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SEMI-COMPATIBILITY AND FIXED POINT THEOREM IN MENGER SPACE USING IMPLICIT RELATION

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ABSTRACT. In this paper the concept of semi-compatibility has been introduced in Menger space and it has been applied to prove results on existence of unique common fixed point of four self maps satisfying an implicit relation. It results in a generalization of Banach contraction principle established by Sehgal and Bharucha-Reid in [8] All the result presented in this paper are new.

1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [3]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [7] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [8] obtained Banach contraction principle in a complete Menger space, which is a milestone in developing fixed point theory in Menger space. Cho, Sharma and Sahu [1] introduced the concept of semi-compatible maps in a d-topological space. They define a pair of self maps (S, T) to be semi-compatible if conditions (i)$Sy = Ty$ implies $STy = TSy$ (ii)$\{Sx_n\} \to x, \{Tx_n\} \to x$ implies $STx_n \to Tx$, as $n \to \infty$, hold. However, in Menger space (ii) implies (i), taking $x_n = y$ and $x = Ty = Sy$. So, we define a semi-compatible pair of self mappings in Menger space by condition (ii)

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only. Saliga [6] and Sharma et. al [9] and Popa [5] proved interesting fixed point results using implicit real functions and semi-compatibility in d-complete topological spaces. Recently, Jungck and Rohades [2] termed a pair of self maps to be coincidentally commuting or equivalently weak compatible if they commute at their coincidence points. The novelty of this paper is to obtain fixed point theorems in the setting of a Menger space using implicit relation with weak compatibility and semi-compatibility of maps. In the sequel we derive a characterization of such implicit relation if it is in linear form and use the same for obtaining some results on fixed points. This leads to a generalization of Banach contraction principle given in Sehgal and Bharucha-Reid [8]. For the sake of completeness, following [4] and [8] we recall some definitions and known results in Menger space.

2. Preliminaries

DEFINITION 2.1. *:A mapping $F : R \to R^+$ is called a distribution if it is non-decreasing left continuous with $inf\{F(t) : t \in R\} = 0$ and $sup\{F(t) : t \in R\} = 1$. We shall denote by L the set of all distribution functions while H will always denote the specific distribution function defined by*

$$H(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

DEFINITION 2.2. *:A Probabilistic metric space (PM-space) is an ordered pair (X, F), where X is an abstract set of elements and $F : X \times X \to L$, defined by $(p, q) \longmapsto F_{p,q}$, where L is the set of all distribution functions i.e.$L = \{F_{p,q} | p, q \in X\}$, if the functions $F_{p,q}$ satisfy:*

(a) $F_{p,q}(x) = 1$, for all $x > 0$, if and only if $p = q$;
(b) $F_{p,q}(0) = 0$;
(c) $F_{p,q} = F_{q,p}$;
(d) If $F_{p,q}(x) = 1$ and $F_{q,r}(y) = 1$ then $F_{p,r}(x + y) = 1$.

DEFINITION 2.3. *:A mapping $t : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if*

(e) $t(a, 1) = a$;

(f) $t(a,b) = t(b,a)$;
(g) $t(c,d) \geq t(a,b)$ *for* $c \geq a, d \geq b$;
(h) $t(t(a,b),c) = t(a,t(b,c))$,
for all $a,b,c,d \in [0,1]$.

DEFINITION 2.4. : *A Menger space is a triplet* (X,F,t) *where* (X,F) *is PM-space and* t *is a t-norm such that* $\forall p,q,r \in X$ *and* $\forall x,y \geq 0$

$$F_{p,r}(x+y) \geq t(F_{p,q}(x), F_{q,r}(y)).$$

Schweizer and Sklar [7] proved that if (X,F,t) is a Menger space with $sup_{\ 0<x<1}\ t(x,x) = 1$, then (X,F,t) is a Hausdorff topological space in the topology induced by the family of (ϵ,λ)-neighborhoods,
$\{U_p(\epsilon,\lambda) : p \in X, \epsilon > 0, \lambda > 0\}$,
where $U_p(\epsilon,\lambda) = \{x \in X : F_{x,p}(\epsilon) > 1-\lambda\}$.

DEFINITION 2.5. : *Let* (X,F,t) *be a Menger space with* $sup_{\ 0<x<1}$ $t(x,x) = 1$. *A sequence* $\{p_n\}$ *in* X *is said to converge to a point* p *in* X *(written as* $p_n \to p$*) if for every* $\epsilon > 0$ *and* $\lambda > 0, \exists$ *an integer* $M(\epsilon,\lambda)$ *such that* $F_{p_n,p}(\epsilon) > 1-\lambda, \forall n \geq M(\epsilon,\lambda)$. *Further, the sequence is said to be a Cauchy sequence if for each* $\epsilon > 0$ *and* $\lambda > 0, \exists$ *an integer* $M(\epsilon,\lambda)$ *such that* $F_{p_n,p_m}(\epsilon) > 1-\lambda, \forall n,m \geq M(\epsilon,\lambda)$. *A Menger space* (X,F,t) *is said to be complete if every Cauchy sequence in it converges to a point of it.*

A complete metric space can be treated as a complete Menger space in the following way.

PROPOSITION 2.6. *If* (X,d) *is a metric space then the metric* d *induces a mapping* $X \times X \to L$, *defined by* $F_{p,q}(x) = H(x - d(p,q)), \forall p,q \in X$ *and* $x \in R$. *Further, if* $t : [0,1 \times [0,1] \to [0,1]$ *is defined by* $t(a,b) = min\{a,b\}$, *then* (X,F,t) *is a Menger space. It is complete if* (X,d) *is complete.*
The space (X,F,t) *so obtained is called the induced Menger space.*

PROPOSITION 2.7. : *In a Menger space* (X,F,t), *if* $t(x,x) \geq x, \forall x \in [0,1]$ *then* $t(a,b) = min\{a,b\}, \forall a,b \in [0,1]$.

DEFINITION 2.8. *:Self mappings A and S of a Menger space (X, F, t) are said to be weak compatible if they commute at their coincidence points i.e. $Ax = Sx$ for $x \in X$ implies $ASx = SAx$.*

DEFINITION 2.9. *: Self mappings A and S of a Menger space (X, F, t) are called compatible if $F_{ASp_n,SAp_n}(x) \to 1, \forall x > 0$, whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to u$, for some $u \in X$, as $n \to \infty$.*

Here we introduce the notion of semi-compatible mappings in Menger space.

DEFINITION 2.10. *: Self-mappings A and S of a Menger space (X, F, t) are called semi-compatible if $F_{ASp_n,Su}(x) \to 1 \forall x > 0$, whenever $\{p_n\}$ is a sequence in X such that $Ap_n, Sp_n \to u$, for some $u \in X$, as $n \to \infty$.*

PROPOSITION 2.11. *: If self-mappings A and S of a Menger space (X, F, t) are semi-compatible then they are weak compatible.*

PROPOSITION 2.12. [11]*: Let S and T be two self maps on a Menger space (X, F, t) with $t(a, a) \geq a, \forall a \in [0, 1]$ of which T is continuous. Then (S, T) is a semi-compatible if and only if (S, T) is compatible.*

In [11] it has been shown that the semi-compatibility of (A, S) need not imply the semi-compatibility (S, A). Further, an example of pair of self maps is given, which is commuting (hence compatible, weak compatible) yet it is not semi-compatible. For a detailed discussion of semi-compatibility we refer to [11], [12] and [14].

LEMMA 2.13. *[10]: Let $\{p_n\}$ be a sequence in a Menger space (X, F, t) with continuous t-norm $t(x, x) \geq x, \forall x \in [0, 1]$. If $\exists\ k \in (0, 1)$ such that for all $x > 0$ and $n \in N$,*
$F_{p_n,p_{n+1}}(kx) \geq F_{p_{n-1},p_n}(x)$.
Then $\{p_n\}$ is a Cauchy sequence in X.

A Class of Implicit Relation

Let Φ be set of all real continuous functions $\phi : (R^+)^4 \to R$, non-decreasing in first argument and satisfying the following conditions:
(i) For $u, v \geq 0, \phi(u, v, v, u) \geq 0$ or $\phi(u, v, u, v) \geq 0$ imply $u \geq v$.
(ii) $\phi(u, u, 1, 1) \geq 0$ implies $u \geq 1$.

EXAMPLE 2.14. *:Define* $\phi(t_1, t_2, t_3, t_4) = 15t_1 - 13t_2 + 5t_3 - 7t_4$*.Then* $\phi \in \Phi$.

3. MAIN RESULTS

THEOREM 3.1. *:Let* A, B, S *and* T *be self mappings of a complete Menger space* (X, F, Min) *satisfying :*

(3.11) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

(3.12) *The pair* (A, S) *is semi-compatible and* (B, T) *is weak compatible;*

(3.13) *One of* A *or* S *is continuous;*
For some $\phi \in \Phi$ *there exists* $k \in (0, 1)$ *such that* $\forall p, q \in X$ *and* $t > 0$,
(3.14) $\phi(F_{Ap,Bq}(kt), F_{Sp,Tq}(t), F_{Ap,Sp}(t), F_{Bq,Tq}(kt)) \geq 0$ *and*
(3.15) $\phi(F_{Ap,Bq},(kt), F_{Sp,Tq}(t), F_{Ap,Sp}(kt), F_{Bq,Tq}(t) \geq 0$.
Then A, B, S *and* T *have a unique common fixed point in* X.

Proof. :Let $x_0 \in X$ be any arbitrary point as $A(X) \subsetneq T(X)$ and $B(X) \subsetneq S(X)$, $\exists\ x_1, x_2 \in X$ such that $Ax_0 = Tx_1, Bx_1 = Sx_2$. Inductively, construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1} = Ax_{2n} = Tx_{2n+1}, y_{2n+2} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \ldots$. Now using (3.14) with $p = x_{2n}, q = x_{2n+1}$ we get,
$\phi\left(\ F_{Ax_{2n},Bx_{2n+1}}(kt), F_{Sx_{2n},Tx_{2n+1}}(t), F_{Ax_{2n},Sx_{2n}}(t), F_{Bx_{2n+1},Tx_{2n+1}}(kt)\ \right) \geq 0$,
i.e. $\phi\left(\ F_{y_{2n+1},y_{2n+2}}(kt), F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n}}(t), F_{y_{2n+2},y_{2n+1}}(kt)\ \right) \geq 0$.
Using (i) we get,
$F_{y_{2n+2},y_{2n+1}}(kt) \geq F_{y_{2n+1},y_{2n}}(t)$
Similarly, putting $p = x_{2n+2}$ and $q = x_{2n+1}$ in (3.15) we have,
$\phi\left(\ F_{y_{2n+3},y_{2n+2}}(kt), F_{y_{2n+1},y_{2n+2}}(t), F_{y_{2n+3},y_{2n+2}}(kt), F_{y_{2n+1},y_{2n+2}}(t)\ \right) \geq 0$
Using (i) we get,
$F_{y_{2n+3},y_{2n+2}}(kt) \geq F_{y_{2n+1},y_{2n+2}}(t)$.
Thus, for any n and t we have,
$F_{y_n,y_{n+1}}(kt) \geq F_{y_{n-1},y_n}(t)$.
Hence by Lemma 2.13, $\{y_n\}$ is a Cauchy sequence in X, which is

complete. Therefore $\{y_n\}$ converges to $u \in X$. Its subsequences

(1) $$\{Ax_{2n}\} \to u \quad \text{and} \quad \{Bx_{2n+1}\} \to u.$$

(2) $$\{Sx_{2n}\} \to u \quad \text{and} \quad \{Tx_{2n+1}\} \to u.$$

Case I: S is continuous
In this case
$SAx_{2n} \to Su, S^2x_{2n} \to Su,$
and semi-compatibility of the pair (A, S) gives $lim_{n\to\infty} ASx_{2n} = Su$.
Step I: Putting $p = Sx_{2n}$ and $q = x_{2n+1}$ in (3.14) we get,
$\phi(F_{ASx_{2n},Bx_{2n+1}}(kt), F_{SSx_{2n},Tx_{2n+1}}(t), F_{ASx_{2n},SSx_{2n}}(t), F_{Bx_{2n+1},Tx_{2n+1}}(kt)) \geq 0.$
Taking limit as $n \to \infty$ and using (1)and(2) we get,
$\phi(F_{Su,u}(kt), F_{Su,u}(t), F_{Su,Su}(t), F_{u,u}(kt)) \geq 0,$
i.e. $\phi(F_{Su,u}(kt), F_{Su,u}(t), 1, 1) \geq 0,$
i.e. $\phi(F_{Su,u}(t), F_{Su,u}(t), 1, 1) \geq 0.$
Using (ii) we get that $F_{Su,u}(t) \geq 1$, which gives $F_{Su,u}(t) = 1$, i.e. $Su = u$.
Step II : Putting $p = u, q = x_{2n+1}$ in condition (3.14) we get,
$\phi(F_{Au,Bx_{2n+1}}(kt), F_{Su,Tx_{2n+1}}(t), F_{Au,Su}(t), F_{Bx_{2n+1},Tx_{2n+1}}(kt)) \geq 0.$
Taking limit as $n \to \infty$ and using results of step I and (2) we get,
$\phi(F_{Au,u}(kt), F_{Su,u}(t), F_{Au,Su}(t), F_{u,u}(kt)) \geq 0,$
i.e. $\phi(F_{Au,u}(kt), 1, F_{Au,u}(t), 1, 1) \geq 0,$
i.e. $\phi(F_{Au,u}(t), 1, F_{Au,u}(t), 1, 1) \geq 0.$
Using (i) we get that $F_{Au,u}(t) \geq 1$, which gives $u = Au$. Hence $u = Au = Su$.
As $A(X) \subseteq T(X), \exists\ w \in X$ such that $Au = Tw$ for some $w \in X$.
Therefore $u = Au = Su = Tw$
Step III : Putting $p = x_{2n}, q = w$ in condition (3.14) we get,
$\phi(F_{Ax_{2n},Bw}(kt), F_{Sx_{2n},Tw}(t), F_{Ax_{2n},Sx_{2n}}(t), F_{Bw,Tw}(kt)) \geq 0.$
Taking limit as $n \to \infty$ and using the results from above steps we get,
$\phi(F_{u,Bw}(kt), 1, 1, F_{Bw,u}(kt)) \geq 0.$
Using (i) we get that $F_{u,Bw}(kt) \geq 1$.
Hence, $u = Bw$. Therefore $Bw = Tw = u$. As (B, T) is weak compatible we get that $TBw = BTw$, i.e. $Bu = Tu$.
Step IV: Put $p = u, q = u$ in condition (3.14) and using results from above steps we have

$\phi(F_{Au,Bu}(kt), F_{Su,Tu}(t), F_{Au,Su}(t), F_{Bu,Tu}(kt)) \geq 0$,
i.e. $\phi(F_{Au,Bu}(kt), F_{Au,Bu}(t), 1, 1 \geq 0$,
i.e. $\phi(F_{Au,Bu}(t), F_{Au,Bu}(t), 1, 1 \geq 0$.
Using (ii) we get, $F_{Au,Bu}(t) \geq 1$. Hence $Bu = Au$. Therefore $u = Au = Su = Bu = Tu$, i.e. u is a common fixed point of the four self mappings A, B, S and T in this case.

Case II: A is continuous
As A is continuous we have $ASx_{2n} \to Au$ and semi-compatibility of (A, S) gives $ASx_{2n} \to Su$. By uniqueness of limit in Menger Space we get $Au = Su$.

Step V : Putting $p = u, q = x_{2n+1}$ in condition (3.14) we get,
$\phi(F_{Au,Bx_{2n+1}}(kt), F_{Su,Tx_{2n+1}}(t), F_{Au,Su}(t), F_{Bx_{2n+1},Tx_{2n+1}}(kt)) \geq 0$.
Taking limit as $n \to \infty$ and using (2) we get,
$\phi(F_{Au,u}(kt), F_{Su,u}(t), 1, F_{u,u}(kt)) \geq 0$,
i.e. $\phi(F_{Au,u}(t), F_{Au,u}(t), 1, 1) \geq 0$.
Using (ii) we get, $F_{Au,u}(kt) \geq 1$, which gives $u = Au$. Hence $u = Au = Su$ and rest of the proof follows from step III onwards of previous case.

Uniqueness : Let z be another common fixed point of A, B, S and T. Then $z = Az = Bz = Sz = Tz$. Putting p = u and q = z in (3.14) we get, $\phi(F_{Au,Bz}(kt), F_{Su,Tz}(t), F_{Au,Su}(t), F_{Bz,Tz}(kt)) \geq 0$,
i.e. $\phi(F_{u,z}(kt), F_{u,z}(t), 1, 1 \geq 0$,
i.e. $\phi(F_{u,z}(t), F_{u,z}(t), 1, 1 \geq 0$.
Using (ii) we get,$F_{u,z}(t) \geq 1$, which gives $u = z$. Therefore u is unique common fixed point of self maps A, B, S and T. □

COROLLARY 3.2. : *Let A, B, S and T be self mappings of a complete Menger space (X, F, Min) satisfying (3.11), (3.14), (3.15) and*
(3.12) *The pairs (A, S) and (B, T) are semi-compatible;*
(3.13) *One of A, B, S or T is continuous.*
Then A, B, S and T have unique common fixed point in X.
Proof : As semi-compatibility implies weak compatibility proof follows from theorem 3.1

Now, taking $S = I$ and $T = I$ in theorem 3.1, the conditions (3.11), (3.12), (3.13) are satisfied trivially and we get:

COROLLARY 3.3. : *Let A and B be self mappings of a complete Menger space (X, F, Min). Suppose that for some $\phi \in \Phi$ there exists*

some $k \in (0,1)$ *such that* $\forall p, q \in X$ *and* $\forall t > 0$,
$\phi(F_{Ap,Bq}(kt), F_{p,q}(t), F_{Ap,p}(t), F_{Bq,q}(kt)) \geq 0$ *and*
$\phi(F_{Ap,Bq}, (kt), F_{p,q}(t), F_{Ap,p}(kt), F_{Bq,q}(t) \geq 0$.
Then A *and* B *have unique common fixed point in* X.

Applications

In the following, a characterization of family of real implicit functions Φ has been derived and it has been applied to obtain a generalization of Banach contraction principle as given in [8].

A characterization of Φ in linear form:

Define $\phi(t_1, t_2, t_3, t_4) = at_1 + bt_2 + ct_3 + dt_4$, where $a, b, c, d \in R$ with $a+b+c+d = 0, a > 0, a+c > 0, a+b > 0$ and $a+d > 0$.Then $\phi \in \Phi$.

Proof. :For $u, v \geq 0$ and $\phi(u, v, v, u) \geq 0$ we have,
$(a+d)u + (b+c)v \geq 0$
i.e. $(a+d)u \geq (a+d)v$. Hence $u \geq v$, since $a+d > 0$.
Again,
$\phi(u, v, u, v) \geq 0$ gives
$(a+c)u + (b+d)v \geq 0$. Thus
$(a+c)u - (a+c)v \geq 0$.
Hence, $u \geq v$ as $(a+c) > 0$.
Also, $\phi(u, u, 1, 1) \geq 0$ gives
$(a+b)u + (c+d) \geq 0$,
i.e. $(a+b)u \geq -(c+d)$,
i.e $(a+b)u \geq (a+b)$, as $a+b+c+d = 0$.
Hence, $u \geq 1$.
As $a > 0$, ϕ is non-decreasing in the first argument and the result follows. □

COROLLARY 3.4. : *Let* A *and* B *be self mappings of a complete Menger space* (X, F, Min) *such that there exists some* $k \in (0,1)$ *satisfying:*
$aF_{Ap,Bq}(kt) + bF_{p,q}(t) + cF_{Ap,p}(t) + dF_{Bq,q}(kt) \geq 0$ *and*
$aF_{Ap,Bq}(kt) + bF_{p,q}(t) + cF_{Ap,p}(kt) + dF_{Bq,q}(t) \geq 0$
$\forall p, q \in X, \forall t > 0$ *and for some fixed* $a, b, c, d \in R$ *such that* $a > 0, a+b > 0, a+c > 0, a+d > 0$ *and* $a+b+c+d = 0$.
Then A *and* B *have a unique common fixed point in* X.

Proof: Using the characterization of Φ in corollary 3.3, the result follows.

THEOREM 3.5. : *Let A and B be self mappings of a complete Menger space (X, F, Min) such that there exists some $k \in (0,1)$ satisfying:*
$F_{Ap,Bq}(kt) \geq b_0 F_{p,q}(t) + c_0 F_{Ap,p}(t), \forall p, q \in X$ *and* $\forall t > 0$, *where* $b_0, c_0 \in (0,1)$ *with* $b_0 + c_0 = 1$.*Then A and B have a unique common fixed point in X.*

Proof. : Choosing $a = 1, d = 0, b = -b_0$ and $c = -c_0, c_0 > 0$, in corollary 3.4 and using the fact that $F_{x,y}(t)$ is a non-decreasing function, the second condition of 3.4 is trivially satisfied and the result follows. □

The study of fixed point in theory of P.M. space was started by Sehgal and Bharucha-Reid in [8]. The following definition and theorem appeared in their paper

Definition [8]: A mapping f of a P.M. space (X, F) into itself is a contraction if there exist $0 < k < 1$ such that for each x and $y \in X$,
$F_{fp,fq}(kt) \geq F_{fp,fq}(t), \forall t > 0$.

Theorem [8]: Let (X, F, t) be a complete Menger space where $t(a, b) = Min\{a, b\}$. If f is any contraction, there exists a unique $p \in X$ such that $f(p) = p$. Moreover, $lim_{n\to\infty} f^n(q) = p$ for each $q \in X$.

The main theme of what follows is to generalize this result substantially. To prove the main results we need the following result established by the authors in [12].
Corollary : (3.2, [12]):Let A, S, L and M are self-maps on a complete Menger space (X, F, t) with $t(x, x) \geq x, \forall x \in [0,1]$ and satisfying:
(a) $L(X) \subseteq S(X), M(X) \subseteq A(X)$;
(b) Either A or L is continuous;
(c) (L, A) is compatible and (M, S) is weak compatible;
(d) There exists $k \in (0,1)$ such that
$F_{Ap,Lq}(kx) \geq Min\{F_{Sp,Lp}(x), F_{Sq,Mq}(x), F_{Sp,LP}(\beta x), F_{Lp,Mq}((2-\beta)x), F_{Ap,Sp}(x)\}$,
for all $p, q \in X, \beta \in (0,2)$ and $x > 0$.

Then A, S, L and M have a unique common fixed point in X.
We derive the following result from it:

COROLLARY 3.6. : *Let A and B be self mappings of a complete Menger space (X, F, Min) such that there exists some $k \in (0,1)$ satisfying any one of the conditions:*
(a) $F_{Ap,Bq}(kt) \geq F_{p,q}(kt), \forall p, q \in X$ and $t > 0$.
(b) $F_{Ap,Bq}(kt) \geq F_{Ap,p} \forall p, q \in X$ and $t > 0$.
Then A and B have a unique common fixed point in X.
Proof: Taking $A = S = I$, the identity map on X in the above result of [12] and restricting the contractive condition to the last factor and to the first factor respectively, the results follow for self maps L and M.

Combining corollary 3.6 and theorem 3.5 we have the following generalization of Banach contraction principle for two self maps in a complete Menger space

THEOREM 3.7. : *Let A and B be self mappings of a complete Menger space (X, F, Min) such that there exists some $k \in (0,1)$ satisfying:*
$F_{Ap,Bq}(kt) \geq b_0 F_{p,q}(t) + c_0 F_{Ap,p}(t), \forall p, q \in X$ and $\forall t > 0$, where $b_0, c_0 \in [0,1]$ with $b_0 + c_0 = 1$. Then A and B have a unique common fixed point in X.

THEOREM 3.8. :*Let $\{A_n\}$ be a sequence of self-maps on a complete Menger space (X, F, Min). Suppose, $\exists k \in (0,1)$ and for each pair $(A_i, A_j), \exists$ constants $b_{i,j}, c_{i,j} \in [0,1]$ with $b_{i,j} + c_{i,j} = 1$ such that*
$F_{A_ip,A_jq}(kt) \geq b_{i,j} F_{p,q}(t) + c_{i,j} F_{A_ip,p}(t), \forall p, q \in X$ and $\forall t > 0$.
For some x_0 in X define sequence $\{x_n\}$ in X by $x_n = A_n x_{n-1}, \forall n \in N$. Then $\{x_n\}$ converges to some point u in X, which is the unique common fixed point of $\{A_n\}$.

Proof. :First of all we shall prove $\{x_n\}$ is a Cauchy sequence in X. Putting $p = x_{n-1}$ and $q = x_n$ in the given contraction we have,

$$\begin{aligned} F_{x_n,x_{n+1}}(kt) &= F_{A_n x_{n-1}, A_{n+1} x_n}(kt) \\ &\geq b_{n,n+1} F_{x_{n-1},x_n}(t) + c_{n,n+1} F_{A_n x_{n-1}, x_{n-1}}(t) \\ &= (b_{n,n+1} + c_{n,n+1}) F_{x_{n-1},x_n}(t) \end{aligned}$$

$= F_{x_{n-1},x_n(t)}, \forall t > 0.$

Hence, by Lemma 2.13, $\{x_n\}$ is a Cauchy sequence in X, which is complete. Hence it converges to some $u \in X$. Again for some fixed A_i, putting $p = u$ and $q = x_{n-1}$ with $j = n$ in the given contraction we have,

$$F_{A_iu,x_n}(kt) = F_{A_iu,A_nx_{n-1}}(kt)$$
$$\geq b_{i,n}F_{u,x_{n-1}}(t) + c_{i,n}F_{A_iu,u}(t).$$

Taking limit as $n \to \infty$ we get,

$$F_{A_iu,u}(kt) \geq b_{i,n}F_{u,u}(t) + c_{i,n}F_{A_iu,u}(t)$$
$$\geq (b_{i,n} + c_{i,n})F_{A_iu,u}(t)$$
$$= F_{A_iu,u}(t), \forall t > 0.$$

Therefore $A_iu = u, \forall i \in N$. Thus u is a common fixed point of $A_n, \forall n$,which is unique by theorem 3.7. □

THEOREM 3.9. : *Let A be self mapping of a complete Menger space (X, F, Min). Suppose for some $k \in (0, 1)$,*
$F_{Ap,Aq}(kt) \geq b_0F_{p,q}(t) + c_0F_{Ap,p}(t), \forall p, q \in X$ and $\forall t > 0$, where b_0, c_0 are some non-negative real numbers such that $b_0 + c_0 = 1$. Then A has unique fixed point u in X. Moreover, $lim_{n\to\infty}A_n(v) = u$, for all v in X.
Proof:The proof follows from theorem 3.8 by taking $A_n = A, \forall n$.

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