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ON THE FEKETE-SZEGÖ PROBLEM FOR STRONGLY α-LOGARITHMIC CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $\mathcal{CS}^\alpha(\beta)$ denote the class of normalized strongly α-logarithmic close-to-convex functions of order β, defined in the open unit disk $\mathbb{U}$ by

$$\left|\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{1-\alpha}\left(\frac{zf'(z)}{g(z)}\right)^{\alpha}\right\}\right| \leq \frac{\pi}{2}\beta, \quad (\alpha, \beta \geq 0)$$

where $g \in \mathcal{S}^*$ the class of normalized starlike functions. In this paper, we prove sharp Fekete-Szegö inequalities for functions $f \in \mathcal{CS}^\alpha(\beta)$.

1. Introduction

Let $\mathcal{S}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are univalent in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $\mathcal{S}^*$ be the subclas of $\mathcal{S}$ consisting of all starlike functions in $\mathbb{U}$.

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A classical result of Fekete and Szegö [4] determines the maximum value of $|a_3 - \mu a_2^2|$, as a function of the real parameter μ, for $f \in \mathcal{S}$. There are also several reults of this type in the literature, each of them dealing with $|a_3 - \mu a_2^2|$ for various classes of functions (see, e. g., [1,6-10]).

Denote by $\mathcal{K}(\beta)$ the class of strongly close-to -convex functions of order β. Thus $f \in \mathcal{K}(\beta)$ if and only if there exists $g \in \mathcal{S}^*$ such that

$$\left|\arg \frac{zf'(z)}{g(z)}\right| \leq \frac{\pi}{2}\beta, \quad (\beta \geq 0;\ z \in \mathbb{U})$$

A great deal of attention has been given in recent years to the class $\mathcal{K}(\beta)$ introduced by Pommerenke [13]. For $0 \leq \beta \leq 1$, the class $\mathcal{K}(\beta)$ is a subclass of close-to-convex functions introduced by Kaplan [6] and hence contains only univalent functions. However, Goodman [5] showed that $\mathcal{K}(\beta)$ can contain functions with infinite valence for $\beta > 1$. The Fekete-Szegö problems for $\mathcal{K}(1)$ and $\mathcal{K}(\beta)$ has been also solved by Keogh and Merkes [7] and London[10], respectively. We now introduce a new class which covers the class $\mathcal{K}(\beta)$ in terms of powers as follows:

Definition. A function $f \in \mathcal{S}$, given by (1.1) is said to be strongly α-logarithmic close-to-convex of order β if there exists a function $g \in \mathcal{S}^*$such that

$$\left|\arg\left\{\left(\frac{f(z)}{g(z)}\right)^{1-\alpha}\left(\frac{zf'(z)}{g(z)}\right)^{\alpha}\right\}\right| \leq \frac{\pi}{2}\beta, \quad (\alpha, \beta \geq 0;\ z \in \mathbb{U}). \quad (1.2)$$

We denote by $\mathcal{CS}^{\alpha}(\beta)$ the class of strongly α-logarithmic close-to-convex functions of order β. We note that $\mathcal{CS}^1(\beta) = \mathcal{K}(\beta)$. In particular, $\mathcal{CS}^0(1)$ is the class of close-to-star functions introduced by Reade [15].

The purpose of the present paper is to prove sharp Fekete-Szegö inequalities of the functions belonging to the class $CS^{\alpha}(\beta)$, which extend the results by Abdel-Gawad and Thomas [1], Keogh and Merkes [7] and London [10].

2. Results

In proving our main result, we need the following lemma.

LEMMA. *Let p be analytic in $\mathbb{U}$ and satisfying* Re $\{p(z)\} > 0$ *for* $z \in \mathbb{U}$, *with* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$.
Then

$$\mid p_n \mid \ \leq 2 \tag{2.1}$$

and

$$\left| p_2 - \frac{{p_1}^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \tag{2.2}$$

The inequlaity (2.1) can be first proved by Carathéodory [2] (also see Duren [3], p.41) and the inequality (2.2) can be found in [14, p.166].

With the help of Lemma, we now derive

THEOREM. *Let $f \in \mathcal{CS}^\alpha(\beta)$ and be given by* (1.1). *Then for* $\alpha \geq 0$ *and* $\beta \geq 0$, *we have*

$$(1+2\alpha)|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} 1 + \frac{2(1+\beta)^2((1+3\alpha)-2(1+2\alpha)\mu)}{(1+\alpha)^2} & \text{if } \mu \leq \frac{(1+\beta)(1+3\alpha)-(1+\alpha)^2}{2(1+\beta)(1+2\alpha)}, \\ 1 + 2\beta + \frac{2((1+3\alpha)-2(1+2\alpha)\mu)}{(1+\alpha)^2-\beta((1+3\alpha)-2(1+2\alpha)\mu)} & \text{if } \frac{(1+\beta)(1+3\alpha)-(1+\alpha)^2}{2(1+\beta)(1+2\alpha)} \leq \mu \leq \frac{1+3\alpha}{2(1+2\alpha)}, \\ 1 + 2\beta & \text{if } \frac{1+3\alpha}{2(1+2\alpha)} \leq \mu \leq \frac{(1+\beta)(1+3\alpha)+(1+\alpha)^2}{2(1+\beta)(1+2\alpha)}, \\ -1 + \frac{2(1+\beta)^2(2(1+2\alpha)\mu-(1+3\alpha))}{(1+\alpha)^2} & \text{if } \mu \geq \frac{(1+\beta)(1+3\alpha)+(1+\alpha)^2}{2(1+\beta)(1+2\alpha)}. \end{cases}$$

For each μ, there are functions in $\mathcal{CS}^\alpha(\beta)$ such that equality holds in all cases.

Proof. Let $f \in \mathcal{CS}^\alpha(\beta)$. Then it follows from (1.2) that we may write

$$\left(\frac{f(z)}{g(z)}\right)^{1-\alpha}\left(\frac{zf'(z)}{g(z)}\right)^{\alpha}=p^{\beta}(z),$$

where g is starlike and p has positive real part. Let $g(z)=z+b_2z^2+b_3z3+\cdots$, and let be given as in Lemma. Then by equating coefficients, we obtain

$$(\alpha+1)a_2=b_2+\beta p_1$$

and

$$\begin{aligned}(1+2\alpha)a_3=b_3&+\frac{\alpha(1-\alpha)}{2(1+\alpha)^2}b_2^2+\frac{\beta(1+3\alpha)}{(1+\alpha)^2}p_1b_2\\&+\frac{\beta(\beta(1+3\alpha)-(1+\alpha)^2)}{2(1+\alpha)^2}p_1^2+\beta p_2.\end{aligned}$$

So, with

$$x=\frac{(1+3\alpha)-2(1+2\alpha)\mu}{(1+\alpha)^2},$$

we have

$$\begin{aligned}(1+2\alpha)(a_3-\mu a_2^2)&=b_3+\frac{1}{2}(x-1)b_2^2\\&\quad+\beta(p_2+\frac{1}{2}(\beta x-1)p_1^2)+\beta xp_1b_2.\end{aligned}\tag{2.3}$$

Since rotations of f also belong to $\mathcal{CS}^{\alpha}(\beta)$, we may assume, without loss of generality, that $a_3-\mu a_2^2$ is positive. Thus we now estimate $\mathrm{Re}(a_3-\mu a_2^2)$.

For some functions $h(z)=1+k_1z+k_2z^2+\cdots(z\in\mathbb{U})$ with positive real part, we have $zg'(z)=g(z)h(z)$. Hence, by equating coefficients, $b_2=k_1$ and $b_3=(k_2+k_1^2)/2$. So, by using Lemma and letting $k_1=2\rho e^{i\phi}(0\le\rho\le1,\ 0\le\phi\le2\pi)$ and $p_1=2re^{i\theta}(0\le r\le 1,\ 0\le\theta\le2\pi)$ in (2.3), we obtain

$$\begin{aligned}\operatorname{Re}(1+2\alpha)(a_3-\mu a_2^2) \leq 1-\rho^2 &+ (1+2x)\rho^2\cos 2\phi \\ &+2\beta(1-r^2)+2\beta^2 xr^2\cos 2\theta \\ &+4\beta xr\rho\cos(\theta+\phi)\end{aligned} \tag{2.4}$$

and we now proceed to maximize the right-hand side of (2.4). This function will be denote ψ whenever all parameters except x are held constant.

Assume that

$$\frac{(1+\beta)(1+3\alpha)-(1+\alpha)^2}{2(1+\beta)(1+2\alpha)} \leq \mu \leq \frac{1+3\alpha}{2(1+2\alpha)},$$

so that $0 \leq x \leq 1/(1+\beta)$. The expression $-t^2+t^2\beta x\cos 2\theta + 2xt$ is the largest when $t + x/(\beta x\cos 2\theta)$, we have

$$\begin{aligned}\psi(x) &\leq 1+2x+2\beta\left(1+\frac{x^2}{1-\beta x}\right) \\ &= 1+2\beta+\frac{2((1+3\alpha)-2(1+2\alpha)\mu)}{(1+\alpha)^2-\beta((1+3\alpha)-2(1+2\alpha)\mu)}\end{aligned}$$

and with (2.4) this estiablishes the second inequality in the theorem. Equality occurs if and only if

$$p_1 = \frac{2((1+3\alpha)-2(1+2\alpha)\mu)}{(1+\alpha)^2-\beta((1+3\alpha)-2(1+2\alpha)\mu)},\ p_2=b_2=2,\ b_3=3$$

and the corresponding function f is defined by

$$f(z)^{1-\alpha}(zf'(z))^{\alpha} = \frac{z}{(1-z)^2}\left(\lambda\frac{1+z}{1-z}+(1-\lambda)\frac{1-z}{1+z}\right)^{\beta},$$

where

$$\lambda = \frac{(1+\alpha)^2 + (1-2\beta)((1+3\alpha) - 2(1+2\alpha)\mu)}{2((1+\alpha)^2 - \beta((1+3\alpha) - 2(1+2\alpha)\mu))}.$$

We now to prove the first inequlity. Let

$$\mu \le \frac{(1+\beta)(1+3\alpha) - (1+\alpha)^2}{2(1+\beta)(1+2\alpha)},$$

so that $x \ge 1/(1+\beta)$. With $x_0 = 1/(1+\beta)$, we have

$$\begin{aligned}\psi(x) &\le \psi(x_0) + 2(x - x_0)(1+\beta)^2 \\ &\le 1 + \frac{2(1+\beta)^2((1+3\alpha) - 2(1+2\alpha)\mu)}{(1+\alpha)^2},\end{aligned}$$

as required. Equality occurs only if $p_1 = p_2 = 2,\ b_2 = 2,\ b_3 = 3$ and the corresponding function f is defined by

$$(f(z))^{1-\alpha}(zf'(z))^\alpha = \frac{z}{(1-z)^2}\left(\frac{1+z}{1-z}\right)^\beta.$$

Let $x_1 = -1/(1+\beta)$. We note that $\psi(x_1) \le 1 + 2\beta$. Then $\psi(x)$ satisfies

$$\begin{aligned}\psi(x) &\le \psi(x_1) + 2|x - x_1|(1+\beta)^2 \\ &\le -1 + \frac{2(1+\beta)^2(2(1+2\alpha)\mu - (1+3\alpha))}{(1+\alpha)^2},\end{aligned}$$

if $x \le x_1$, that is,

$$\mu \ge \frac{(1+\beta)(1+3\alpha) + (1+\alpha)^2}{2(1+\beta)(1+2\alpha)}.$$

Equality occurs only if $p_1 = 2i,\ \ p_2 = -2,\ \ b_2 = 2i,\ \ b_3 = -3$ and the corresponding function f is defined by

$$(f(z))^{1-\alpha}(zf'(z))^\alpha = \frac{z}{(1-iz)^2}\left(\frac{1+iz}{1-iz}\right)^\beta.$$

Finally, since

$$\psi(\lambda x_1) = \lambda\psi(x_1) + (1-\lambda)\psi(0) \le 1 + 2\beta$$

for $0 \le \lambda \le 1$, we obtain $\psi(x) \le 1 + 2\beta$ for $x_1 \le x \le 0$, i.e.,

$$\frac{1+3\alpha}{2(1+2\alpha)} \le \mu \le \frac{(1+\beta)(1+3\alpha)+(1+\alpha)^2}{2(1+\beta)(1+2\alpha)}.$$

Equality occurs only if $p_1 = b_2 = 0,\ p_2 = 2,\ b_3 = 1$ and the corresponding function f is defined by

$$(f(z))^{1-\alpha}(zf'(z))^{\alpha} = \frac{z(1+z^2)^{\beta}}{(1-z^2)^{1+\beta}}.$$

Therefore we complete the proof of Theorem.

From Theorem, we have immediately the following

COROLLARY. *Let $f \in \mathcal{CS}^0(\beta)$ and be given by* (1.1). *Then for $\beta \ge 0$, we have*

$$|a_3 - \mu a_2^2| \le \begin{cases} 1 + 2(1+\beta)^2(1-2\mu) & \text{if } \mu \le \frac{\beta}{2(1+\beta)}, \\ 1 + 2\beta + \frac{2(1-2\mu)}{1-\beta(1-2\mu)} & \text{if } \frac{\beta}{2(1+\beta)} \le \mu \le \frac{1}{2}, \\ 1 + 2\beta & \text{if } \frac{1}{2} \le \mu \le \frac{2+\beta}{2(1+\beta)}, \\ -1 + 2(1+\beta)^2(2\mu-1) & \text{if } \mu \ge \frac{2+\beta}{2(1+\beta)}. \end{cases}$$

For each μ, there are functions in $\mathcal{CS}^0(\beta)$ such that equality holds in all cases.

REMARK. If we take $\alpha = 1$ in Theorem, then we have the result by London [10], which covers the results of Keogh and Merkes [7] and Abdel-Gawad and Thomas [1].

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