) = 0,$$

which implies that r is constant, where r denotes the scalar curvature of the manifold M^n. Thus we can state the following:

THEOREM 5. *If a Kenmotsu manifold M^n has η-parallel Ricci tensor, then the scalar curvature is constant.*

6. η-Einstein Kenmotsu manifold

In [1], it is proved that if a Kenmotsu manifold is η-Einstein, then $a + b = -(n-1)$. Here we prove that a and b are both constant.

Let us consider an η-Einstein Kenmotsu manifold. Then putting $X = Y = e_i$ in (2.12), $i = 1, 2, ..., n$ and taking summation for $1 \le i \le n$, we have

$$r = an + b, \tag{6.1}$$

where r is the scalar curvature. On the other hand, putting $X = Y = \xi$ in (2.12) and then using (2.7) and (2.1) we get

$$a + b = -(n-1). \tag{6.2}$$

Hence it follows from (6.1) and (6.2) that

$$a = \frac{r}{n-1} + 1, \ b = \frac{-r}{n-1} - n. \tag{6.3}$$

Thus we have

LEMMA 6. *The Ricci tensor of an η-Einstein Kenmotsu manifold is given by*

$$S(X,Y) = \left\{\frac{r}{n-1} + 1\right\} g(X,Y) + \left\{\frac{-r}{n-1} - n\right\} \eta(X)\eta(Y). \tag{6.4}$$

Now we consider an η-Einstein Kenmotsu manifold $M^n (n > 3)$ which is not an Einstein one. Then differentiating (6.4) covariantly along Z and using (2.11) we get

$$\begin{aligned} &(\nabla_Z S)(X,Y) \\ &= \left\{\frac{dr(Z)}{n-1}\right\}\{g(X,Y) - \eta(X)\eta(Y)\} \\ &+ \left\{\frac{-r}{n-1} - n\right\}\{g(Z,X)\eta(Y) - g(Z,Y)\eta(X)\}. \end{aligned} \tag{6.5}$$

Putting $X = Y = e_i$ in (6.5) and taking summation for $1 \leq i \leq n$, we have

$$(n-2)dr(Z) = 0.$$

If $n > 3$, we get r is constant. Thus a and b are both constants. This leads to the following:

THEOREM 7. *If a Kenmotsu manifold is η-Einstein, then a and b are both constants.*

From Corollary 9 of Proposition 8 of Kenmotsu [1] we get the following:

COROLLARY 3. *An η-Einstein Kenmotsu manifold is an Einstein one.*

7. Some transformations in Kenmotsu manifold

We now consider a transformation μ which transform a Kenmotsu structure (ϕ, ξ, η, g) into another Kenmotsu structure $(\overline{\phi}, \overline{\xi}, \overline{\eta}, \overline{g})$. We denote by the notation "bar" the geometric objects which are transformed by the transformation μ.

We first suppose that in a Kenmotsu manifold the Riemannian curvature tensor is invariant with respect to the transformation μ. Thus we have

$$\overline{R}(X,Y)Z = R(X,Y)Z, \tag{7.1}$$

for all X, Y and Z. This gives $\eta(\overline{R}(X,Y)Z) = \eta(R(X,Y)Z)$, and hence by virtue of (3.2) we get

$$\eta(Y)g(X,Z) - \eta(X)g(Y,Z) = \eta(\overline{R}(X,Y)Z). \tag{7.2}$$

Putting $Y = \overline{\xi}$ in (7.2) and then using (2.8) we obtain

$$\eta(\overline{\xi})g(X,Z) - \eta(X)g(\overline{\xi},Z) = \eta(\overline{\xi})\overline{g}(X,Z) - \overline{\eta}(Z)\eta(X). \tag{7.3}$$

Interchanging X and Z in (7.3), it follows

$$\eta(\overline{\xi})g(X,Z) - \eta(Z)g(\overline{\xi},X) = \eta(\overline{\xi})\overline{g}(X,Z) - \overline{\eta}(X)\eta(Z). \tag{7.4}$$

Subtracting (7.4) from (7.3) we obtain

$$\eta(Z)g(\overline{\xi},X) - g(\overline{\xi},Z)\eta(X) = \overline{\eta}(X)\eta(Z) - \overline{\eta}(Z)\eta(X). \tag{7.5}$$

Substituting ξ for Z in (7.5), we obtain by using (2.1)

$$g(\overline{\xi},X) - g(\overline{\xi},\xi)\eta(X) = \overline{\eta}(X) - \overline{\eta}(\xi)\eta(X). \tag{7.6}$$

Also from (7.1) we have

$$\overline{S}(X,Y) = S(X,Y)$$

and hence

$$\overline{S}(\xi,\overline{\xi}) = S(\xi,\overline{\xi}).$$

This gives by virtue of (2.7) that

$$\overline{\eta}(\xi) = \eta(\overline{\xi}). \tag{7.7}$$

Using (7.7) in (7.6) and since $\eta(\overline{\xi}) = g(\overline{\xi},\xi)$ we get

$$\overline{\eta}(X) = g(\overline{\xi},X). \tag{7.8}$$

By virtue of (7.8) we get from (7.4)

$$\{g(X,Z) - \overline{g}(X,Z)\}\eta(\overline{\xi}) = 0.$$

This implies

$$g(X,Z) = \overline{g}(X,Z)$$

for all X and Z, if $\eta(\overline{\xi}) \neq 0$. Hence we can state the following:

THEOREM 8. *In a Kenmotsu manifold (M^n,g), the transformation μ which leaves the curvature tensor invariant and $\eta(\overline{\xi}) \neq 0$ is an isometry.*

Again a vector field V on a contact manifold with contact form η is said to be an *infinitesimal contact transformation* [10] if V satisfies

$$(L_V\eta)X = \sigma\eta(X), \tag{7.9}$$

for a scalar function σ, where L_V denotes the Lie differentiation with respect to V. Especially, if σ vanishes identically, then it is called an *infinitesimal strict contact transformation* [10].

Let us now suppose that in a Kenmotsu manifold, the infinitesimal contact transformation leaves the Ricci tensor invariant. Then we have

$$(L_VS)(X,Y) = 0,$$

which gives

$$(L_VS)(X,\xi) = 0. \tag{7.10}$$

We have

$$(L_VS)(X,\xi) = L_V(S(X,\xi)) - S(L_VX,\xi) - S(X,L_V\xi). \tag{7.11}$$

By virtue of (2.7) and (7.10) we get from (7.11)

$$(n-1)(L_V\eta)(X) + S(X,L_V\xi) = 0. \tag{7.12}$$

Using (7.9) in (7.12), we obtain

$$S(X,L_V\xi) = -(n-1)\sigma\eta(X). \tag{7.13}$$

Substituting ξ for X in (7.13) and using (2.7), we get

$$\eta(L_V\xi) = \sigma. \tag{7.14}$$

Again substituting ξ for X in (7.9), we have $(L_V\eta)(\xi) = \sigma$, that is,

$$L_V(\eta(\xi)) - \eta(L_V\xi) = \sigma. \tag{7.15}$$

By virtue of (7.14) and (7.15) we get

$$\sigma = 0.$$

Thus we can state the following:

THEOREM 9. *In a Kenmotsu manifold, the infinitesimal contact transformation which leaves the Ricci tensor invariant is an infinitesimal strict contact transformation.*

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