A Mathematical Program for Gifted Students in High Schools

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In this note we propose a 10 hour-program for gifted students. A special feature of this program is to utilize the computer for the topics on cryptography. During the program the students are required to read some books. It is possible for some parts of the program may not be attractive to the students who are not interested in computers.

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ZDM Classification: U54, U64
MSC2000 Classification: 97U60, 97U70

INTRODUCTION

There might be various issues in gifted education. Two of them are developing teaching/learning materials and educating teachers for gifted students. In Shin & Han (2000), we roughly proposed some teaching programs for gifted students. Here in this note, we present a little elaborated program based on there. Some programs were introduced in Shin (2004). For complete description of the programs we include them here again.

The education and professional development of mathematics teachers were discussed in Park & Shin (2004). The teachers at institutes for gifted education need to have strong mathematical content knowledge and understand the characteristics of gifted students. However, the education of teachers for the institutes for gifted students doesn't seem to be very different from that for ordinary students. In this note, however, we don't discuss teacher education for gifted students.

The program introduced in this note is for 10 hours in classroom. The students are required to read the recommended books during the course. Depending on students or classroom situation, some parts of the program can be assigned as home-works. It is possible for some parts of the program may not be attractive to the students who are not interested in computers.

DESCRIPTION OF THE PROGRAM

I. The first course

The first 1 hour course consisting of six (1, 2, 3, 4, 5 and 6) problems is for enhancing flexibility, fluency and originality in thinking. No prerequisite knowledge is assumed for this course. Reading Abbot (1991), Abbot (2002), Greene (1999), Hoffman (1998) and Schechter (1998) are recommended.

Problem 1.

(1) Show a picture that presents the following question: An 8-year old boy wrote "M" and asked his sister (7-year old) sitting in front of him to read it.

(2) Question: What do you think was her answer?

(3) Now encourage the students to discuss the answer "W."

Problem 2.

(1) Show a picture that presents the situation of the following question: A 6-year old girl who has never learned English claimed that she could read "BASEBALL."

(2) Question: What do you think was her reading?

(3) Now encourage the students to discuss the answer "77483548."

Problem 3.

(1) Show a picture that presents the following question: Two trains are on the same t rack, heading for a collision. The trains are exactly a mile apart and each is traveling at 30 miles per hour, when a very speedy fly perched on the front of one takes off toward the other train at 60 miles per hour. When the fly lands on the other train it instantly turns around and flies back. It keeps this up, flying from train to train, until it is crushed in the inevitable noisy collision. How far does the fly travel (Schechter 1998, p.115)?

(2) Let the students find various solutions for it and discuss them.

Problem 4.

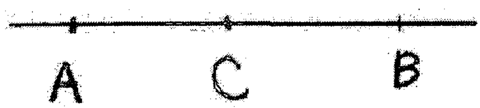
(1) Show a picture that presents the following question: There are two 1-liter jars with

milk and tea, respectively. A spoon of milk from milk jar was put into tea jar. After churning sufficiently, a spoon of liquid from tea jar was put into milk jar. Compare the amounts of milk in tea jar and tea in milk jar.

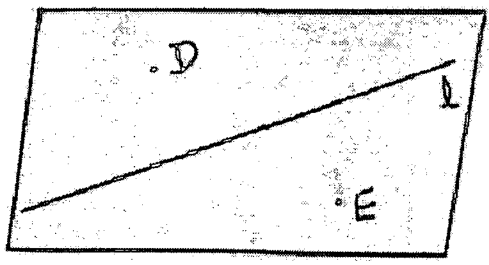
(2) Let the students find various solutions for it and discuss them.

Problem 5.

(1) In a lineland (1 dimensional world), is it possible for A to go to B without passing through C?
(2) What if we are allowed one more dimension?
(3) What is the shortest distance between A and B?
(4) What if the world is curved?



(5) In a flat land (2 dimensional world), is it possible for D to go to E without touching the line *l?*
(6) What if we are allowed one more dimension?
(7) What is the shortest distance between D and E?
(8) What if the world is curved?



(9) What is the shortest distance between Seoul and New York?
(10) What if our universe is curved?
(11) The point F is inside a ball (a 3 dimensional world), and point G is outside of the ball. Is it possible for F to go to G without touching the surface of the ball?
(12) What if we are allowed one more dimension for the space?
(13) What is the dimension of this universe? (It may be a good place for you to mention

Newtonian physics (classical physics), Einstein's physics (the general theory of relativity), and super-string theory.)

(14) What can we see from the above?

Problem 6.

(1) Let the students solve the following problem (Kilpatrick, Swafford & Findell 2001, p.126): There are 36 bicycles and tricycles. Collectively there are 80 wheels. How many tricycles are there?

(2) Now encourage the students to discuss the following solution: $80 - 36 \times 2 = 8$.

II. The second course

The purpose of the second course with the following problem (7) is to inform the students that an argument which is valid in some special cases might not be so in general. This is for 1 hour and doesn't assume any prerequisite knowledge.

Problem 5.

Challenge the students with the following questions (Hamming 1986).

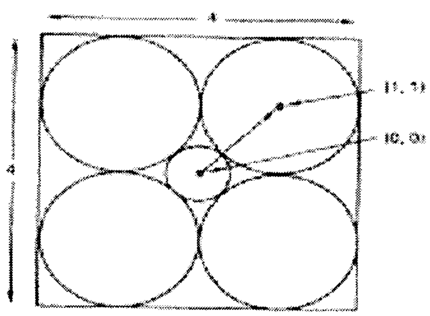
(1) Consider the following figure of a 4×4 square. Find the radius of the circle at the center.

(2) Consider the same problem in 3-dimensional case. In this case, we have a $4 \times 4 \times 4$ cube with eight unit spheres in the corners. Find the radius of the inner sphere.

(3) Now consider the similar problem in n-dimension. We have a $4 \times 4 \times \cdots \times 4$ cube with 2^n unit spheres in the corners, each one touching all its n neighboring spheres. Find the radius of the inner sphere.

(4) What happen if n are 10?

(5) Refer to Hamming (1986) to see the following paradoxical conclusion: The inner sphere becomes arbitrarily larger than the volume of the cube which contains all the 2^n unit spheres in the corners.



(6) What can we see from the above?
(7) Is there any mathematical theorem which holds in 2-dimensional but not in 3 dimensional case (Sharygin 2000)?

III. The course of 3 topics

The course consisting of the following 3 topics (8, 9 and 10) may be helpful for the students not only to understand the characteristic of mathematics and the connection of mathematics and physics but also to see the gap between intuitional observation and reality. This is for 3 hours and requires the students to understand the concepts of dimension, cardinality, and continuous map. Reading Feynman (1985), Greene (1999), Hoffman (1998) and Schechter (1998) are recommended.

Problem 8.
(1) Explain the following examples:
  (a) A confusion of Cantor: 'I see it, but I don't believe it.' The cardinalities of (0, 1) and (0, 1) × (0, 1) are same.
  (b) Space-filling curve by Peano or by Hilbert. There exists a continuous map from [0, 1] onto [0, 1] × [0, 1].
(2) Question: Does your concept of dimension help you clearly understand the above two situations?

Problem 9.
(1) Explain the following: Cantor set, Sierpinski gasket, Menger sponge. In fact, Cantor set is a non-denumerable set with measure 0. Sierpinski gasket can be considered as the 2-dimensional version of Cantor set. The circumference of this set is of infinite length but with area zero. Menger sponge can be considered as the 3-dimensional version of Cantor set. The outer surface of this set is of infinite area but with volume zero.
(2) Question: What can you imagine the 4-dimensional version of Cantor set?

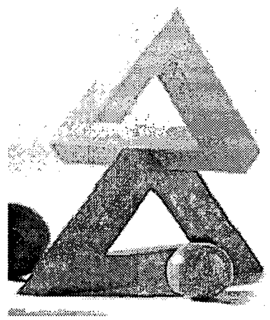
Problem 10.
(1) The above situations might motivate various mathematical or physical questions. Explain briefly the various definitions of dimension in following cases. It'd be effective to utilize some proper photos or figures.
    In linear algebra
    In topology

In fractal geometry

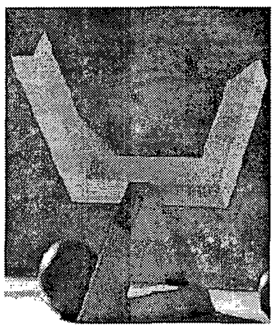
(2) Explain briefly the connection between mathematics and physics:

Euclidean geometry and Newtonian physics
Minkowski geometry and Special Relativity of Einstein
Riemannian geometry and General Relativity of Einstein
Algebraic geometry and super-string theory

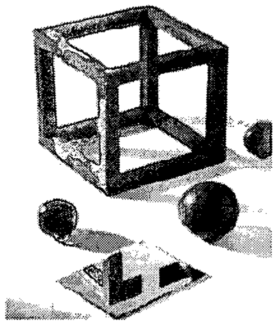
(3) Discuss the following photos[1]. From these photos we can conclude that the situations that seem to be contradictory in the second dimensional world can be very natural if one more dimension is allowed. Furthermore we can expect that some conflicting phenomena in this 3 dimensional world could be well explained in four or more dimensional world. It is worthwhile to remark that the super-string theory claims that our universe is of 11 dimensions.



The upper triangle in the above photo seems to be the Penrose Triangle. Noting that this is not a drawing but a photo, it is quite contradictory. The real situation can be seen in the photo below. The contradictory situation in two dimensional world could be very natural in three dimensional world.



[1] The author doesn't know the original source of this question. Being informed of it, he will appreciate it.



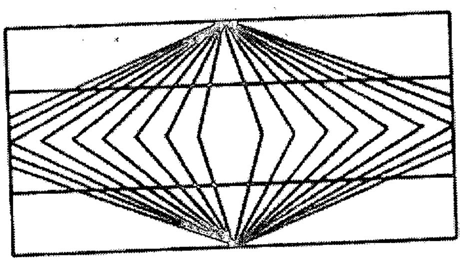
The upper box seems to be contradictory. The mirror in the lower part makes it clear what the situation is. As same as before, the contradictory situation in two dimensional world could be very natural in three dimensional world.

IV. The course of 3 problems

The course consisting of the following 3 problems (11, 12 and 13) may help the students to see the limit of science and mathematics. In particular, the students may see the inconsistence between practical experience and mathematics. This is for 2 hours and assuming no prerequisite knowledge.

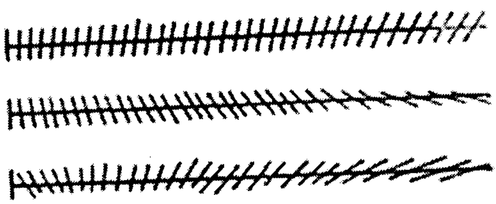
Problem 11.

Capability of human sensors is quite limited. To understand it, discuss the following drawings:



Even though the two lines in the middle are in parallel, they don't seem to be so. Our sense of sight may be affected by surrounding information.

Even though the above segment is divided into 7 equal subsections, it doesn't seem to be so. This is another example to show that our sense of sight may be affected by surrounding information. The following is one more example. Note that the three lines are in parallel.



Problem 12.

(1) The capability of human reasoning is also quite limited. To understand it, consider the following cognitively conflicting situations: Painting of Menger Sponge. How much paint will be needed to paint all the walls of Menger Sponge? As we can see easily, the outer surface of this set is of infinite area but its volume is zero.

(2) Banach-Tarski Paradox: Note how conflicting the claim of this paradox is with our general instinct. It is remarkable that this is not a paradox but a theorem if we accept the Axiom of Choice.

(3) Algebraic numbers and transcendental numbers: Ask every student in your class to write a number on her/his sheet of paper. Check how many of them are transcendental numbers. Compare the result with the fact that the theoretical probability of getting an algebraic number by random choice is zero. From this experiment we can see the difference between theoretical probability and practical probability.

Problem 13.

To see that the capacity of mathematics is very limited, explain Goedel's Incompleteness Theorem: In any consistent formal system of mathematics sufficiently strong to allow one to do basic arithmetic, one can construct a statement about natural numbers that can be neither proven nor disproven within that system. The consistency and completeness of axioms need to be also explained. For this, incidence geometries (3-point, 4-point and 5-point) are useful.

V. The course of 4 problems

The course consisting of the following 4 problems (14, 15, 16 and 17) may be effective for the students to see the usefulness as well as the difficulty of the concept of probability. This is for 3 hours and requires the students to read Flannery & Flannery (2000), Hoffman (1998) and Schechter (1998). A little knowledge in the elementary number theory is assumed.

Problem 14.
The concept of the probability is frequently conflicting with our intuition. Challenge the students with the following questions:
Monty Hall dilemma (Hoffman 1998, p.250)
Two-envelope paradox (Chalmers 2001)

Problem 15.
(1) The concept of probability, however, is useful for hiding information. Challenge the students with the following question as a typical example of probabilistic justification: A bag contains two GO pebbles. There are three possibilities: Two whites, one white and one black, or two blacks. I have a bag with two whites. How can I prove that my bag has two whites without showing the contents?
(2) Ask the students to compare the probabilistic arguments with deterministic ones.

Problem 16[2].

- The following experiment will be helpful for the students to see the usefulness of the probabilistic arguments. From the experiment it'd be clear that, in computational mathematics, primality test is not connected with factoring.
- Write two programs (using BASIC or LOGO) for factoring and for expressing the input natural number in the binary system.
- Running above program, express $2^k - 1$, $1 \leq k \leq 10$, in binary system.
- Find a necessary condition for $2^k - 1$, $k \in N$, to be prime numbers.
- Test the primality for $2^3 - 1$, $2^5 - 1$, $2^7 - 1$, $2^{11} - 1$.
- Use MATHEMATICA or some other tool to test the primality of $2^{19} - 1$, $2^{23} - 1$.
- Visit http://www.mersenne.org/primes.htm for Mersenne primes.
- Consider the similar problems for repunit numbers.

2 This problem can be offered as homework. If some students feel uncomfortable with writing a program, they can be supplied with it.

Problem 15.

The probabilistic arguments can be interwoven with some number theories to design protocols for hiding information.

(1) Explain some basic facts in number theory:

Euler-Fermat Theorem

Quadratic residue

Discrete logarithm

(2) Explain the computationally one-way functions:

Factoring

Primality test is feasible[3].

Factoring is not feasible.

Extracting square roots

Squaring is feasible.

Extracting square roots is probabilistically feasible with prime modulus.

Extracting square roots is not feasible with composite modulus[4].

Discrete logarithm

Raising a number to any power is feasible.

Computing the index is not feasible.

RSA function

(3) Explain a Rabin's result that factoring and extracting square roots are computationally equivalent.

(4) Present some interesting protocols:

Coin flipping by telephone: The typical protocol utilizes the following properties.

One-wayness of extracting of square roots

One-wayness of factoring

Key distribution: The typical protocol utilizes the following property.

One-wayness of discrete logarithm

RSA system: The typical protocol utilizes the following property.

One-wayness of RSA function

Non-deterministic schemes of verification

Probabilistic arguments

Zero-knowledge proof: The typical protocol utilizes the following properties.

One-wayness of extracting of square roots

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[3] Recently deterministic and polynomial time algorithm for primality test has been introduced (Agrawal; Kayal & Saxena 2002; Bornemann 2003).

[4] This modulus is a product of two big prime numbers (of 100 digits, say) of same sizes.

One-wayness of factoring

CONCLUSION

It is good for the teachers to supply many interesting problems that are carefully designed. It is also effective to recommend some proper books to stimulate the students' mathematical talent.

It is worthwhile to note that a student who is very good at number theory may have no interest in analysis. This means that programs need to be student-centered.

Algebraic coding theory is another good source for developing materials for gifted students.

ACKNOWLEDGMENT

Some problems and photos in this note don't belong to the author. Even though he has tried to refer to the original sources, he was not successful. It'd be appreciated if he is informed of the sources.

REFERENCES

Abbot, E. A. (1991): *Flatland.* Princeton: Princeton University Press.

Abbot, E. A. (2002): *The Annotated Flatland: A Romance of Many Dimensions (Introduction and Notes by Ian Stewart).* Massachusetts: Perseus Publishing And Cambridge.

Agrawal, M.; Kayal, N. & Saxena, N. (2002): PRIMES is in P, IIT Kanpur preprint; http://www.cse.iitk.ac.in/news/primality.html.

Bornemann, F. (2003): PRIMES is in P: a breakthrough for "Everyman." *Notices of the American Mathematical Society* **50(5)**, 545–552.

Chalmers, D. J. (2001): The Two-Envelope Paradox: A Complete Analysis? http://www.u.arizona.edu/~chalmers/papers/envelope.html.

Feynman, R. P. (1985): QED: The Strange Theory of Light and Matter, Princeton University Press, Princeton.

Flannery, S. & Flannery, D. (2000): In Code, Profile Books.

Greene, B. (1999): the elegant universe, W. W. Norton & Company, New York, London.

Hamming, R. W. (1986): Coding and Information Theory, Prentice-Hall, INC.

Hoffman, P. (1998): The Man Who Loved Only Numbers, Hyperion, New York.

Kilpatrick, J.; Swafford, J. & Findell, B. (2001): Adding it up. National Research Council.

Park, H. S. & Shin, H. (2004): *Education and Professional Development of Mathematics Teachers in Korea*. A Regular Talk. ICME10, Copenhagen, Denmark.

Schechter, B. (1998): My brain is open, Simon & Schuster, New York.

Sharygin, I. (2000): Two articles and two hundred problems, manuscript.

Shin, H. (2004): A Study on Necessity and Possibility of Teaching Verification Schemes in School Mathematics, Communications of the Korean Mathematical Society, **19(4)**, *Korean Mathematical Society*, 585–599.

Shin, H. & Han, I. (2000): Mathematics Education for Gifted Students in Korea, A Regular Talk. ICME9, Tokyo, Japan.