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ON GENERALIZED WRIGHT'S HYPERGEOMETRIC FUNCTIONS AND FRACTIONAL CALCULUS OPERATORS

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ABSTRACT. In the present paper we first establish some basic results for a substantially more general class of functions defined below. The results include simple differentiation and fractional calculus operators (integration and differentiation of arbitrary orders) for this class of functions. These results are then invoked in determining similar properties for the generalized Wright's hypergeometric functions. Further, norm estimate of a certain class of integral operators whose kernel involves the generalized Wright's hypergeometric function, and its composition (and other related properties) with the fractional calculus operators are also investigated.

1. Introduction and preliminaries

The generalized hypergeometric function ${}_p\psi_q\,[x]$ which was introduced by Wright [6] (see also [3, Section 4]) is the extended form of the more familiar generalized hypergeometric function ${}_pF_q\,[x]$, and is defined by

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$$
\begin{aligned}
{}_p\psi_q\,[x] &= {}_p\psi_q\left[\left.\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right| x\right]\\
&= {}_p\psi_q\left[\left.\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right| x\right]\\
&= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{x^k}{k!},
\end{aligned}
\tag{1.1}
$$

provided that $p,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}; a_i,b_j\in\mathbb{C};\alpha_i,\beta_j\in\mathbb{R};\alpha_i,\beta_j\neq 0; i=1,....p; j=1,...,q$.

The series (1.1) is absolutely convergent if $\Delta>-1,\forall\ x\in\mathbb{C}$; and if $\Delta=-1$, then the series (1.1) is absolutely convergent for $|x|<\delta$ (and if $|x|=\delta$, then $\mathrm{Re}(\mu)>-1/2$), where Δ,δ, and μ are given by

$$
\Delta=\sum_{j=1}^{q}\beta_j-\sum_{i=1}^{p}\alpha_i>-1\ , \tag{1.2}
$$

$$
\delta=\left(\prod_{i=1}^{p}|\alpha_i|^{-\alpha_i}\right)\left(\prod_{j=1}^{q}|\beta_j|^{\beta_j}\right)\ , \tag{1.3}
$$

and

$$
\mu=\sum_{j=1}^{q}b_j-\sum_{i=1}^{p}a_j+\frac{p-q}{2}\ . \tag{1.4}
$$

We assume here and throught this paper that the aforementioned conditions of existence hold true for the function ${}_p\psi_q\,[x]$. The generalized Wright's hypergeometric function contains in its fold the well known generalized hypergeometric function ${}_pF_q\,[x]$, Mittag-Leffler function $E_{\rho,\mu}(x)$, and its mild generalization $E^{\gamma}_{\rho,\mu}(x)$, as well as, its extended form $E_\rho[(\beta_1,\alpha_1),...,(\beta_m,\alpha_m);\mathrm{x}]$. These special cases have been studied by several authors, and we refer for their details to [1] and [3].

In a recent paper [2], several interesting properties were investigated for the function $E^{\gamma}_{h,\lambda}(x)$(which is a generalization of the classical Mittag-Leffler function $E_{h,\lambda}(x)$, and the Kummer function $\Phi(\gamma;\lambda;x)$), and is defined by

$$E_{h,\lambda}^{\gamma}(x) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(hk+\lambda)} x^k \quad (h, \lambda, \gamma \in \mathbb{C}; \mathrm{Re}(h) > 0)\,, \tag{1.5}$$

where $(\gamma)_k$ is the familiar Pochhammer symbol defined by

$$(\gamma)_0 = 1; (\gamma)_k = \prod_{i=1}^{k} (\gamma + i - 1) \quad (k \in \mathbb{N})\,. \tag{1.6}$$

The fractional calculus operators, viz. the Riemann-Liouville fractional integral operator I_{a+}^{α} of order α, and the fractional derivative operator $\mathcal{D}_{a+}^{\alpha}$ of order α, are respectively, defined by ([4]; see also [3])

$$\left(I_{a+}^{\alpha}\varphi\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \varphi(t)dt \quad (\alpha \in \mathbb{C}\,, \mathrm{Re}(\alpha) > 0)\ , \tag{1.7}$$

and

$$\left(\mathcal{D}_{a+}^{\alpha}\varphi\right)(x) = \left(\frac{d}{dx}\right)^n \left(I_{a+}^{n-\alpha}\varphi\right)(x)$$
$$(\alpha \in \mathbb{C}\,, \mathrm{Re}(\alpha) > 0; n = [\mathrm{Re}(\alpha)] + 1)\,. \tag{1.8}$$

The purpose of this paper is to obtain some of the properties established in [2] for a substantially more general class of functions defined below. Our results include the simple differentiation and integration as well as the arbitrary orders of integration and differentiation for this class of functions. We also consider the applications of the basic results to the Wright's type function defined above by (1.1), and obtain composition properties (and other related properties) of the fractional calculus operators with a certain class of integral operators whose kernel involves the generalized Wright's hypergeometric function.

2. Basic results

In this section we present various basic results in compact forms which may very well be attributed to the function $E_{h,\lambda}^{\gamma}(x)$ defined by (1.5).

For a bounded arbitrary sequence $\sigma(k)$ of real (or complex) numbers, let us define a function $\mathcal{F}_{\rho,\lambda}(x)$ by

$$\mathcal{F}_{\rho,\lambda}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k \quad (\rho,\lambda\in\mathbb{C}\,(\mathrm{Re}(\rho)>0)\,;\,|x|<R)\,, \quad (2.1)$$

where R is the set of real numbers, and consider the integral operator

$$\left(\mathcal{J}_{\rho,\lambda,a+;\omega}\varphi\right)(x)=\int_a^x(x-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}\left[\omega(x-t)^{\rho}\right]\varphi(t)dt\ (x>a)\ , \quad (2.2)$$

where $a\in\mathbb{R}_+\,(x>a)\,;\ \lambda,\rho,\omega\in\mathbb{C};(\mathrm{Re}(\lambda)>0,\mathrm{Re}(\rho)>0),\ \varphi(t)$ is such that the integral on the right side exists.

Making use of (2.1), and differentiating term-wise the right-side (which is permissible provided the series converges uniformly in any compact set of $\mathbb{C}$), we readily obtain

$$\left(\frac{d}{dx}\right)^n x^{\lambda-1}\mathcal{F}_{\rho,\lambda}(\omega x^{\rho})=x^{\lambda-n-1}\mathcal{F}_{\rho,\lambda-n}(\omega x^{\rho}), \quad (2.3)$$

where $\rho,\lambda,\omega\in\mathbb{C}\ \ (\mathrm{Re}(\rho)>0,\mathrm{Re}(\lambda)>0)\,;\,n\in\mathbb{N}$.

Similarly, one can easily derive the following result (involving the function $\mathcal{F}_{\rho,\lambda}(x)$):

$$\int_0^x\ldots\int_0^x t^{\lambda-1}\mathcal{F}_{\rho,\lambda}(\omega t^{\rho})\,(dt)^n=x^{\lambda+n-1}\mathcal{F}_{\rho,\lambda+n}(\omega x^{\rho}), \quad (2.4)$$

where $\rho,\lambda,\omega\in\mathbb{C}\ \ (\mathrm{Re}(\rho)>0,\mathrm{Re}(\lambda)>0)\,;\,n\in\mathbb{N}$.

Next, we consider the fractional integral (and derivative) operators of the function $\mathcal{F}_{\rho,\lambda}(x)$. In view of (1.7) and (2.1), and implementing again the term-wise fractional integral operator I_{0+}^{α}, and in the process using the formula [4, p. 40, (2.44)]:

$$\left(I_{a+}^{\alpha}(t-a)^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda+\alpha)}(x-a)^{\lambda+\alpha-1}$$
$$(\alpha,\lambda\in\mathbb{C};\mathrm{Re}(\alpha)>0,\mathrm{Re}(\lambda)>0)\,, \quad (2.5)$$

we get for $x > a$:

$$\begin{aligned}
&\left(I_{a+}^{\alpha}(t-a)^{\lambda-1}\mathcal{F}_{\rho,\lambda}\left[\omega(t-a)^{\rho}\right]\right)(x)\\
&=\left(I_{a+}^{\alpha}\left\{\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}\omega^{k}(t-a)^{\rho k+\lambda-1}\right\}\right)(x)\\
&=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}\omega^{k}\left(I_{a+}^{\alpha}\left\{(t-a)^{\rho k+\lambda-1}\right\}\right)(x)\\
&=(x-a)^{\lambda+\alpha-1}\mathcal{F}_{\rho,\lambda+\alpha}\left[\omega(x-a)^{\rho}\right].
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(I_{a+}^{\alpha}(t-a)^{\lambda-1}\mathcal{F}_{\rho,\lambda}[\omega(t-a)^{\rho}]\right)(x)\\
&\qquad\qquad=(x-a)^{\lambda+\alpha-1}\mathcal{F}_{\rho,\lambda+\alpha}\left[\omega(x-a)^{\rho}\right],
\end{aligned}\tag{2.6}$$

where $a \in \mathbb{R}_+\,(x > a)\,;\,\alpha,\lambda,\rho,\omega \in \mathbb{C}\,(\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\lambda) > 0, \mathrm{Re}(\rho) > 0)$.

Using (1.8), (2.1), (2.3) and (2.6), we obtain the following result:

$$\begin{aligned}
&\left(\mathcal{D}_{a+}^{\alpha}(t-a)^{\lambda-1}\mathcal{F}_{\rho,\lambda}[\omega(t-a)^{\rho}]\right)(x)\\
&\qquad\qquad=(x-a)^{\lambda-\alpha-1}\mathcal{F}_{\rho,\lambda-\alpha}\left[\omega(x-a)^{\rho}\right],
\end{aligned}\tag{2.7}$$

where $a \in \mathbb{R}_+\,(x > a)\,;\,\alpha,\lambda,\rho,\omega \in \mathbb{C}\,(\mathrm{Re}(\lambda) > \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\rho) > 0)$.

We now determine the composition properties of the operator $(\mathcal{J}_{\rho,\lambda,a+;\omega}\varphi)(x)$ defined by (2.2) with the fractional integral operator I_{0+}^{α} and fractional derivative operator $\mathcal{D}_{0+}^{\alpha}$.

Using (1.7) and (2.2), we obtain

$$\begin{aligned}
&\left(\, I_{a+}^{\alpha}\quad(\mathcal{J}_{\rho,\lambda,a+;\omega}\varphi)\,\right)(x)\\
&=\quad\frac{1}{\Gamma(\alpha)}\int_a^x(x-u)^{\alpha-1}du\int_a^u(u-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}\left[\omega(u-t)^{\rho}\right]\varphi(t)dt\\
&=\quad\int_a^x\frac{1}{\Gamma(\alpha)}\left[\int_t^x(x-u)^{\alpha-1}(u-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}\left[\omega(u-t)^{\rho}\right]du\right]\varphi(t)dt\\
&=\quad\int_a^x\frac{1}{\Gamma(\alpha)}\left[\int_0^{x-t}(x-t-\tau)^{\alpha-1}\tau^{\lambda-1}\mathcal{F}_{\rho,\lambda}\left[\omega\tau^{\rho}\right]d\tau\right]\varphi(t)dt\\
&=\quad\int_a^x I_{0+}^{\alpha}\left(\tau^{\lambda-1}\mathcal{F}_{\rho,\lambda}\left[\omega\tau^{\rho}\right]\right)(x-t)\varphi(t)dt.
\end{aligned}$$

Applying the assertion (2.6) on the right-side, we arrive at the following result:

$$\begin{aligned}&\left(I_{a+}^{\alpha}\left(\mathcal{J}_{\rho,\lambda,a+;\omega}\varphi\right)\right)(x)\\ &\quad= \int_a^x (x-t)^{\lambda+\alpha-1}\mathcal{F}_{\rho,\lambda+\alpha}\left[\omega(x-t)^{\rho}\right]\varphi(t)dt \ (x>a),\end{aligned} \tag{2.8}$$

provided the integral on the right side exists.

In an analogous manner, (1.8) and (2.2) yield the results

$$\begin{aligned}&\left(\mathcal{D}_{a+}^{\alpha}\cdot\left(\mathcal{J}_{\rho,\lambda,a+;\omega}\varphi\right)\right)(x)\\ &\quad= \int_a^x (x-t)^{\lambda-\alpha-1}\mathcal{F}_{\rho,\lambda-\alpha}\left[\omega(x-t)^{\rho}\right]\varphi(t)dt \qquad (x>a),\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}&\left(\left(\tfrac{d}{dx}\right)^{r}\left(\mathcal{J}_{\rho,\lambda,a+;\omega}\ \varphi\right)\right)(x)\\ &\qquad\qquad = \int_a^x (x-t)^{\lambda-r-1}\mathcal{F}_{\rho,\lambda-r}\left[\omega(x-t)^{\rho}\right]\varphi(t)dt\\ &\qquad\qquad\quad \left(x>a; \mathrm{Re}(\lambda)>r;\ r\in N\right),\end{aligned} \tag{2.10}$$

provided the integrals on the right sides of (2.9) and (2.10) exist.

Remark 1. If we put $\sigma(k)=(\gamma)_k$ in (2.1), then by virtue of (1.5), we obtain the relationship $\mathcal{F}_{\rho,\lambda}(x)=E_{\rho,\lambda}^{\gamma}(x)$. With these substitutions, we observe that the results (2.3), (2.4), (2.6), (2.7), (2.8), (2.9) and (2.10) correspond to the results given in [2].

3. Applications to Wright's function

In order to consider the applications of the basic results obtained in Section 2 above, we prefer to involve some general recognizable functions stemming from the function $\mathcal{F}_{\rho,\lambda}(x)$, by appropriately, selecting the arbitrary sequence $\sigma(k)$ which essentially defines the function by means of (2.1). The generalized Wright's function defined by (1.1) is the one which we would deduce from (2.1).

Setting

$$\sigma(k) = \frac{\Gamma(\rho k + \lambda)}{k!} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)}, \tag{3.1}$$

and also replacing x by x^h ($h \in \mathbb{R}_+$) in (2.1), we observe that

$$\mathcal{F}_{\rho,\lambda}(x^h) = {}_p\psi_q \left[x^h\right]. \tag{3.2}$$

Applying (3.1) and (3.2) in (2.1), then (2.3), (2.4), (2.6) and (2.7) yield the following results involving the generalized Wright's function ${}_p\psi_q\left[x\right]$.

Theorem 1. *Let* $h, \lambda, \omega, a_i, b_j \in \mathbb{C}; \alpha_i, \beta_j \in \mathbb{R}$ $(\mathrm{Re}(h) > 0; i = 1, ..., p; j = 1, ..., q)$ *and* $n \in \mathbb{N}$. *Then*

$$\left(\frac{d}{dx}\right)^n x^{\lambda-1}{}_p\psi_q \left[\omega x^h\right] = x^{\lambda-n-1}{}_{p+1}\psi_{q+1} \left[\left.{}^{(a_i,\alpha_i)_{1,p},(\lambda,h)}_{(b_i,\beta_i)_{1,q},(\lambda-n,h)}\right| \omega x^h\right]. \tag{3.3}$$

Theorem 2. *Let* $h, \lambda, \omega, a_i, b_j \in \mathbb{C}$ *and* $\alpha_i, \beta_j \in \mathbb{R}$ $(\mathrm{Re}(h) > 0, \mathrm{Re}(\lambda) > 0; i = 1, ..., p; j = 1, ..., q)$. *Then*

$$\int_0^x \cdots \int_0^x t^{\lambda-1}{}_p\psi_q \left[\omega t^h\right] (dt)^n = x^{\lambda}{}_{p+1}\psi_{q+1} \left[\left.{}^{(a_i,\alpha_i)_{1,p},(\lambda,h)}_{(b_i,\beta_i)_{1,q},(\lambda+n,h)}\right| \omega x^h\right]. \tag{3.4}$$

Theorem 3. *Let* $a \in \mathbb{R}_+; h, \lambda, \omega, a_i, b_j \in \mathbb{C}$ *and* $\alpha_i, \beta_j \in \mathbb{R}$ $(\mathrm{Re}(h) > 0; i = 1, ..., p; j = 1, ..., q)$. *Then for* $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\lambda) > 0,$ *and* $x \in (a, \infty)$, *there exist the following relations:*

$$\begin{aligned} &\left(I^{\alpha}_{a+} (t-a)^{\lambda-1}{}_p\psi_q \left[\omega(t-a)^h\right]\right)(x) \\ &= (x-a)^{\lambda+\alpha-1}{}_{p+1}\psi_{q+1} \left[\left.{}^{(a_i,\alpha_i)_{1,p},(\lambda,h)}_{(b_i,\beta_i)_{1,q},(\lambda+\alpha,h)}\right| \omega(x-a)^h\right], \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} &\left(\mathcal{D}^{\alpha}_{a+} (t-a)^{\lambda-1}{}_p\psi_q \left[\omega(t-a)^h\right]\right)(x) \\ &= (x-a)^{\lambda-\alpha-1}{}_{p+1}\psi_{q+1} \left[\left.{}^{(a_i,\alpha_i)_{1,p},(\lambda,h)}_{(b_i,\beta_i)_{1,q},(\lambda-\alpha,h)}\right| \omega(x-a)^h\right]. \end{aligned} \tag{3.6}$$

Putting $p=2, q=1; a_1=b, a_2=c, \alpha_1=1, \alpha_2=\tau; b_1=d, \beta_1=\tau$ in (1.1), we get

$$ {}_2\psi_1\left[\left.\begin{matrix}(b,1),(c,\tau)\\(d,\tau)\end{matrix}\right| x\right] = {}_2R_1^\tau(x), \tag{3.7} $$

where ${}_2R_1^\tau(x)$ is the function introduced and studied by Virchenko [5], Theorem 3 gives then the following results:

COROLLARY 1. *Let $a\in\mathbb{R}_+; h,\lambda,\omega,b,c,d\in\mathbb{C}$ and $\tau\in\mathbb{R}$ $(\mathrm{Re}(h)>0)$. Then for $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\lambda)>0$, and $x\in(a,\infty)$, there exist the following relations:*

$$ \begin{aligned} &\left(I_{a+}^{\alpha}(t-a)^{\lambda-1}{}_2R_1^\tau(x)\right)(x)\\ &\quad= \frac{\Gamma(d)}{\Gamma(b)\Gamma(c)}(x-a)^{\lambda+\alpha-1}{}_3\psi_2\left[\left.\begin{matrix}(b,1),(c,\tau),(\lambda,h)\\(d,\tau),(\lambda+\alpha,h)\end{matrix}\right|\omega(x-a)^h\right], \end{aligned} \tag{3.8} $$

and

$$ \begin{aligned} &\left(\mathcal{D}_{a+}^{\alpha}(t-a)^{\lambda-1}{}_2R_1^\tau(x)\right)(x)\\ &\quad= \frac{\Gamma(d)}{\Gamma(b)\Gamma(c)}(x-a)^{\lambda-\alpha-1}{}_3\psi_2\left[\left.\begin{matrix}(b,1),(c,\tau),(\lambda,h)\\(d,\tau),(\lambda-\alpha,h)\end{matrix}\right|\omega(x-a)^h\right]. \end{aligned} \tag{3.9} $$

Next, if we put $p=1, q=2; a_1=1, \alpha_1=1$ in (1.1), so that

$$ {}_1\psi_2\left[\left.\begin{matrix}(1,1)\\(b_1,\beta_1),(b_2,\beta_2)\end{matrix}\right| x\right] = \Phi_{\beta_1,\beta_2}(x;b_1,b_2), \tag{3.10} $$

where the function $\Phi_{\beta_1,\beta_2}(x;b_1,b_2)$ was introduced by Djzrbashian (see [5, (E.33)]). Theorem 3 in this special case yields the results which are given by the following:

COROLLARY 2. *Let $a\in\mathbb{R}_+; h,\lambda,\omega,b_j\in\mathbb{C}$ and $\beta_j\in\mathbb{R}$ $(\mathrm{Re}(h)>0; j=1,2)$. Then $\mathrm{Re}(\alpha)>0, \mathrm{Re}(\lambda)>0$, and $x\in(a,\infty)$, there exist the following relations:*

$$ \begin{aligned} &\left(I_{a+}^{\alpha}(t-a)^{\lambda-1}\Phi_{\beta_1,\beta_2}\left(\omega(t-a)^h;b_1,b_2\right)\right)(x)\\ &\quad= (x-a)^{\lambda+\alpha-1}{}_2\psi_3\left[\left.\begin{matrix}(1,1),(\lambda,h)\\(b_1,\beta_1),(b_2,\beta_2)(\lambda+\alpha,h)\end{matrix}\right|\omega(x-a)^h\right], \end{aligned} \tag{3.11} $$

and

$$\begin{aligned}&\left(\mathcal{D}_{a+}^{\alpha}(t-a)^{\lambda-1}\Phi_{\beta_1,\beta_2}\left(\omega(t-a)^h;b_1,b_2\right)\right)(x)\\ &= (x-a)^{\lambda-\alpha-1}{}_2\psi_3\left[\left.{}_{(b_1,\beta_1),(b_2,\beta_2)(\lambda-\alpha,h)}^{(1,1),(\lambda,h)}\right|\omega(x-a)^h\right].\end{aligned} \tag{3.11}$$

4. Convolution operator involving Wright's function

In view of (3.1) and (3.2), the integral operator $\left(\mathcal{J}_{h,\lambda,a+;\omega}\varphi\right)(x)$ defined by (2.2) yields a class of convolution type of integral operators involving the generalized Wright's hypergeometric function (1.1), which will be represented by

$$\left(\mathcal{H}_{\omega,a+:(b_q,\beta_q)}^{\lambda,h:(a_p,\alpha_p)}\varphi\right)(x) = \int_a^x (x-t)^{\lambda-1}{}_p\psi_q\left[\omega(x-t)^h\right]\varphi(t)dt \quad (x>a), \tag{4.1}$$

where $\lambda,\omega,h,a_i,b_j \in \mathbb{C}\left(\mathrm{Re}(\lambda)>0,\mathrm{Re}(h)>0\right);\alpha_i,\beta_j\in\mathbb{R},\forall\ i = 1,...,p; j=1,...,q;$

$$\Delta = \sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1$$

(or

$$\Delta = -1,\ \left|\omega(x-t)^h\right| < \delta = \prod_{i=1}^{p}|\alpha_i|^{-\alpha_i}\prod_{j=1}^{q}|\beta_j|^{\beta_j},$$

and if $\left|\omega(x-t)^h\right| = \delta$, then $\mathrm{Re}(\mu) > -1/2); \Delta,\ \delta$ and μ are respectively, given by (1.2), (1.3) and (1.4)

By invoking the generalized Wright's function as a kernel, the convolution type integral operator defined by (4.1) is now used to investigate its boundedness property on $L(a,b)$ by calculating its norm estimate. This property and the estimate are given by the following:

THEOREM 4. *Let* $\lambda, \omega, h, a_i, b_j \in \mathbb{C}\,(\mathrm{Re}(\lambda) > 0, \mathrm{Re}(h) > 0)\,; \alpha_i, \beta_j \in \mathbb{R}, \forall\, i = 1, ..., p; j = 1, ..., q;$ *be such that* $\Delta^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -\mathrm{Re}(h)$. *Then the operator* $\mathcal{H}\,{}^{\lambda,h:(a_p,\alpha_p)}_{\omega,a+:(b_q,\beta_q)}$ *is bounded on* $L(a,b)(a<b)$, *and*

$$\left\| \mathcal{H}\,{}^{\lambda,h,(a_p,\alpha_p)}_{\omega,a+,(b_q,\beta_q)}\,\varphi \right\| \leq \Omega^* \left\| \varphi \right\|, \tag{4.2}$$

where

$$\Omega^* = (b-a)^{\mathrm{Re}(\lambda)} \sum_{k=0}^{\infty} \frac{\left| \prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) \right|}{\left| \prod_{j=1}^{q} \Gamma(b_j + \beta_j k) \right|} \frac{\left| \omega (b-a)^{\mathrm{Re}(h)} \right|^k}{\left[\mathrm{Re}(h) k + \mathrm{Re}(\lambda) \right] k!}. \tag{4.3}$$

Proof. If we denote (for convenience) the kth term of the series (4.3) by C_k, then using the estimates:

$$\Gamma(a_i + \alpha_i k) \sim A_i \left(\tfrac{k}{e}\right)^{\alpha_i k} \alpha_i^{\alpha_i k} k^{a_i - 1/2} \left(A_i = \sqrt{2\pi} \alpha_i^{a_i - 1/2} e^{-a_i} \right) \quad \text{as } k \to \infty \ \ (i = 1, ..., p), \tag{4.4}$$

and

$$\Gamma(b_j + \beta_j k) \sim B_j \left(\tfrac{k}{e}\right)^{\beta_j k} \beta_j^{\beta_j k} k^{b_j - 1/2} \left(B_j = \sqrt{2\pi} \beta_j^{b_j - 1/2} e^{-b_j} \right) \quad \text{as } k \to \infty \ \ (j = 1, ..., q), \tag{4.5}$$

we obtain

$$\begin{aligned} \frac{C_{k+1}}{C_k} = \frac{|\omega|}{k+1} & \frac{\prod_{i=1}^{p} |\Gamma(a_i + \alpha_i + \alpha_i k)|}{\prod_{i=1}^{p} |\Gamma(a_i + \alpha_i k)|} \frac{\prod_{i=1}^{q} |\Gamma(b_i + \beta_i k)|}{\prod_{i=1}^{q} |\Gamma(b_i + \beta_i + \beta_i k)|} \\ & \cdot \frac{[\mathrm{Re}(h) k + \mathrm{Re}(\lambda)]}{[\mathrm{Re}(h) k + \mathrm{Re}(h) + \mathrm{Re}(\lambda)]} (b-a)^{\mathrm{Re}(h)}. \end{aligned} \tag{4.6}$$

After an elementary simplification, this gives

$$\frac{C_{k+1}}{C_k} \sim \frac{A\,|\omega|\,(b-a)^{\mathrm{Re}(h)}}{k+1} \left(\tfrac{k}{e}\right)^{-\Delta^* k} \to 0, \ \text{as } k \to \infty,$$

where

$$\Delta^* = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i + \mathrm{Re}(h) > 0, \tag{4.7}$$

and

$$A = [\mathrm{Re}(h)]^{-\mathrm{Re}(h)} \left(\prod_{i=1}^{\mathrm{p}} \alpha_{\mathrm{i}}^{\alpha_{\mathrm{i}}} \right) \left(\prod_{j=1}^{q} \beta_{\mathrm{j}}^{-\beta_{\mathrm{j}}} \right). \tag{4.8}$$

In view of the conditions imposed with the theorem, it follows that the series occurring in (4.3) is a convergent series, and so Ω^* is finite.

Applying the norm definition given, for instance, in [2, p. 34, (1.15)] to the operator (4.1), and changing the order of integrations, we have

$$\begin{aligned} &\left\| \mathcal{H}^{\lambda,h,(a_p,\alpha_p)}_{\omega,a+,(b_q,\beta_q)} \varphi \right\|_1 \\ &= \int_a^b \left| \int_a^x (x-t)^{\lambda-1} {}_p\psi_q \left[\omega(x-t)^h \right] \varphi(t) dt \right| dx \\ &\leq \int_a^b \left\{ \int_a^x (x-t)^{\mathrm{Re}(\lambda)-1} \left| {}_p\psi_q \left[\omega(x-t)^h \right] \right| dx \right\} |\varphi(t)| \, dt \qquad (4.9) \\ &= \int_a^b \left\{ \int_a^{b-t} u^{\mathrm{Re}(\lambda)-1} \left| {}_p\psi_q \left[\omega u^h \right] \right| du \right\} |\varphi(t)| \, dt \\ &\leq \int_a^b \left\{ \int_a^{b-a} u^{\mathrm{Re}(\lambda)-1} \left| {}_p\psi_q \left[\omega u^h \right] \right| du \right\} |\varphi(t)| \, dt. \end{aligned}$$

Using the definition of the Wright's function (1.1), and integrating term-wise, the inner integral in (4.9) becomes

$$\begin{aligned} &\int_a^{b-a} u^{\Re(\lambda)-1} \left| {}_p\psi_q \left[\omega u^h \right] \right| du \\ &\leq \sum_{k=0}^{\infty} \frac{\left| \prod_{i=1}^{p} \Gamma(a_i + \alpha_i k) \right|}{\left| \prod_{i=1}^{q} \Gamma(b_i + \beta_i k) \right|} \frac{|\omega|^k}{k!} \int_0^{b-a} u^{\mathrm{Re}(\lambda)+k\mathrm{Re}(h)-1} du = \Omega^*, \end{aligned} \tag{4.10}$$

and thus (4.9) and (4.10) yield the desired result (4.2).

Making use of the composition relations (2.8), (2.9) and (2.10) together with the substitutions mentioned in (3.1), (3.2) and (4.1), we obtain the following composition properties of the fractional calculus operators I_{0+}^{α} and $\mathcal{D}_{0+}^{\alpha}$ with the integral operator $\left(\mathcal{H}\,{}_{\omega,a+:(b_q,\beta_q)}^{\lambda,h:(a_p,\alpha_p)}\,\varphi\right)(x)$. $\square$

THEOREM 5. *Let $\alpha \in \mathbb{C}\,(\mathrm{Re}(\alpha)) > 0$, and suppose that $\lambda, \omega, h, a_i, b_j \in \mathbb{C}$ are such that $\mathrm{Re}(\lambda) > 0, \mathrm{Re}(h) > 0; \alpha_i, \beta_j \in \mathbb{R}, \forall i = 1, ..., p; j = 1, ..., q$ then*

$$I_{a+}^{\alpha}\mathcal{H}\,{}_{\omega,a+:(b_q,\beta_q)}^{\lambda,h:(a_p,\alpha_p)}\,\varphi = \mathcal{H}\,{}_{\omega,a+:(b_q,\beta_q),(\lambda+\alpha,h)}^{\lambda+\alpha,h:(a_p,\alpha_p),(\lambda,h)}\,\varphi, \tag{4.11}$$

where $\varphi \in L(a,b)$.

THEOREM 6. *Let $\alpha, \lambda, \omega, h, a_i, b_j \in \mathbb{C}(\mathrm{Re}(\lambda) > \mathrm{Re}(\alpha) > 0; \mathrm{Re}(h) > 0)$ and $\alpha_i, \beta_j \in \mathbb{R}, \forall\, i = 1, ..., p; j = 1, ..., q$ then for $\varphi \in L(a,b)$:*

$$\mathcal{D}_{a+}^{\alpha}\mathcal{H}\,{}_{\omega,a+:(b_q,\beta_q)}^{\lambda,h:(a_p,\alpha_p)}\,\varphi = \mathcal{H}\,{}_{\omega,a+:(b_q,\beta_q),(\lambda-\alpha,h)}^{\lambda-\alpha,h:(a_p,\alpha_p),(\lambda,h)}\,\varphi, \tag{4.12}$$

holds true for any continuous function $\varphi \in \mathcal{C}(a,b)$.

Also, for $r \in \mathbb{N}$ and $\Re(\lambda) > r$:

$$\left(\frac{d}{dx}\right)^{r}\mathcal{H}\,{}_{\omega,a+:(b_q,\beta_q)}^{\lambda,h:(a_p,\alpha_p)}\,\varphi = \mathcal{H}\,{}_{\omega,a+:(b_q,\beta_q),(\lambda-r,h)}^{\lambda-r,h:(a_p,\alpha_p),(\lambda,h)}\,\varphi. \tag{4.13}$$

REMARK 2. Corresponding to the Remark 1, if we employ the parametric substitutions $p = q = 1, a_1 = \gamma, \alpha_1 = 1, b_1 = \lambda, \beta_1 = h$ in (1.1), then the Wright's generalized function reduces to the function $\Gamma(\gamma)E_{h,\lambda}^{\gamma}(x)$ where $E_{h,\lambda}^{\gamma}(x)$ is the generalized Mittag-Leffler function studied in [2], so Theorems 1-6 correspond to similar assertions investigated in [2]. One can deduce several other results from those presented in this paper by suitably augmenting the parameters of the Wright's function ${}_p\psi_q\,[x]$ (see, for example [1], [3] and [5] for the various special cases of the Wright's function).

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