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SOLUTION OF AN UNSOLVED PROBLEM IN BCK-ALGEBRA

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ABSTRACT. In this paper we introduced Semi-neutral BCK-algebra and investigate some of its properties. The notions of ideals and subalgebras coincide in Semi-neutral BCK-algebras. We also show that if the number of nonzero elements in a Semi-neutral BCK-algebra is n, then the number of ideals/subalgebras in it is 2^n. Further, we solved an open problem posed by W.A. Dudek in [2].

1. Introduction

In 1966 Y. Imai and K. Iseki introduced two classes of abstract algebras BCK-algebras and BCI-algebras [3,4]. BCI-algebras are a generalization of BCK-algebras. Various researchers have studied these algebras extensively and as a result a lot of literature has emerged. W.A. Dudek ([2]) has posed the open problem:

Open Problem

Describe the class of BCK-algerbas in which every subset containing o is a Subalgebra (an ideal).

In this paper we define Semi-neutral BCK-algebra and investigate some of its properties. The notions of ideals and subalgebras coincide in Semi-neutral BCK-algebras. We also show that if the number of nonzero elements in a Semi-neutral BCK-algebra is n, then the number of subalgebras (ideals) in it is 2^n.

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Further, in Lemma 3 we answered the problem posed by W.A. Dudek and proved that if X is a Semi-neutral BCK-algebra, then every subset A in X containing o is a subalgebra (ideal) in X.

2. Preliminaries

A BCK-algebra X is an abstract algebra $(X, *, o)$ of type $(2, 0)$, where * is a binary operation, o is a constant which is the smallest element in X, satisfying the following conditions; for all $x, y, z \in X$,

1.1 $((x^*y)^*(x^*z))^*(z^*y) = o$
1.2 $(x^*(x^*y))^*y = o$
1.3 $x^*x = o$
1.4 $x^*y = o = y^*x \Rightarrow x = y$
1.5 $o^*x = o$ where $x^*y = o \Leftrightarrow x \leq y$
Moreover, the following properties hold in every BCK/BCI-algebra ([5, 7]):
1.6 $(x^*y)^*z = (x^*z)^*y$
1.7 $x \leq y \Rightarrow x * z \leq y^*z$ and $z^*y \leq z^*x$
1.8 $x^*o = x$
1.9 An implicative BCK-algebra is commutative and positive implicative. [7]
1.10 If A is an ideal in a BCK-algebra X, then the quotient algebra $X/A = \{{}^AC_x : x \in X\}$, where ${}^AC_x = \{y \in X : x^*y, y^*x \in A\}$, is a BCK-algebra. [6]

1.11 Definition [5] Let X be a BCI-algebra and S be a nonempty subset of X, S is known as a subalgerba of X if for $x, y \in S,\ x^*y \in S$.

1.12 Definition [6,7] Let X be a BCK-algerba and I be a nonempty subset of X. I is known as an ideal in X if

(i) $o \in I$
(ii) $x^*y, y \in I \Rightarrow x \in I$

1.13 Definition [6] A nonempty subset I of a BCK-algebra X is called an implicative ideal, if

(i) $o \in I$
(ii) $(y^*x)^*z, x^*z \Rightarrow y^*z \in I$

1.14 Definition [1] Let X be a BCI-algebra, and $x, y \in X$. Then x, y are said to be comparable if and only if $x^*y = o$ or $y^*x = o$. Further, we shall say that x proceeds y and y succeeds x if and only if $x^*y = o$ and denote it by $x \rightarrow y$ or $x \leq y$.
Similarly in BCK-algebras, if $x^*y = o$ or $y^*x = o$, then x and y are comparable.

1.15 Definition [7] A BCK-algebra X is said to be commutative if $\mathbf{y}^*(\mathbf{y}^*\mathbf{x}) = \mathbf{x}^*(\mathbf{x}^*\mathbf{y})$ holds for all $x, y \in X$.

1.16 Definition [7] i) A BCK-algebra X is said to be implicative if $\mathbf{x}^*(\mathbf{y}^*\mathbf{x}) = \mathbf{x}$ holds for all $x, y \in X$.
ii) If every ideal of a BCK-algebra M is implicative then M is implicative. [6]

1.17 Definition [7] A BCK-algebra X is said to be positive implictive if $(\mathbf{x}^*\mathbf{y})^*\mathbf{z} = (\mathbf{x}^*\mathbf{z})^*(\mathbf{y}^*\mathbf{z})$ holds for all $x, y, z \in X$.
In [7] it is also shown that a BCK-algebra X is positive implicative if and only if $x^*y = (x^*y)^*y$.

1.18 Definition Let X be a BCK-algebra. An element x_0 in X is said to be a **Semi-neutral** element in X if and only if for all $x \neq x_0$, $x^*x_0 = x$ and $x_0^*x = x_0$.

The set of all **Semi-neutral** elements is denoted as $S(X)$ and is known as the semi-neutral part of the BCK-algebra X. Obviously $S(X)$ is nonempty, because X is a BCK-algebra, therefore $o^*x = o$ and $x^*o = x$, so $o \in S(X)$.

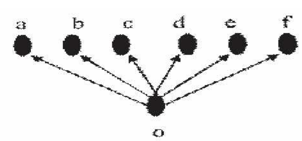
Note that any nonzero element x of a BCK-algebra X such that $x \leq y$ for some $y \in X$ (or, $y \leq x$ for some $y(\neq 0) \in X$) cannot be a semi-neutral element of X.

1.19 Definition A BCK-algebra X is said to be a **Semi-neutral BCK-algebra** if it satisfies: for all $x, y \in X, x \neq y \Rightarrow x^*y = x$.
Equivalently, if $X = S(X)$, then we say that X is a Semi-neutral BCK-algebra.
Note that the BCK-algebra of order 2 is semi neutral.

Example 1 Let $X = \{o, a, b, c, d, e, f\}$ be a BCK-algebra. The multiplication table (Table 1) and the tree diagram representing the BCK-algebra are known as follows:

Table 1

*	o	a	b	c	d	e	f
o	o	o	o	o	o	o	o
a	a	o	a	a	a	a	a
b	b	b	o	b	b	b	b
c	c	c	c	o	c	c	c
d	d	d	d	d	o	d	d
e	e	e	e	e	e	o	e
f	f	f	f	f	f	f	o

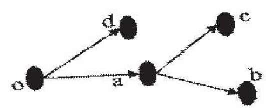


By routine calculations, we know that o, a, b, c, d, e, f are the **Semi-neutral** elements. Here $S(X) = \{o, a, b, c, d, e, f\} = X$, therefore X is a Semi-neutral BCK-algebra.

Example 2 Let $X = \{o, a, b, c, d\}$ be a BCK-algebra. The multiplication table (Table 2) and the tree diagram representing the BCK-algebra are shown as follows:

Table 2

*	o	a	b	c	d
o	o	o	o	o	o
a	a	o	o	o	a
b	b	b	o	b	b
c	c	c	c	o	c
d	d	d	d	d	o

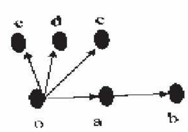


Note that for all $x \neq 0 \in X, x^*d = x$ and $d^*x = d$. Hence d is the only nonzero Semi-neutral element in X. Hence $S(X) = \{o, d\}$.

Example 3 Let $X = \{o, a, b, c, d, e\}$ be a BCK-algebra. The multiplication table (Table 3) and the tree diagram representing the BCK-algebra are shown as follows:

Table 3

*	o	a	b	c	d	e
o	o	o	o	o	o	o
a	a	o	o	a	a	a
b	b	a	o	b	b	b
c	c	c	c	o	c	c
d	d	d	d	d	o	d
e	e	e	e	e	e	o

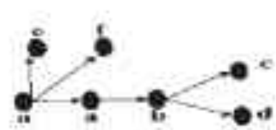


By routine calculations, we know that c, d, e are the nonzero Semi-neutral elements. However $S(X) = \{o, c, d, e\}$.

Example 4 Let $X = \{o, a, b, c, d, e, f\}$ be a BCK-algebra. The multiplication table (Table 4) and the tree diagram representing the BCK-algebra are shown as follows:

Table 4

*	o	a	b	c	d	e	f
o	o	o	o	o	o	o	o
a	a	o	o	o	o	a	a
b	b	b	o	o	o	b	b
c	c	c	b	o	b	c	c
d	d	c	a	a	o	d	d
e	e	e	e	e	e	o	e
f	f	f	f	f	f	f	o



By routine calculations, we know that e, f are the nonzero Semi-neutral elements. Hence $S(X) = \{o, e, f\}$.

Lemma 1 The semi-neutral part of a BCK-algebra X is an ideal in X.

Proof Obviously $o \in S(X)$
Let $x^*y, y \in S(X)$. Since $y \in S(X)$, therefore by definition 1.18, for all $x \neq y$.

$$y^*x = y \tag{1}$$

and

$$x^*y = x \tag{2}$$

because of equation (2), $x^*y \in S(X) \Rightarrow x \in S(X)$. Hence $S(X)$ is an ideal in X.

Lemma 2 Each subalgebra of a Semi-neutral BCK-algebra X is an ideal in X.

Proof Let X be a Semi-neutral BCK-algebra and S be a subalgebra of X. Then, $x, y \in S \Rightarrow x^*y \in S$. Since X is a Semi-neutral, therefore $X = S(X)$, so each element in X is a semi-neutral element. Therefore each element in S is also a semi-neutral element. Let x, y be any two elements in S. Then

$$x^*y = x$$

and

$$y^*x = y$$

Now

$$y^*x,\ x \in S \Rightarrow y \in S$$

and

$$x^*y,\ y \in S \Rightarrow x \in S$$

which implies that S is an ideal in X.
From above Lemma it follows that in case of Semi-neutral BCK-algebras the notions of ideals and sub-algebras coincide with each other.

Lemma 3 Let X be a Semi-neutral BCK-algebra, then every subset containing o is a subalgebra/an ideal of X.

Proof Let A be any subset of X containing o. As each element in X is a Semi-neutral element therefore each element in A is also Semi-neutral.
Let x be any element in A. Then

$$x^*o = x \tag{1}$$

and

$$o^*x = o \tag{2}$$

Also by definition 1.18 $x^*y = x$ holds for all $x, y \in A$.
Further for $x, y, z \in A$

$$(x^*y)^*(x^*z) = x^*x = o \leq z = z^*y \text{ (becasue } 0 \leq z)$$

i.e.

$$(x^*y)^*(x^*z) \leq z^*y \tag{3}$$

and

$$x^*(x^*y) = x^*x = o \leq y \text{ (becasue } o \leq y)$$

i.e.

$$x^*(x^*y) \leq y \tag{4}$$

Since $x \in A \subseteq X$, therefore

$$x^*x = o \tag{5}$$

holds for all $x \in A$.
From (1),.....,(5) it follows that A is a Semi-neutral BCK-algebra. Hence A is a sub algebra of X. As A is an arbitrary subset of X containing O, therefore each subset of X containing 0 is a sub algebra of X. Because of Lemma 2, each subset of X containing 0 is an ideal of X.

Lemma 4 Let n be the number of nonzero elements in a Semi-neutral BCK-algebra X. Then X has 2^n ideals/subalgebras.

Proof Consider those ideals/subalgebras of X that have k nonzero elements each, where $k = 0,\ 1, 2, \cdots, n$. Since number of ways in which k nonzero elements can be chosen out of n nonzero elements is ${}^nC_k = n!/k!(n-k)!$, therefore the number of ideals/subalgebras in X having k nonzero elements each in ${}^nC_k$. Hence the total number of ideals/subalgebras in X is

$$\sum_{k=0}^{n} ({}^nC_k)$$

Now

$$(1+x)^n = \sum_{k=0}^{n} ({}^nC_k)x^k$$

put $x = 1$ in above equation and get

$$(1+1)^n = \sum_{k=0}^{n} ({}^nC_k) = 2^n$$

Hence the proof.

Lemma 5 If X is a semi-neutral BCK-algebra of finite order, then it is unique.

Proof Let X be a Semi-neutral BCK-algerba and assume that $o(X) = n$. So $X = S(X)$ and by the definition of 1.19, $x * y = x$ holds for all $x, y \in X$. Thus, each cell of the Calay's table representing the semi-neutral BCK-algebra has a unique value. Hence, there exists only a unique BCK-algebra of order n.

Lemma 6 If X is a BCK-chain, then $S(X) = \{o\}$.

Proof Straight Forward.

Lemma 7 Let X be a Semi-neutral BCK-algebra. Then the quotient algebra X/A is a Semi-neutral BCK-algerba, for A being an ideal in X.

Proof Let X be a BCK-algebra which is Semi-neutral and A is an ideal in X, then by 1.10 X/A is a BCK-algebra. We show that X/A is Semi-neutral. Let $X/A = \{{}^AC_X : x \in X\}$ be a quotient algebra, where

$$^AC_x = C_x = \{y \in X : \ x^*y, \ y^*x \in A\}$$

under the binary operation * defined as follows:

$$C_x^* C_y = C_{x^* y}$$

for $C_x, C_y \in X/A$
Let $C_x, C_y \in X/A$, for $x, y \in X$. Then

$$\begin{aligned} C_x^* C_y &= \mathbf{C}_{X^* y} \\ &= C_x \text{ (because of 1.19, } x^*y = x, \text{ for } x, y \in X) \end{aligned}$$

Also

$$\begin{aligned} C_y,^* C_x &= \mathbf{C}_{y^* X} \\ &= C_y \text{ (because of 1.19, } y^*x = y, \text{ for } x, y \in X) \end{aligned}$$

$\Rightarrow$ each class C_x of X/A is a Semi-neutral element.
Hence X/A is a Semi-neutral BCK-algebra.

Lemma 8 A Semi-neutral BCK-algebra is commutative, positive implicative and implicative.

Proof Assume that X is a Semi-neutral BCK-algebra. Then $X = S(X)$.
Let $x, y \in S(X)$. Then by definition of Semi-neutral BCK-algebra

$$x * y = x \tag{1}$$

holds $\forall x, y \in X$.
Now using (1)

$$x * (x * y) = x * x = 0 \tag{2}$$

and

$$y * (y * x) = y * y = 0 \tag{3}$$

From (2) and (3) it follows that

$$x * (x * y) = y * (y * x)$$

Which implies that X is commutative.
Further for x, y and $z \in X$, using (1)

$$(x * y) * z = x * z = x \tag{4}$$

and

$$(x * z) * (y * z) = x * y = x \tag{5}$$

From (4) and (5) it follows that

$$(x * y) * z = (x * z) * (y * z)$$

Which implies that X is positive implicative.
Also using (1)

$$x^*(y^*x) = x^*y = x \Rightarrow x^*(y^*x) = x$$

By 1.16(i), X is an implicative BCK-algebra.

Lemma 9 Every ideal in a Semi-neutral BCK-algebra is an implicative ideal.

Proof Let A be any ideal in a Semi-neutral BCK-algebra X. Then

(i) $o \in A$
(ii) $y^*x,\ x \in A \Rightarrow y \in A$

Because X is a semi neutral BCK-algebra, therefore by definition 1.19. $x^*y = x$ holds $\forall x, y \in X$, therefore

$$y^*x = (y^*x)^*z,\ x = x^*z,\ y = y^*z$$

Thus (ii) implies

$$(y^*x)^*z,\ x^*z \in A \Rightarrow y^*z \in A$$

Hence A is an implicative ideal.

From Lemma 9, because of 1.16 (ii), it follows that the Semi-neutral BCK-algebra is an implicative BCK-algebra.

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