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GENERALIZED TOEPLITZ ALGEBRAS OF SEMIGROUPS

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ABSTRACT. We analyze the structure of C^*-algebras generated by left regular isometric representations of semigroups.

1. Introduction

For a given group G, we consider the unitary representations of G which correspond to the representations of the full group C^*-algebra $C^*(G)$. Specially, the left regular unitary representation of G gives rise to the reduced group C^*-algebra, which has been an important object in the theory of C^*-algebras.

In analogy with a group case one can associate isometric representations with semigroups. The left regular isometric representations of semigroups may be considered as the counterpart of the left regular unitary representations of groups, so they are very interesting non-unitary isometric representations of semigroups. In this paper we are going to study C^*-algebras generated by left regular isometric representations for left cancellative semigroups, called reduced semigroup C^*-algebras from the point of view of crossed products.

The theory of crossed products of C^*-algebras by semigroups has been studied by many authors in [1, 2, 3, 4, 7, 8, 9, 12, 13]. We are

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concerned with the two kinds of crossed products of C^*-algebras by semigroups, the one is the full crossed product by the semigroup of automorphisms introduced by G. J. Murphy in [11] and the other is the reduced crossed product by the semigroup of automorphisms [5] for a C^*-dynamical system $(\mathcal{A}, M, \alpha)$ with a left-cancellative semigroup M. If these constructions applied to the trivial C^*-dynamical system with the left-cancellative semigroup M, they give rise to the full semigroup C^*-algebra $C^*(M)$ and the reduced semigroup C^*-algebra $C^*_{red}(M)$, respectively. The full semigroup C^*-algebra $C^*(M)$ is the universal C^*-algebra generated by enveloping of all isometric representations of M and effect isometric representation theories of M. On the other hand reduced semigroup C^*-algebras which are generated by left regular isometric representations give interesting examples of the theory of C^*-algebras and have been studied much in the another name in [1, 2, 3, 6, 11] among others. As we can see from the fact that the typical model of reduced semigroup C^*-algebras is the Toeplitz algebra, there are many interesting problems in the theory of the reduced semigroup C^*-algebras.

For a left-cancellative semigroup M, let $\mathcal{L} : M \to l^2(M)$ denote the left regular isometric representation of M on the Hilbert space $l^2(M)$ of all square-summable maps on M. The followings have been key questions in the theory of the reduced semigroup C^*-algebras:

(1) The uniqueness property, that is, when is $C^*_{red}(M)$ isomorphic to $C^*(M)$?
(2) If $\mathcal{Z}(C^*_{red}(M))$ is the ideal of $C^*_{red}(M)$ generated by $I - \mathcal{L}_x\mathcal{L}_x^*$ for all $x \in M$, when is $\mathcal{Z}(C^*_{red}(M))$ simple ?
(3) When does $C^*_{red}(M)$ contain the algebra $\mathcal{K}(l^2(M))$ of compact operators on $l^2(M)$?
(4) When is $C^*_{red}(M)$ prime ?

(1) and (2) of the above problems were solved partially [1,2,3,11,6] and (1), (3) and (4) also were partially proved [9, 10].

Besides the above problems, the computation of K-groups, and stable rank of $C^*_{red}(M)$ and $\mathcal{Z}(C^*_{red}(M))$ are also interesting problems. In this paper we have some results of the uniqueness property of $C^*_{red}(M)$ and analyze the structure of $\mathcal{Z}(C^*_{red}(M))$.

2. Preliminaries

Throught this paper M denotes a left-cancellative discrete semigroup with unit e. Let $\mathcal{B}$ be a unital C^*-algebra. A map $W : M \to \mathcal{B}, x \mapsto W_x$ is called an *isometric homomorphism* if $W_e = 1$, W_x is an isometry and $W_{xy} = W_x W_y$ for all $x, y \in M$. If $\mathcal{B}$ is the algebra $\mathcal{B}(H)$ of all bounded linear operators of a non-zero Hilbert space H, we call (H, W) an *isometric representation* of M.

Let H be a non-zero Hilbert space and $l^2(M, H)$ denote the Hilbert space of all norm square-summable maps f from M to H. The left regular isometric representation $\mathcal{L}$ of M on $l^2(M, H)$ is a map $\mathcal{L} : M \to l^2(M, H), x \mapsto \mathcal{L}_x$ defined by the equation

$$(\mathcal{L}_x f)(z) = \begin{cases} f(y), & \text{if } z = xy \text{ for some } y \in M, \\ 0, & \text{if } z \notin xM, \end{cases}$$

for each $f \in l^2(M, H)$. If we define $\widetilde{\xi}^{(x)} \in l^2(M, H)$ by setting

$$\widetilde{\xi}^{(x)}(y) = \begin{cases} \xi, & \text{if } y = x, \\ 0, & \text{otherwise,} \end{cases}$$

for $\xi \in H$ and $x \in M$, then $\mathcal{L}_y(\widetilde{\xi}^{(x)}) = \widetilde{\xi}^{(yx)}$ for all $x,\ y \in M$

If $\mathcal{A}$ is a C^*-algebra and $\alpha : x \to \alpha_x$ is a homomorphism from M into the group $Aut(\mathcal{A})$ of automorphisms of $\mathcal{A}$, then a triple $(\mathcal{A}, M, \alpha)$ is called a C^*-dynamical system. For a given C^*-dynamical system $(\mathcal{A}, M, \alpha)$, G J. Murphy constructed the full crossed product $\mathcal{A} \rtimes_\alpha M$ by the semigroup M under the action α, which has the universal property of the covariant homomorphisms of $(\mathcal{A}, M, \alpha)$. The full crossed product by the semigroup corresponding to the trivial C^*-dynamical system is called the full semigroup C^*-algebra and denoted by $C^*(M)$. By the universal property of the full crossed product by the semigroup, $C^*(M)$ is the universal C^*-algebra generated by enveloping of all isometric representations of M [10].

Let (π_u, H_u) be the universal representation of $\mathcal{A}$ and $\mathcal{L}$ be the left regular isometric representation of M on $l^2(M, H_u)$. Then π_u induces

a covariant representation $(\bar{\pi}_u, \mathcal{L})$ of $(\mathcal{A}, M, \alpha)$ on $l^2(M, H_u)$ where $(\bar{\pi}_u(a))(f)(x) = \pi_u(\alpha_x^{-1}(a))(f(x))$ for $a \in \mathcal{A}$, $f \in l^2(M, H_u)$, and $x \in M$. Since $\mathcal{A} \rtimes_\alpha M$ has the universal property of the covariant homomorphisms, there exists a unique *-homomorphism $\bar{\pi}_u \times \mathcal{L} : \mathcal{A} \rtimes_\alpha M \to \mathcal{B}(l^2(M, H_u))$. We call $(\bar{\pi}_u \times \mathcal{L})(\mathcal{A} \rtimes_\alpha M)$ *the reduced crossed product* of $\mathcal{A}$ by the semigroup M under the action α and denote it by $\mathcal{A} \rtimes_{\alpha r} M$. In fact, $\mathcal{A} \rtimes_{\alpha r} M$ is generated by $\{a\mathcal{L}_x | a \in \mathcal{A}, x \in M\}$. In the case of the trivial C^*-dynamical system, $\mathbb{C} \rtimes_{\alpha r} M$ is generated by the left regular isometric representation $\mathcal{L}$ of M on $l^2(M)$. We denote $\mathbb{C} \rtimes_{\alpha r} M$ by $C^*_{red}(M)$ and call it the *reduced semigroup C^*-algebra* of M [5].

3. The Structure of Reduced Semigroup C^*-algebras

We are going to say that if an element x in M is contained in yM for some element $y \in M$, then x and y are comparable and we denote it by $y \leq x$. This relation defines a pre-order on M. If the unit of M is the only invertible element of M, the above relation on M becomes a partial order on M. And we can say a maximal and a minimal element in M in the following sense ; An element $a_0 \in M$ is maximal if and only if $a_0 \leq x$ implies $x = a_0$ and an element a_1 is minimal if and only if $x \leq a_1$ implies $a_1 = x$ for $x \in M$.

Since $C^*_{red}(M)$ is generated by $\{\mathcal{L}_x \mid x \in M\}$ and $\mathcal{L}_x$ is isometry, $C^*_{red}(M)$ is the closed linear span of $\{\mathcal{L}_{x_1}\mathcal{L}^*_{x_2} \cdots \mathcal{L}^*_{x_{2n}}\mathcal{L}_{x_{2n+1}} \mid x_i \in M\}$. Isometries $\mathcal{L}_x$'s induce two projections $\mathcal{L}_x\mathcal{L}^*_x$ and $1 - \mathcal{L}_x\mathcal{L}^*_x$ which play an important role in the theory of the structure of reduced semigroup C^*-algebras. We denote them by P_x and Q_x, respectively.

PROPOSITION 3.1. *If any two elements x and y in M are comparable, then $C^*_{red}(M)$ is the closed linear span of $\{\mathcal{L}_x\mathcal{L}^*_y \mid x, y \in M\}$.*

Proof. Since $\{\mathcal{L}_{x_1}\mathcal{L}^*_{x_2} \cdots \mathcal{L}^*_{x_{2n}}\mathcal{L}_{x_{2n+1}} \mid x_i \in M\}$ is total in $C^*_{red}(M)$, it is enough to show that $\mathcal{L}_{x_1}\mathcal{L}^*_{x_2} \cdots \mathcal{L}^*_{x_{2n}}\mathcal{L}_{x_{2n+1}}$ can be changed into the element of the form $\mathcal{L}_x\mathcal{L}^*_y$ for $x, y \in M$. Since any two elements in M are comparable, we can assume that either $x_{2n} = x_{2n+1}z_n$ or $x_{2n+1} = x_{2n}z_n'$ for some z_n and $z_n' \in M$. First, if $x_{2n} = x_{2n+1}z_n$,

then

$$\begin{aligned}\mathcal{L}_{x_1}\mathcal{L}_{x_2}^*\cdots\mathcal{L}_{x_{2n}}^*\mathcal{L}_{x_{2n+1}} &= \mathcal{L}_{x_1}\mathcal{L}_{x_2}^*\cdots\mathcal{L}_{z_n}^*\mathcal{L}_{x_{2n+1}}^*\mathcal{L}_{x_{2n+1}}\\ &= \mathcal{L}_{x_1}\mathcal{L}_{x_2}^*\cdots\mathcal{L}_{x_{2n-1}}\mathcal{L}_{z_n}^*.\end{aligned}$$

Next, we compare z_n with x_{2n-1}. If we continue this process, then $\mathcal{L}_{x_1}\mathcal{L}_{x_2}^*\cdots\mathcal{L}_{x_{2n}}^*\mathcal{L}_{x_{2n+1}}$ becomes $\mathcal{L}_x\mathcal{L}_y^*$ for some $x, y \in M$. We can get the same result by the similar way when $x_{2n+1} = x_{2n}z_n^{'}$ for some $z_n^{'} \in M$. □

Group C^*-algebras and reduced group C^*-algebras give lots of examples of simple C^*-algebras. However, semigroup C^*-algebras and reduced semigroup C^*-algebras are rarely simple. In fact, if M is not a group, $C_{red}^*(M)$ has an interesting non-trivial ideal $\mathcal{Z}(C_{red}^*(M))$ generated by $1 - \mathcal{L}_x\mathcal{L}_x^*$ for $x \in M$. Sometimes $\mathcal{Z}(C_{red}^*(M))$ plays an important role to analyze the structure of $C_{red}^*(M)$ (cf. [6,10]). Furthermore if M is abelian, $\mathcal{Z}(C_{red}^*(M))$ is the commutator ideal of $C_{red}^*(M)$. Let $\mathcal{I}$ denote the closed two-sided ideal of $C_{red}^*(M)$ generated by $\mathcal{L}_{x_1}\mathcal{L}_{y_1}^*\cdots\mathcal{L}_{x_n}\mathcal{L}_{y_n}^* - \mathcal{L}_{\sum_i x_i}\mathcal{L}_{\sum_j y_j}^*$ for $x_i, y_i \in M$.

PROPOSITION 3.2. *Let M be an abelian semigroup. Let $Q_x = 1 - \mathcal{L}_x\mathcal{L}_x^*$ for each $x \in M$. Then $\{Q_x \mid x \in M\}$ is an approximate unit for $\mathcal{I}$.*

Proof. Let $S = \mathcal{L}_{x_1}\mathcal{L}_{y_1}^*\cdots\mathcal{L}_{x_n}\mathcal{L}_{y_n}^* - \mathcal{L}_{\sum_i x_i}\mathcal{L}_{\sum_j y_j}^*$. We can choose an element $z \in M$ such that $z \geq \sum_{j=1}^n y_j$. Then we have for each $x \in M$

$$\begin{aligned}S\mathcal{L}_z(\delta_x) &= \mathcal{L}_{x_1}\mathcal{L}_{y_1}^*\cdots\mathcal{L}_{x_n}\mathcal{L}_{y_n}^*\mathcal{L}_z(\delta_x) - (\mathcal{L}_{\sum_i x_i} - \mathcal{L}_{\sum_j y_j}^*)\mathcal{L}_z(\delta_x)\\ &= \delta_{z+(\sum_i^n x_i - \sum_i^n y_i)+x} - \delta_{z+(\sum_i^n x_i - \sum_i^n y_i)+x}\\ &= 0.\end{aligned}$$

Therefore $S\mathcal{L}_z\mathcal{L}_z^* = S(1 - Q_z) = 0$. It follows from the above equation that if $T \in \mathcal{I}$, then $\lim TQ_y = T$. Thus $\{Q_x \mid x \in M\}$ is an approximate unit for $\mathcal{I}$.

□

PROPOSITION 3.3. *Let M be an abelian semigroup. Then $\mathcal{I}$ is equal to the commutator ideal $\mathcal{Z}(C^*_{red}(M))$.*

Proof. Since $Q_x = 1 - \mathcal{L}_x\mathcal{L}_x^*$ is contained in $\mathcal{I}$ for each $x \in M$, $C^*_{red}(M)/\mathcal{I}$ is abelian. Hence we have $\mathcal{I} \supseteq \mathcal{Z}(C^*_{red}(M))$. Furthermore since $\{Q_x \mid x \in M\}$ is an approximate unit for $\mathcal{I}$ and $\{Q_x \mid x \in M\}$ is contained in $\mathcal{Z}(C^*_{red}(M))$, $\mathcal{I} \subseteq \mathcal{Z}(C^*_{red}(M))$. □

We can get the following result from the above propositions.

THEOREM 3.4. *Let M be an abelian semigroup. Then $\{Q_x \mid x \in M\}$ is an approximate unit for $\mathcal{Z}(C^*_{red}(M))$.*

4. Generalized Toeplitz Algebras

When M is the semigroup $\mathbb{N}$ of natural numbers, the Grothendieck group of $\mathbb{N}$ is the integer group $\mathbb{Z}$ and the character group of $\mathbb{Z}$ is the cicle group $\mathbb{T}$. We consider the normalized Haar measure on the circle group $\mathbb{T}$, denoted by $d\lambda$. For each integer n, the function $\epsilon_n : \mathbb{T} \to \mathbb{T},\ \lambda \mapsto \lambda^n$ is continuous. Then $\{\epsilon_n \mid n \in \mathbb{Z}\}$ is the orthonormal basis of $L^2(\mathbb{T})$. For $f \in L^p(\mathbb{T})$. $\widehat{f}(n)$ is denoted by the n - th Fourier coefficient of f. $H^p = \{f \in L^p(\mathbb{T}) \mid \widehat{f}(n) = 0 (n \geq 0)\}$ is called the Hardy space for $p \in [1, +\infty]$. If $\phi \in L^\infty(\mathbb{T})$, then the multiplication operator M_ϕ on $L^2(\mathbb{T})$ is defined by

$$M_\phi(f) = \phi f$$

for $f \in L^2(\mathbb{T})$. The restriction T_ϕ of M_ϕ on the Hardy space $H^2(\mathbb{T})$ is called the Toeplitz operator with symbol ϕ for $\phi \in L^\infty(\mathbb{T})$. Then M_{ϵ_1} is the bilateral shift on the basis $\{\epsilon_n \mid n \in \mathbb{Z}\}$ of the Hilbert space $L^2(\mathbb{T})$. The restriction T_{ϵ_1} of M_{ϵ_1} on the Hardy space $H^2(\mathbb{T})$ is the unilateral shift on the basis $\{\epsilon_n \mid n \in \mathbb{N}\}$ of the Hardy space

$H^2(\mathbb{T})$. The C^*-algebra generated by all Toeplitz operators T_ϕ with continuous symbol ϕ is called the Toeplitz algebra $\mathcal{T}$, but in fact the Toeplitz algebra $\mathcal{T}$ is generated by T_{ϵ_1}.

Let M be a cancellative abelian semigroup and G be the Grothendieck enveloping group of M. Since M is cancellative, we can identify every element in M with its image in the Grothendieck enveloping group.

For each $x \in G$, define the evaluation homomorphism $\epsilon_x : \widehat{G} \to \mathbb{T}$ by setting $\epsilon_x(\gamma) = \gamma(x)$ for $\gamma \in \widehat{G}$. Clearly $\{\epsilon_x \mid x \in G\}$ becomes an orthonormal basis of the Hilbert space $L^2(\widehat{G})$. The Hilbert subspace generated by $\{\epsilon_x \mid x \in M\}$ is denoted by $H^2(\widehat{G})$. Put P be the orthogonal projection onto $H^2(\widehat{G})$. For $\phi \in L^\infty(\widehat{G})$ we define a map $T_\phi \in \mathcal{B}(H^2(\widehat{G}))$ by $T_\phi(f) = P(\phi f)$ for $f \in H^2(\widehat{G})$. We denote by $\mathcal{T}(M)$ the C^*-subalgebra $\mathcal{B}(H^2(\widehat{G}))$ generated by all T_φ for $\varphi \in C(\widehat{G})$, the space of all continuous functions on $\widehat{G}$. We call it the generalized Toeplitz algebras.

L.A. Coburn proved his well known theorem in [1] that if v is a non-unitary isometry in a unital C^*-algebra B, then there exists a unique isometric *-homomorphism $\phi : \mathcal{T} \to B$ such that $\phi(T_{\epsilon_1}) = v$.

PROPOSITION 4.1. *Let M be a cancellative abelian semigroup and G be its Grothendieck enveloping group. Then $\mathcal{T}(M)$ is generated by T_{ϵ_x} for all $x \in M$.*

Proof. Let $P(\widehat{G})$ denote a linear span of $\{\epsilon_x \mid x \in G\}$. If $\varphi \in P(\widehat{G})$, we can have $\varphi = \sum_{i=1}^n \lambda_i \varepsilon_{y_i}$ for some $y_i \in G$ and $\lambda_i \in \mathbb{C}$. We can choose elements z_i^1 and z_i^2 in M such that $y_i = z_i^1 - z_i^2$ for each $i \in M$. Hence we have $T_\varphi = \sum_i \lambda_i T_{\epsilon_{y_i}} = \sum_i \lambda_i T_{\epsilon^*_{z_i^2}} T_{\epsilon_{z_i^1}}$. Since $P(\widehat{G})$ is norm dense in $C(\widehat{G})$, $\{T_{\varepsilon_x} \mid x \in M\}$ generates $\mathcal{T}(M)$. □

Furthermore we can see that $\mathcal{T}(M)$ is unitarily isomorphic to $C^*_{red}(M)$. Let U be a unitary from $H^2(\widehat{G})$ onto $l^2(M)$ defined by

$$U(f) = \xi_f$$

where $\xi_f(x)$ is the Fourier coefficient $\hat{f}(x)$ of f with respect to the basis $\{\epsilon_x \mid x \in M\}$ of $H^2(\widehat{G})$ for each $x \in M$. Then we have

$$(UT_{\varepsilon_x}U^*)(\xi)(z) = \mathcal{L}_x(\xi)(z)$$

Since $C^*_{red}(M)$ and $\mathcal{T}(M)$ are generated by $\{\mathcal{L}_x \mid x \in M\}$ and $\{T_{\epsilon_x} \mid x \in M\}$, respectively, $C^*_{red}(M) = U\mathcal{T}(M)U^*$.

Now we can consider a reduced semigroup C^*-algebra $C^*_{red}(S)$ for $S = \{0, 4, 5, 8, 9, 10, 12, 13, 14, ..\}$. Then S is a really simple, but not quasi-lattice ordered semigroup. If we consider S as a subsemigroup of $\mathbb{N}$, then it generates $\mathbb{N}$. Though it is generated by two elements 4 and 5, the following theorem shows that $C^*_{red}(S)$ is generated by only one element.

THEOREM 4.2. *$C^*_{red}(S)$ is generated by only one single element.*

Proof. We define a compact operator K_0

$$K_0(\delta_n) = \begin{cases} \delta_4, & n = 0, \\ 0, & \text{otherwise.} \end{cases}$$

And then we define a compact operator L_l for $l = 1, 2$ such as

$$L_1(\delta_n) = \begin{cases} \delta_8, & n = 5, \\ 0, & \text{otherwise.} \end{cases}$$

$$L_2(\delta_n) = \begin{cases} \delta_{12}, & n = 10, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\mathcal{L}$ be the left regular isometric representation on $l^2(S)$ and put $U = \mathcal{L}_4^*\mathcal{L}_5 + K_0 + L_1 + L_2$. The compact operator algebra $\mathcal{K}(l^2(S))$

is contained in $C^*_{red}(S)$ because $C^*_{red}(S)$ acts irreducibly on $l^2(S)$, and thus U is contained in $C^*_{red}(S)$. We can see that

$$U(\delta_0) = \mathcal{L}_4^*\mathcal{L}_5(\delta_0) + K_0(\delta_0) + L_1(\delta_0) + L_2(\delta_0) = \delta_4.$$

Similiarly we have that

$$U(\delta_4) = \delta_5, \quad U(\delta_5) = \delta_8.$$

Furthermore, since $K_0(\delta_n) = 0$ and $L_1(\delta_n) = 0$ for $n > 5$, we have that

$$U(\delta_n) = \delta_{n+1}.$$

Therefore the operator U translates the elements of the canonical orthonormal basis $\{\delta_n \mid n \in S\}$ of $l^2(S)$ to the left, one by one. If we put the C^*-algebra $\mathcal{U}$ of $C^*_{red}(S)$ generated by U, then $\mathcal{U}$ is isomorphic to the Toeplitz algebra.

Eventually, $\mathcal{L}_4$ and $\mathcal{L}_5$ generates $C^*_{red}(S)$, so it is enough to show that $\mathcal{L}_4$ and $\mathcal{L}_5$ can be written as U + {suitable operators in $\mathcal{U}$} in order to say that U generates $C^*_{red}(S)$. So we consider U^4 and U^5. Since the terms of U^4 containing K_0 are removed,

$$U^4 = (\mathcal{L}_4^*\mathcal{L}_5)^4 + \sum (\mathcal{L}_4^*\mathcal{L}_5)^{s_1} L_i^{s_2} (\mathcal{L}_4^*\mathcal{L}_5)^{s_3} \cdots L_j^{s_q}$$

where $s_1 + \cdots + s_q = 3$ and s_i may be zero. In order to make up the gaps of $(\mathcal{L}_4^*\mathcal{L}_5)^4$ we define compact operators M_l as follows;

$$M_0(\delta_n) = \begin{cases} \delta_4, & n = 0, \\ 0, & \text{otherwise}, \end{cases} \qquad M_4(\delta_n) = \begin{cases} \delta_8, & n = 4, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$M_5(\delta_n) = \begin{cases} \delta_9, & n = 5, \\ 0, & \text{otherwise}, \end{cases} \qquad M_8(\delta_n) = \begin{cases} \delta_{12}, & n = 8, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$M_9(\delta_n) = \begin{cases} \delta_{13}, & n = 9, \\ 0, & \text{otherwise}, \end{cases} \qquad M_{10}(\delta_n) = \begin{cases} \delta_{14}, & n = 10, \\ 0, & \text{otherwise}. \end{cases}$$

Due to the compact operators M_i, we have

$$\mathcal{L}_4 = U^4 + M_l - \sum (\mathcal{L}_4^* \mathcal{L}_5)^{s_1} L_1^{s_2} (\mathcal{L}_4^* \mathcal{L}_5)^{s_3} \cdots L_1^{s_q}.$$

Similarly, $\mathcal{L}_5 = U^5 + T$ for a suitable compact operator T. Therefore, U generates $C^*_{red}(S)$. □

By the Coburn's result we can get the result.

COROLLARY 4.3 *$C^*_{red}(S)$ is isomorphic to the Toeplitz algebta.*

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