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L^p NORM INEQUALITIES FOR AREA FUNCTIONS WITH APPROACH REGIONS

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Abstract. In this paper we first introduce a space of homogeneous type X, and then consider a kind of generalized upper half-space $X \times (0, \infty)$. We are mainly considered with inequalities for the L^p norms of area functions associated with approach regions in $X \times (0, \infty)$.

1. Introduction

Recently enormous progress in harmonic analysis has been made. This paper will announce a problem related to harmonic analysis: In this paper we first introduce a space of homogeneous type X, which is a more general setting than $\mathbb{R}^n$, and we also consider a kind of generalized upper half-space $X \times (0, \infty)$. Suppose that for each boundary point $x \in X$ we are given an approach region $\Gamma_\alpha(x) \subset X \times (0, \infty)$. Let f be a measurable function defined on $X \times (0, \infty)$. Then we define an area function $A^{(\alpha)}(f)$ associated with $\Gamma_\alpha(x)$.

The purpose of this paper is to study inequalities for the L^p norms of area functions $A^{(\alpha)}(f)$ and $A^{(1)}(f)$ for $\alpha > 1$.

2. Preliminaries and Notations

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We begin by introducing the notion of a space of homogeneous type (see Coifman and Weiss [2]): Let X be a topological space endowed with Borel measure μ. Assume that d is a pseudo-metric on X, that is, a non-negative function on $X \times X$ satisfying

(i) $d(x,x) = 0; d(x,y) > 0$ if $x \neq y$,
(ii) $d(x,y) = d(y,x)$, and
(iii) $d(x,z) \leq K[d(x,y)+d(y,z)]$, where K is some fixed constant.

Assume further that

(a) the balls $B(x,\rho) = \{y \in X : d(x,y) < \rho\}$, $\rho > 0$, form a basis of open neighborhoods at $x \in X$,

and that μ satisfies the doubling property:

(b) $0 < \mu(B(x,2\rho)) \leq A\mu(B(x,\rho)) < \infty$, where A is some fixed constant.

Then we call (X, d, μ) a *space of homogeneous type*.

Property (iii) will be referred to as the "triangle inequality."

We consider the space $X \times (0,\infty)$, which is a kind of generalized upper half-space over X. We then introduce the analogue of non-tangential or conical regions. For $x \in X$ and $\alpha > 0$, set

$$\Gamma_\alpha(x) = \{(y,t) \in X \times (0,\infty) : x \in B(y, \alpha t)\}.$$

Throughout this paper, we put $\Gamma(x) = \Gamma_1(x)$ for simplicity.

For any closed subset $F \subset X$ and $\alpha > 0$, $\mathcal{R}^{(\alpha)}(F)$ will be the union of approach regions with boundary points in F, that is,

$$\mathcal{R}^{(\alpha)}(F) = \bigcup_{x \in F} \Gamma_\alpha(x).$$

We now introduce an area function associated with an approach region. Let f be a measurable function defined on $X \times (0,\infty)$. For $x \in X$ and $\alpha > 0$, we define an arera function $A^{(\alpha)}(f)$ by

$$A^{(\alpha)}(f)(x) = \left\{ \int_{\Gamma_\alpha(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{\sigma+1}} \right\}^{1/2}, \quad \sigma \in \mathbb{R} \tag{1}$$

whenever the integral exists.

Let f be a locally integrable function on X. Then for $x \in X$, we define

$$M(f)(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y),$$

where the supremum is taken over all balls B containing x. Then we call $M(f)$ the *Hardy-Littlewood maximal function* of f.

We need the notion of points of density. Suppose F is a closed subset of X whose complement has finite measure. Let γ be a fixed parameter, $0 < \gamma < 1$. Then we say that a point $x \in X$ has *global γ-density* with respect to F, if

$$\frac{\mu(F \cap B)}{\mu(B)} \geq \gamma$$

for all balls B centered at x in X. Let F^* be the set of points of global γ-density with respect to F; then F^* is closed, $F^* \subset F$, and

$${}^cF^* = \{x \in X : M(\chi_{{}^cF})(x) > 1 - \gamma\},$$

where $\chi_{{}^cF}$ is the characteristic function of the open set ${}^cF$.

3. L^p estimate for $A^{(\alpha)}(f)$ and $A^{(1)}(f)$, $\alpha > 1$

We state the four lemmas we need:

LEMMA 1. *Let (X, d, μ) be a space of homogeneous type. Assume F is a closed subset of X. Then there is a constant C_γ such that*

$$\mu({}^cF^*) \leq C_\gamma \mu({}^cF),$$

where F^* is the set of points of global γ-density with respect to F.

Proof. See Suh [6].

∎

LEMMA 2. *Let (X, d, μ) be a space of homogeneous type. Suppose $\alpha > 0$ is given. Then there is a constant C_α so that whenever F is a closed subset of X and $S(y,t)$ is a non-negative measurable function on $X \times (0,\infty)$, then*

$$\int_F \left\{ \int_{\Gamma_\alpha(x)} S(y,t)d\mu(y)dt \right\} d\mu(x) \leq C_\alpha \int_{\mathcal{R}^{(\alpha)}(F)} S(y,t)t^\sigma d\mu(y)dt,$$

where σ is given as (1).

Proof. Fubini's theorem gives

$$\begin{aligned}&\int_F \left\{ \int_{\Gamma_\alpha(x)} S(y,t)d\mu(y)dt \right\} d\mu(x)\\ &= \int_{X\times(0,\infty)} S(y,t) \left\{ \int_F \chi_{B(y,\alpha t)}(x)d\mu(x) \right\} d\mu(y)dt,\end{aligned}$$

where $\chi_{B(y,\alpha t)}$ is the characteristic function of the ball $B(y, \alpha t)$. Thus for given $(y,t) \in \mathcal{R}^{(\alpha)}(F)$, it will suffice to show that there is a constant C_α so that

$$\int_F \chi_{B(y,\alpha t)}(x)d\mu(x) \leq C_\alpha t^\sigma.$$

In fact, let $(y,t) \in \mathcal{R}^{(\alpha)}(F)$. Then

$$\begin{aligned}\int_F \chi_{B(y,\alpha t)}(x)d\mu(x) &\leq \int_X \chi_{B(y,\alpha t)}(x)d\mu(x)\\ &= C_\alpha t^\sigma,\end{aligned}$$

as desired. The proof is therefore complete.

□

LEMMA 3. *Let (X, d, μ) be a space of homogeneous type. Suppose $\alpha > 0$ is given. Then there are constants $C_{\alpha,\gamma}$ and γ, $0 < \gamma < 1$, sufficiently close to 1, so that whenever F is a closed subset of X whose complement has finite measure and $S(y,t)$ is a non-negative measurable function on $X \times (0,\infty)$, then*

$$\int_{\mathcal{R}^{(\alpha)}(F^*)} S(y,t)t^{\sigma}d\mu(y)dt \le C_{\alpha,\gamma}\int_F \left\{\int_{\Gamma(x)} S(y,t)d\mu(y)dt\right\}d\mu(x),$$

where F^* is the set of points of global γ-density with respect to F, and σ is given as in (1).

Proof. See Suh [6].

□

LEMMA 4. *Let (X, d, μ) be a space of homogeneous type. If f is a non-negative function defined on X, and $M(f)$ is the Hardy-Littlewood maximal function of f. Suppose*

$$\Phi_t(f)(x) = \frac{1}{t^{\sigma}}\int_X \chi_{B(x,t)}(y)f(y)d\mu(y), \quad t > 0,$$

where $\chi_{B(x,t)}$ is the characteristic function of the ball $B(x,t)$, and σ is given as in (1). Then for $\alpha > 0$, there is a constant C_α such that

$$\Phi_{\alpha t}(f) \le C_\alpha \Phi_t(M(f)). \tag{2}$$

Proof. We observe that if f is a non-negative function on X, then

$$\Phi_{\alpha t}(f) \le C'_\alpha \Phi_t(\Phi_{\alpha t}(f)).$$

Because $\Phi_{\alpha t}(f) \le CM(f)$, we get (2). The proof is therefore complete.

□

The main result of this paper is now the following:

THEOREM 5. *Let (X, d, μ) be a space of homogeneous type. Suppose $0 < p < \infty$ and $\alpha > 1$. Then there is a constant $C_{\alpha,p}$ such that*

$$||A^{(\alpha)}(f)||_{L^p(d\mu)} \le C_{\alpha,p}||A^{(1)}(f)||_{L^p(d\mu)}.$$

Proof. Assume first that $0 < p < 2$. For each $\lambda > 0$, we define the open set O by

$$O =^c F = \{x \in X : A^{(1)}(f)(x) > \lambda\}.$$

Then we take $F^* \subset F$ to be the set of points of global γ-density with respect to F, with $O^* =^c F^*$. Apply Lemma 2 with $S(y,t) = |f(y,t)|^2 t^{-\sigma-1}$ (and F^* in place of F), and we obtain

$$\int_{F^*} (A^{(\alpha)}(f)(x))^2 d\mu(x) \le C_\alpha \int_{\mathcal{R}^{(\alpha)}(F^*)} |f(y,t)|^2 \frac{d\mu(y)dt}{t}. \tag{3}$$

Next apply Lemma 3, again with $S(y,t) = |f(y,t)|^2 t^{-\sigma-1}$, and we obtain

(4)

$$\int_{\mathcal{R}^{(\alpha)}(F^*)} |f(y,t)|^2 \frac{d\mu(y)dt}{t} \le C'_{\alpha,\gamma} \int_F \left\{ \int_{\Gamma(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{\sigma+1}} \right\} d\mu(x).$$

Then (3) and (4) imply that

$$\int_{F^*} (A^{(\alpha)}(f)(x))^2 d\mu(x) \le C'_{\alpha,\gamma} \int_F (A^{(1)}(f)(x))^2 d\mu(x). \tag{5}$$

Thus it follows from Lemma 1 and (5) that

(6)

$$\begin{aligned}
&\mu(\{x \in X : A^{(\alpha)}(f)(x) > \lambda\}) \\
\le&\mu(O^*) + \frac{C'_{\alpha,\gamma}}{\lambda^2} \int_F (A^{(1)}(f)(x))^2 d\mu(x) \\
\le&C'_\alpha \left(\mu(O) + \frac{1}{\lambda^2} \int_F (A^{(1)}(f)(x))^2 d\mu(x) \right) \\
=&C'_\alpha \left(\mu(\{x \in X : A^{(1)}(f)(x) > \lambda\}) + \frac{1}{\lambda^2} \int_F (A^{(1)}(f)(x))^2 d\mu(x) \right).
\end{aligned}$$

If we multiply both sides of (6) by λ^{p-1} and integrate, then we get that

$$||A^{(\alpha)}(f)||_{L^p(d\mu)} \leq C_{\alpha,p}||A^{(1)}(f)||_{L^p(d\mu)}$$

for $0 < p < 2$.

Assume second that $2 \leq p < \infty$. Then observe that

$$||A^{(\alpha)}(f)||^2_{L^p(d\mu)} = \sup_\psi \int_X (A^{(\alpha)}(f)(x))^2\psi(x)d\mu(x),$$

where the supremum is taken over all non-negative ψ which belong to the space $L^q(X)$ with q dual to $p/2$, and $||\psi||_{L^q(d\mu)} \leq 1$. Then it follows from Lemma 4 that

(7)

$$\begin{aligned}
&\int_X (A^{(\alpha)}(f)(x))^2\psi(x)d\mu(x)\\
&\quad = \int_{X\times(0,\infty)} |f(y,t)|^2 \left\{\int_X \chi_{B(y,\alpha t)}(x)\psi(x)d\mu(x)\right\}\frac{d\mu(y)dt}{t^{\sigma+1}}\\
&\quad = \alpha^\sigma \int_{X\times(0,\infty)} |f(y,t)|^2 \Phi_{\alpha t}(\psi)(y)\frac{d\mu(y)dt}{t}\\
&\quad \leq C''_\alpha \int_{X\times(0,\infty)} |f(y,t)|^2 \Phi_t(M(\psi))(y)\frac{d\mu(y)dt}{t}\\
&\quad = C''_\alpha \int_X (A^{(1)}(f)(x))^2 M(\psi)(x)d\mu(x)\\
&\quad \leq C''_\alpha ||A^{(1)}(f)||^2_{L^p(d\mu)}||M(\psi)||_{L^q(d\mu)}\\
&\quad \leq C''_\alpha ||A^{(1)}(f)||^2_{L^p(d\mu)}||\psi||_{L^q(d\mu)}\\
&\quad \leq C''_\alpha ||A^{(1)}(f)||^2_{L^p(d\mu)}.
\end{aligned}$$

Taking the supremum over all allowable ψ in (7) gives us then

$$||A^{(\alpha)}(f)||_{L^p(d\mu)} \leq C_{\alpha,p}||A^{(1)}(f)||_{L^p(d\mu)}$$

for $2 \leq p < \infty$. The proof of the theorem is therefore complete. □

Remark. In the above proof, the limitation $p < \infty$ arises since the maximal inequality $||M(\psi)||_{L^q(d\mu)} \leq C_q||\psi||_{L^q(d\mu)}$ requires $q > 1$.

REFERENCES

[1] P. Ahern and A. Nagel, *Strong L^p estimates for maximal functions with respect to singular measures; with applications to exceptional sets*, Duke Math. J. **53** (1986), 359–393.

[2] R. R. Coifman and G. Weiss, *Analyse Harmonique Non-commutative sur Certains Espaces Homogènes*, Lecture Notes in Math., Vol. 242, Springer-Verlag, Berlin, 1971.

[3] ———, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. **83** (1977), 569–645.

[4] A. Korányi and S. Vági, *Singular integrals on homogeneous spaces and some problems of classical analysis*, Ann. Scuola Norm. Sup. Pisa **25** (1971), 575–648.

[5] J. Suerio, *On maximal functions and Poisson-Szegö integrals*, Trans. Amer. Math. Soc. **298** (1986), 653–669.

[6] C. -S. Suh, *A decomposition into atoms of tent spaces in the context of spaces of homogeneous type*, Comm. Korean. Math. Soc. **14** (1999), 735–746.

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