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SOME RESULTS ASSOCIATED WITH CERTAIN ANALYTIC AND UNIVALENT FUNCTIONS INVOLVING FRACTIONAL DERIVATIVE OPERATORS

H. IRMAK AND R. K. RAINA

ABSTRACT. This paper investigates some results (Theorems 2.1-2.3, below) concerning certain classes of analytic and univalent functions, involving the familiar fractional derivative operators. We state interesting consequences arising from the main results by mentioning the cases connected with the starlikeness, convexity, close-to-convexity and quasi-convexity of geometric function theory. Relevant connections with known results are also emphasized briefly.

1. Introduction and Definitions

Let $\mathcal{A}_n$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots (n \in \mathbb{N} \equiv \{1, 2, 3, \cdots\}), \quad (1.1)$$

which are *analytic and univalent* in the *open unit disc* $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We denote by $\mathcal{S}_n^*(\alpha)$, $\mathcal{K}_n(\alpha)$, $\mathcal{C}_n(\alpha, \beta)$ and $\mathcal{C}_n^*(\alpha, \beta)$, the subclasses of the class $\mathcal{A}_n$ consisting of functions which are, respectively, *starlike of order* α, *convex of order* α, *close-to-convex of order* β *and type* α, and *quasi-convex of order* β *and type* α in $\mathbf{U}$, *where* $0 \leq \alpha < 1$ and

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$0 \le \beta < 1$. The analytic characterizations of these subclasses, as we know, are respectively, defined in the following forms:

$$\mathcal{S}_n^*(\alpha) := \left\{ f \in \mathcal{A}_n \ : \ \Re e \left(\frac{zf'(z)}{f(z)} \right) > \alpha \right\},$$

$$\mathcal{K}_n(\alpha) := \left\{ f \in \mathcal{A}_n \ : \ \Re e \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \right\},$$

$$\mathcal{C}_n(\alpha, \beta) := \left\{ f \in \mathcal{A}_n \ : \ \Re e \left(\frac{zf'(z)}{g(z)} \right) > \beta \ ; \ \ g \in \mathcal{S}_n^*(\alpha) \right\},$$

and

$$\mathcal{C}_n^*(\alpha, \beta) := \left\{ f \in \mathcal{A}_n \ : \ \Re e \left(\frac{[zf'(z)]'}{g'(z)} \right) > \beta \ ; \ \ g \in \mathcal{K}_n(\alpha) \right\},$$

where $0 \le \alpha < 1$, $0 \le \beta < 1$, and $z \in \mathbf{U}$. For more details of the above definitions, one may refer to [1], [2] and [9] (see also [11], [15]).

It is evident from the aforementioned definitions that

$$f(z) \in \mathcal{K}_n(\alpha) \ \Leftrightarrow \ zf'(z) \in \mathcal{S}_n^*(\alpha),$$

and

$$f(z) \in \mathcal{C}_n^*(\alpha, \beta) \ \Leftrightarrow \ zf'(z) \in \mathcal{C}_n(\alpha, \beta),$$

where $0 \le \alpha < 1$, $0 \le \beta < 1$, and $f(z) \in \mathcal{A}_n$.

We first need the following definitions of fractional calculus operators which will be used in Section 2 ([10], [12]; see also, e.g., [3], [4]).

DEFINITION 1.1. *Let a function* $\kappa(z)$ *be analytic in a simply-connected region of the z-plane containing the origin. The fractional derivative of order* μ $(0 \le \mu < 1)$ *is defined by*

$$D_z^\mu\{\kappa(z)\} = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \kappa(\xi)(z-\xi)^{-\mu} d\xi, \tag{1.2}$$

where the multiplicity of $(z-\xi)^{-\mu}$ *involved in (1.2) is removed by requiring* $\log(z-\xi)$ *to be real when* $z - \xi > 0$.

Definition 1.2. *Under the hypotheses of Definition 1, the fractional derivative of order $m+\mu$ is defined by*

$$D_z^{m+\mu}\{\kappa(z)\} = \frac{d^m}{dz^m}\{D_z^{\mu}\{\kappa(z)\}\} \quad (m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; 0 \leq \mu < 1). \tag{1.3}$$

It follows from (1.1), (1.2) and (1.3) that

$$D_z^{m+\mu}\{f(z)\} = \frac{z^{1-\mu-m}}{\Gamma(2-\mu-m)} + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\mu-m+1)} a_k z^{k-\mu-m+1}, \tag{1.4}$$

$$(m < 2-\mu; m \in \mathbb{N}_0; f(z) \in \mathcal{A}_n).$$

Making use of the fractional calculus operators defined by (1.2) and (1.3), we establish certain results by applying the well known lemmas of Jack [7] and Nunokawa [8]. As a consequence of the main results, we point out the relationship of the results with geometrically important subclasses of functions which are, respectively, starlike of order α, convex of order α, close-to-convex of order β and type α, quasi-convex of order β and type α, where $0 \leq \alpha < 1$ and $0 \leq \beta < 1$.

We require the following lemmas:

Lemma 1.1. ([7]) *Let a function $w(z)$ be non-constant and analytic in $\mathbf{U}$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at the point z_0, then*

$$\left.\frac{zw'(z)}{w(z)}\right|_{z=z_0} = c \quad (c \geq 1).$$

Lemma 1.2. ([8]) *Let $p(z)$ be an analytic function in $\mathbf{U}$ with $p(0) = 1$. If there exists a point $z_0 \in \mathbf{U}$ such that*

$$\Re e\,\{p(z)\} > 0 \ (|z| < |z_0|), \quad \Re e\,\{p(z_0)\} = 0 \ \text{ and } \ p(z_0) \neq 0,$$

then

$$p(z_0) = ia \ \text{ and } \ \left.\frac{zp'(z)}{p(z)}\right|_{z=z_0} = i\frac{c}{2}\left(a + \frac{1}{a}\right) \quad (a \neq 0;\ c \geq 1).$$

2. The Main Results

We first begin by proving the following result.

THEOREM 2.1. *Let the functions $f, g \in \mathcal{A}_n$ $(f \neq g)$, and also let the function $\mathcal{H}$ be defined by*

$$\mathcal{H}(z) := z\left(\frac{\mathcal{D}_z^{1+\mu}\{f(z)\}}{\mathcal{D}_z^{\mu}\{f(z)\}} - \frac{\mathcal{D}_z^{1+\nu}\{g(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}}\right). \tag{2.1}$$

If

$$\Re e\{\mathcal{H}(z)\} < \nu - \mu + \frac{1}{2}, \tag{2.2}$$

is satisfied, then

$$\Re e\left(z^{\mu-\nu} \cdot \frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}}\right) > 0, \tag{2.3}$$

where

$$0 \leq \mu < 1,\ 0 \leq \nu < 1,\ z \in \mathbf{E} := \left\{ \begin{array}{cc} \mathbf{U} & when\ \ \mu - \nu \geq 0 \\ \mathbf{D} := \mathbf{U} - \{0\} & when\ \ \mu - \nu < 0 \end{array} \right\}.$$

Proof. Let the functions $f(z) \in \mathcal{A}_n$ be given by (1.1) and $g(z) \in \mathcal{A}_n$ be defined by

$$g(z) = z + b_{n+1}z^{n+1} + b_{n+2}z^{n+2} + \ldots \quad (n \in \mathbb{N}), \tag{2.4}$$

with $f(z) \neq g(z)$. From (1.1) and (2.4) in conjunction with the representation (1.4), it follows that

$$\frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} = \frac{\frac{z^{1-\mu}}{\Gamma(2-\mu)}\left(1 + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k-\mu+1)} a_k z^{k-1}\right)}{\frac{z^{1-\nu}}{\Gamma(2-\nu)}\left(1 + \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\nu)}{\Gamma(k-\nu+1)} b_k z^{k-1}\right)}.$$

We now define a function $w(z)$ by

$$\frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} = \frac{\Gamma(2-\nu)}{\Gamma(2-\mu)} \cdot z^{\nu-\mu} \cdot [1 + w(z)] \quad (0 \leq \mu < 1; 0 \leq \nu < 1), \tag{2.5}$$

so that the function $w(z)$ from (2.5) is explicitly given by

$$w(z) = z^{\mu-\nu} \cdot \frac{\Gamma(2-\mu)}{\Gamma(2-\nu)} \cdot \frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} - 1 \quad (0 \leq \mu < 1; 0 \leq \nu < 1). \tag{2.6}$$

We observe that for $z = 0$, $w(0) = 0$, and further $w(z)$ given by (2.6) is analytic for every z in $\mathbf{E}$.

We then obtain from the logarithmic differentiation of (2.5) that

$$\mathcal{H}(z) \equiv z\left(\frac{\mathcal{D}_z^{1+\mu}\{f(z)\}}{\mathcal{D}_z^{\mu}\{f(z)\}} - \frac{\mathcal{D}_z^{1+\nu}\{g(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}}\right) = \nu - \mu + \frac{zw'(z)}{1+w(z)}. \quad (2.7)$$

Assume now that there exists a point $z_0 \in \mathbf{E}$ such that

$$|w(z_0)| = 1, \quad \text{and} \quad |w(z)| < 1 \quad \text{when} \quad |z| < |z_0| \quad (z \in \mathbf{E}).$$

Then, applying Lemma 1.1, we have

$$z_0 w'(z_0) = cw(z_0) \quad \left(c \geq 1;\ w(z_0) = e^{i\varphi} \neq -1\right). \quad (2.8)$$

Thus, (2.7) and (2.8) yield that

$$\Re e\{\mathcal{H}(z_0)\} = \nu - \mu + \Re e\left(\left.\frac{zw'(z)}{1+w(z)}\right|_{z=z_0}\right) = \nu - \mu + \frac{c}{2} \geq \nu - \mu + \frac{1}{2}. \quad (2.9)$$

But the inequality in (2.9) is a contradiction to our assumption in (2.2). Therefore, $|w(z)| < 1$ for all $z \in \mathbf{E}$. Hence, (2.6) immediately yields that

$$\left| z^{\mu-\nu} \cdot \frac{\Gamma(2-\mu)}{\Gamma(2-\nu)} \cdot \frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} - 1 \right| = |w(z)| < 1$$

$$(f, g \in \mathcal{A}_n; 0 \leq \mu < 1; 0 \leq \nu < 1; z \in \mathbf{E}),$$

which implies that

$$\Re e\left(z^{\mu-\nu} \cdot \frac{\Gamma(2-\mu)}{\Gamma(2-\nu)} \cdot \frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}}\right) > 0$$

$$(f, g \in \mathcal{A}_n; 0 \leq \mu < 1; 0 \leq \nu < 1; z \in \mathbf{E}),$$

and the desired assertion (2.3) follows, since $\frac{\Gamma(2-\mu)}{\Gamma(2-\nu)}$ (for $0 \leq \mu < 1$ and $0 \leq \nu < 1$) is always positive. This completes the proof of Theorem 2.1.

Our second result is contained in

Theorem 2.2. *Let the functions $f, g \in \mathcal{A}_n$ $(f \neq g)$, and $\mathcal{H}$ be defined by (2.1). If*

$$\Re e\{\mathcal{H}(z)\} > \nu - \mu + \frac{\alpha - 1}{2(1+\alpha)}, \tag{2.10}$$

is satisfied, then

$$\Re e\left(z^{\mu-\nu} \cdot \frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}}\right) > \frac{(1+\alpha)\Gamma(2-\nu)}{2\Gamma(2-\mu)}, \tag{2.11}$$

where $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $0 \leq \nu < 1$, and $z \in \mathbf{E}$.

Proof. Let us define the function $u(z)$ by

$$\frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} = \frac{\Gamma(2-\nu)}{\Gamma(2-\mu)} \cdot z^{\nu-\mu} \cdot \frac{1+\alpha u(z)}{1+u(z)} \quad (z \in \mathbf{E};\ 0 \leq \alpha < 1), \tag{2.12}$$

then $u(z)$ is an analytic function in $\mathbf{E}$, and $u(0) = 0$. It follows from (2.12) including $u(z)$ that

$$\mathcal{H}(z) \equiv \nu - \mu + \frac{\alpha z u'(z)}{1+\alpha u(z)} - \frac{z u'(z)}{1+u(z)},$$

where $\mathcal{H}(z)$ is given by (2.1).

If we now suppose that there exists a point $z_0 \in \mathbf{E}$ such that

$$\max_{|z| \leq |z_0|} |u(z)| = |u(z_0)| = 1,$$

then Lemma 1.1 gives

$$z_0 u'(z_0) = c u(z_0) \quad \left(c \geq 1;\ u(z_0) = e^{i\varphi} \neq -1\right).$$

Therefore, we have

$$\begin{aligned}\mu - \nu + \Re e\{\mathcal{H}(z_0)\} &= \Re e\left\{\frac{\alpha z_0 u'(z_0)}{1+\alpha u(z_0)} - \frac{z_0 u'(z_0)}{1+u(z_0)}\right\} \\ &= c\, \Re e\left(\frac{\alpha e^{i\varphi}}{1+\alpha e^{i\varphi}} - \frac{e^{i\varphi}}{1+e^{i\varphi}}\right) \leq \frac{\alpha - 1}{2(1+\alpha)},\end{aligned}$$

which contradicts the hypothesis in (2.10) of Theorem 2.2, so that $|u(z)| < 1$ for all $z \in \mathbf{E}$, and (2.12) immediately yields then

$$\left|\frac{\frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} \cdot z^{\mu-\nu} - \frac{\Gamma(2-\nu)}{\Gamma(2-\mu)}}{\frac{\alpha\Gamma(2-\nu)}{\Gamma(2-\mu)} - \frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} \cdot z^{\mu-\nu}}\right| = |u(z)| < 1$$

$$(f, g \in \mathcal{A}_n; 0 \leq \alpha < 1; 0 \leq \mu < 1; 0 \leq \nu < 1; z \in \mathbf{E}),$$

which is equivalent to the assertion (2.11), and the proof of Theorem 2.2 is complete.

Our last result is contained in

THEOREM 2.3. *Let the functions $f, g \in \mathcal{A}_n$ ($f \neq g$), and $\mathcal{H}$ be defined by (2.1). If any one of the following conditions:*

$$\Re e\{\mathcal{H}(z)\}\left\{\begin{array}{lll} > \Psi(\nu,\mu;1-\alpha) & \text{when} & 0 \leq \alpha \leq \frac{1}{2} \\ > \Psi(\nu,\mu;\alpha) & \text{when} & \frac{1}{2} \leq \alpha < 1 \end{array}\right\}, \tag{2.13}$$

is satisfied, then

$$\Re e\left(z^{\mu-\nu}\cdot\frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}}\right) > \frac{\alpha\Gamma(2-\nu)}{\Gamma(2-\mu)}, \tag{2.14}$$

where $0 \leq \alpha < 1$, $0 \leq \mu < 1$, $0 \leq \nu < 1$, $z \in \mathbf{E}$ and

$$\Psi(\nu,\mu;\alpha) := \nu - \mu - \frac{1-\alpha}{2\alpha}. \tag{2.15}$$

Proof. We define a function $p(z)$ (involving the fractional derivative operator (1.2)) by

$$\frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} = \frac{\Gamma(2-\nu)}{\Gamma(2-\mu)}\cdot z^{\nu-\mu}\cdot[\alpha+(1-\alpha)p(z)], \tag{2.16}$$

where $z \in \mathbf{E}$, $0 \leq \alpha < 1$, $0 \leq \mu < 1$ and $0 \leq \nu < 1$. Then, it is easily verified that the function $p(z)$ is analytic in $\mathbf{E}$, with p(0)=1. Upon differentiating (2.16), we obtain that

$$\mathcal{H}(z) \equiv \nu - \mu - \frac{(1-\alpha)zp'(z)}{\alpha+(1-\alpha)p(z)} \quad (z \in \mathcal{U};\ 0 \leq \alpha < 1), \tag{2.17}$$

where the function $\mathcal{H}(z)$ is given by (2.1).

Suppose there exists a point $z_0 \in \mathbf{E}$ such that

$$\Re e\{p(z)\} > 0 \quad (|z| < |z_0|), \quad \Re e\{p(z_0)\} = 0, \quad \text{and} \quad p(z_0) \neq 0 \quad (z \in \mathbf{E}).$$

Then, by using Lemma 1.2, we have

$$p(z_0) = ia \quad \text{and} \quad \frac{z_0p'(z_0)}{p(z_0)} = i\frac{c}{2}\left(a+\frac{1}{a}\right) \quad (a \neq 0;\ c \geq 1). \tag{2.18}$$

Thus, from (2.17) and (2.18), we obtain

$$\Re e\{\mathcal{H}(z_0)\} = \nu - \mu + \Re e\left\{ \frac{(1-\alpha)zp'(z)}{p(z)} \cdot \frac{p(z)}{\alpha+(1-\alpha)p(z)}\bigg|_{z=z_0} \right\}$$

$$= \nu - \mu - \frac{c\alpha(1-\alpha)(1+a^2)}{2[\alpha^2+a^2(1-\alpha)^2]}$$

$$\left\{ \begin{array}{lll} \leq \Psi(\nu,\mu;1-\alpha) & \text{when} & 0 \leq \alpha \leq \frac{1}{2} \\ \leq \Psi(\nu,\mu;\alpha) & \text{when} & \frac{1}{2} \leq \alpha < 1 \end{array} \right\}, \tag{2.19}$$

where $\Psi(\nu,\mu;\alpha)$ is given by (2.15). But, the inequalities in (2.19) contradict our assumptions imposed in (2.13). Hence, $\Re e\{p(z)\} > 0$ for all $z \in \mathbf{E}$. Therefore, (2.16) evidently yields (2.14), and the desired proof of Theorem 2.3 is complete.

3. Some Consequences of the Main Results

The various results of importance in the geometric - function theory can be obtained easily and conveniently from the main results (Theorems 2.1-2.3). These results can be achieved by choosing the parameters and functions, appropriately, in the related theorems. We would in brief mention a scheme of steps below which one may apply to arrive at different known (and new results).

(i) The order of derivatives ν and μ in the defined equation (2.1), and in the inequalities stated in Theorems 2.1-2.3 should be chosen, respectively, as ($\nu = 0$ *and* $\mu = 0$), ($\nu = 0$ *and* $\mu \to 1-$), ($\nu \to 1-$ *and* $\mu = 0$) or ($\nu \to 1-$ *and* $\mu \to 1-$).

(ii) The function $g(z) \in \mathcal{A}_n$ may be selected instead of $f(z) \in \mathcal{A}_n$, or $f(z) \in \mathcal{A}_n$ be chosen instead of $g(z) \in \mathcal{A}_n$, in Theorems 2.1-2.3.

(iii) The function $g(z) \in \mathcal{A}_n$ may be chosen to belong to the classes $\mathcal{S}_n^*(\beta)$ or $\mathcal{K}_n(\beta)$. A similar situation can also be considered for the function $f(z) \in \mathcal{A}_n$.

(iv) The functions $f(z) \in \mathcal{A}_n$ and/or $g(z) \in \mathcal{A}_n$, in the equality (2.1), and in the inequalities given in Theorems 2.1-2.3, may be replaced by $zf'(z)$ $(f(z) \in \mathcal{A}_n)$ and/or $zg'(z)$ $(g(z) \in \mathcal{A}_n)$.

The special cases which would arise from the main results with the help of the above mentioned steps in (i)-(iv) can be compared with the results which have appeared recently in [3], [4], [5], [6], [11], [13] and [14]. To illustrate, we give below some examples.

By taking $\mu = 0$ and $\nu = 0$, in Theorems 2.1-2.3, respectively, replacing f by zf', and choosing $g(z) \in \mathcal{S}_n^*(\beta)$, we arrive at the following corollary.

COROLLARY 3.1. *Let the functions $f(z) \in \mathcal{A}_n$, and $g(z) \in \mathcal{S}_n^*(\beta)$ $(f \neq g)$, and also let the function $\mathcal{G}_1(z)$ be defined by*

$$\mathcal{G}_1(z) := \tfrac{zf''(z)+f'(z)}{f'(z)} - \tfrac{zg'(z)}{g(z)} \quad (z \in \mathbf{U}).$$

Then

(a) $\Re e\,[\mathcal{G}_1(z)] < \frac{1}{2} \;\;\Rightarrow\;\; f(z) \in \mathcal{C}_n(\beta, 0)$,

(b) $\Re e\,[\mathcal{G}_1(z)] > -\frac{3+\alpha}{2(1+\alpha)} \;\;\Rightarrow\;\; f(z) \in \mathcal{C}_n(\beta, (1+\alpha)/2)$,

(c) $\Re e\,[\mathcal{G}_1(z)] \left\{ \begin{array}{ll} > \frac{\alpha}{2(\alpha-1)} & if \;\; 0 \le \alpha \le \frac{1}{2} \\ > \frac{\alpha-1}{2\alpha} & if \;\; \frac{1}{2} \le \alpha < 1 \end{array} \right\} \;\;\Rightarrow\;\; f(z) \in \mathcal{C}_n(\beta, \alpha)$.

If $\mu \to 1-$ and $\nu = 0$ in Theorems 2.1-2.3, and we choose $g(z) \in \mathcal{S}_n^*(\beta)$, we arrive at the following corollary.

COROLLARY 3.2. *Let the functions $f(z) \in \mathcal{A}_n$, and $g(z) \in \mathcal{S}_n^*(\beta)$ $(f \neq g)$, and also let the function $\mathcal{G}_2(z)$ be defined by*

$$\mathcal{G}_2(z) := z\left(\tfrac{f''(z)}{f'(z)} - \tfrac{g'(z)}{g(z)}\right) \quad (z \in \mathbf{U}).$$

Then

(a) $\Re e\,[\mathcal{G}_2(z)] < -\frac{1}{2} \;\;\Rightarrow\;\; f(z) \in \mathcal{C}_n(\beta, 0)$,

(b) $\Re e\,[\mathcal{G}_2(z)] > \frac{1+3\alpha}{2(1+\alpha)} \;\;\Rightarrow\;\; f(z) \in \mathcal{C}_n(\beta, (1+\alpha)/2)$,

(c) $\Re e\,[\mathcal{G}_2(z)] \left\{ \begin{array}{ll} > \frac{2-3\alpha}{2(1-\alpha)} & if \;\; 0 \le \alpha \le \frac{1}{2} \\ > -\frac{1-3\alpha}{2\alpha} & if \;\; \frac{1}{2} \le \alpha < 1 \end{array} \right\} \;\;\Rightarrow\;\; f(z) \in \mathcal{C}_n(\beta, \alpha)$.

Next, if we again set $\mu \to 1-$ and $\nu = 0$ in Theorems 2.1-2.3, and let $f(z) := zf'(z)$ $(f(z) \in \mathcal{A}_n)$ and $g(z) := zg'(z)$ $(g(z) \in \mathcal{K}_n(\beta))$, then we get the following result.

COROLLARY 3.3. *Let the functions $f, g \in \mathcal{A}_n$ $(f \neq g)$, and also let the function $\mathcal{G}_3(z)$ be defined by*

$$\mathcal{G}_3(z) := z\left(\tfrac{zf'''(z)+2f''(z)}{zf''(z)+f'(z)} - \tfrac{zg''(z)+g'(z)}{zg'(z)}\right) \quad (g(z) \in \mathcal{K}_n^*(\beta);\, z \in \mathbf{U}).$$

Then

(a) $\Re e\,[\mathcal{G}_3(z)] < -\frac{1}{2} \;\;\Rightarrow\;\; f(z) \in \mathcal{C}_n^*(\beta, 0)$,

(b) $\Re e\,[\mathcal{G}_3(z)] > \frac{1+3\alpha}{2(1+\alpha)} \quad \Rightarrow \quad f(z) \in \mathcal{C}_n^*(\beta, (1+\alpha)/2)$,

(c) $\Re e\,[\mathcal{G}_3(z)] \left\{ \begin{array}{lll} > \frac{2-3\alpha}{2(1-\alpha)} & \textit{if} & 0 \leq \alpha \leq \frac{1}{2} \\ > -\frac{1-3\alpha}{2\alpha} & \textit{if} & \frac{1}{2} \leq \alpha < 1 \end{array} \right\} \quad \Rightarrow \quad f(z) \in \mathcal{C}_n^*(\beta, \alpha)$.

Making use of Theorem 2.1, we establish the following result:

COROLLARY 3.4. *Let $0 \leq \nu < 1$, $0 \leq \delta < 1$, $f, g \in \mathcal{A}_n$, and also let the function g satisfy the condition:*

$$\Re e \left(\frac{z\mathcal{D}_z^{1+\nu}\{g(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} \right) \geq \delta \quad (0 \leq \delta < 1; z \in \mathbf{U}). \tag{3.1}$$

Then

$$\Re e\,[\mathcal{G}_4(z)] < \nu - \mu + \delta + \frac{1}{2} \quad \Rightarrow \quad \Re e\,[\mathcal{G}_5(z)] > 0, \tag{3.2}$$

where

$$\mathcal{G}_4(z) := \frac{z\mathcal{D}_z^{1+\mu}\{f(z)\}}{\mathcal{D}_z^{\mu}\{f(z)\}} \ \ \textit{and} \ \ \mathcal{G}_5(z) := z^{\mu-\nu} \cdot \frac{\mathcal{D}_z^{\mu}\{f(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}} \quad (z \in \mathbf{E}). \tag{3.3}$$

Proof. Let the function $w(z)$ be defined by (2.6). Then, in view of (2.7), we know that

$$\frac{z\mathcal{D}_z^{1+\mu}\{f(z)\}}{\mathcal{D}_z^{\mu}\{f(z)\}} = \nu - \mu + \frac{zw'(z)}{1+w(z)} + \frac{z\mathcal{D}_z^{1+\nu}\{g(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}},$$

or, equivalently,

$$\mathcal{G}_4(z) = \nu - \mu + \frac{zw'(z)}{1+w(z)} + \mathcal{G}_5(z), \tag{3.4}$$

where $\mathcal{G}_4(z)$ and $\mathcal{G}_5(z)$ are given by (3.3). From the various assumptions in the proof of Theorem 2.1 (in view of of Jack's Lemma (2.1)), together with the condition (3.1), we easily find that

$$\begin{aligned} \Re e\,[\mathcal{G}_4(z_0)] &= \nu - \mu + \Re e\,[\mathcal{G}_5(z_0)] + \Re e \left(\frac{z_0 w'(z_0)}{1+w(z_0)} \right) \\ &= \nu - \mu + \frac{c}{2} + \Re e\,[\mathcal{G}_5(z_0)] \geq \nu - \mu + \delta + \frac{c}{2}, \end{aligned}$$

which contradicts our assumption in (3.2) (when $c = 1$). The desired assertion of the Corollary 3.4 follows now from the definition of the function $w(z)$ given by (2.6).

Lastly, the below mentioned results can be established by following similar steps as outlined in the proof of Corollary 3.4 above, and also using in the process Theorems 3.2 and 3.3, respectively (along with the assumption (3.5) below).

COROLLARY 3.5. *Let* $0 \leq \nu < 1$, $0 \leq \delta < 1$, $f, g \in \mathcal{A}_n$, *and also let the function* g *satisfy the condition:*

$$\Re e\left(\frac{z\mathcal{D}_z^{1+\nu}\{g(z)\}}{\mathcal{D}_z^{\nu}\{g(z)\}}\right) \leq \delta \quad (0 \leq \delta < 1; z \in \mathbf{U}). \tag{3.5}$$

Then

(a) $\Re e\,[\mathcal{G}_4(z)] > \nu - \mu + \delta + \dfrac{\alpha - 1}{2(1+\alpha)}$

$$\Rightarrow \quad \Re e\,[\mathcal{G}_5(z)] > \frac{(1+\alpha)\Gamma(2-\nu)}{2\Gamma(2-\mu)},$$

(b) $\Re e\,[\mathcal{G}_4(z)] \begin{cases} > \delta + \Psi(\nu, \mu; 1-\alpha) & \text{if } 0 \leq \alpha \leq \frac{1}{2} \\ > \delta + \Psi(\nu, \mu; \alpha) & \text{if } \frac{1}{2} \leq \alpha < 1 \end{cases}$

$$\Rightarrow \Re e\,[\mathcal{G}_5(z)] > \frac{\alpha\Gamma(2-\nu)}{\Gamma(2-\mu)},$$

where $\Psi(\nu, \mu; \alpha)$ *and* $\mathcal{G}_i(z)$ $(i = 4, 5)$ *are given by (2.15) and (3.3), respectively.*

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Hüseyin Irmak
Department of Mathematics Education
Başkent University Bağlıca Campus,
Tr-06530, Ankara, Turkey
E-mail: hisimya@<baskent.edu.tr><yahoo.com>

R. K. Raina
Department of Mathematics
College of Technology and Engineering
M.P. University of Agriculture and Technology
Udaipur 313 001, Rajasthan, INDIA
E-mail: rainark_7@hotmail.com