East Asian Math. J. **21** (2005), No. 1, pp. 21–26

ON LIE IDEALS OF PRIME RINGS WITH GENERALIZED JORDAN DERIVATION

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Abstract. The purpose of this paper is to show that every generalized Jordan derivation of prime ring with characteristic not two is a generalized derivation on a nonzero Lie ideal U of R such that $u^2 \in U$ for $\forall u \in U$ which is a generalization of the well-known result of I. N. Herstein.

1. Introduction

Let R be a prime ring with characteristic different from two. An additive mapping $d : R \to R$ is called Jordan derivation if $d(x^2) = d(x)x + xd(x)$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$, for all $u \in U, r \in R$. The notion of generalized derivation of prime ring R was introduced by B. Hvala in [1]. An additive map f of an associative ring R is called a generalized derivation if there is a derivation d of R such that

$$f(xy) = f(x)y + xd(y), \quad \text{for all } x, y \in R.$$

A classical result of I. N. Herstein states that every Jordan derivation of prime ring with characteristic not two is a derivation in [2]. A brief proof of this theorem can be found in [4]. Latter on, this result was generalized on Lie ideals of R such that $u^2 \in U$ for all $u \in U$ in [7] and generalized derivations of prime ring R in [6]. We shown that

Received November 29, 2004.
2000 Mathematics Subject Classification: 16N60, 16U80, 16W25.
Key words and phrases: Prime Rings, Generalized Derivations, Jordan Derivations.

this result holds for generalized derivation on a Lie ideal of R such that $u^2 \in U$ for all $u \in U$.

Throughout this paper, let R be a prime ring with its characteristic not two, U a nonzero Lie ideal of R with $u^2 \in U$ for all $u \in U$ and let $f : R \to R$ be a generalized Jordan derivation of R associated with a derivation d of R such that $f(x^2) = f(x)x + xd(x)$, for all $x \in R$.

In view of the hypothesis the $u^2 \in U$ for all $u \in U$, we get $(u+v)^2 \in U$ and so $(u+v)^2 - u^2 - v^2 = uv + vu \in U$ for all $u, v \in U$. Also $vu - uv \in U$, for all $u, v \in U$. Hence we find that $2vu \in U$, for all $u, v \in U$.

LEMMA 1. *For all $u, v, w \in U$,*

i) $f(uv + vu) = f(u)v + ud(v) + f(v)u + vd(u)$
ii) $f(uvu) = f(u)vu + ud(v)u + uvd(u)$
iii) $f(uvw + wvu) = f(u)vw + ud(v)w + uvd(w) + f(w)vu$
$+wd(v)u + wvd(u)$.

Proof. i) Linearizing, we get

$$\begin{aligned} f((u+v)^2) &= f((u+v)(u+v)) = f(u^2 + uv + vu + v^2) \\ &= f(u^2) + f(uv+vu) + f(v^2) \\ &= f(u)u + ud(u) + f(uv+vu) + f(v)v + vd(v) \text{ for all } u, v \in U. \end{aligned} \tag{1.1}$$

On the other hand,

$$\begin{aligned} f((u+v)^2) &= f(u+v)(u+v) + (u+v)d(u+v) \\ &= f(u)u + f(v)u + f(u)v + f(v)v \\ &+ud(u) + ud(v) + vd(u) + vd(v) \text{ for all } u, v \in U. \end{aligned} \tag{1.2}$$

Comparing (1.1) and (1.2), we have

$$f(uv + vu) = f(u)v + ud(v) + f(v)u + vd(u) \text{ for all } u, v \in U.$$

ii) Replacing v by $uv + vu$ in (i), we get

$$
\begin{aligned}
&f(u(uv+vu)+(uv+vu)u) = f(u^2v+uvu+uvu+vu^2)\\
(1.3)\qquad &= f(u^2v+vu^2)+2f(uvu) = f(u^2)v+u^2d(v)+f(v)u^2\\
&+vd(u^2)+2f(uvu) = f(u)uv+ud(u)v+u^2d(v)\\
&+f(v)u^2+vd(u)u+vud(u)+2f(uvu) \text{ for all } u,v\in U.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&f(u(uv+vu)+(uv+vu)u) = f(u)(uv+vu)+ud(uv+vu)\\
&+f(uv+vu)u+(uv+vu)d(u) = f(u)uv+f(u)vu+ud(uv)\\
(1.4)\qquad &+ud(vu)+f(u)vu+ud(v)u+f(v)u^2+vd(u)u\\
&+uvd(u)+vud(u) = f(u)uv+f(u)vu+ud(u)v+u^2d(v)\\
&+ud(v)u+uvd(u)+f(u)vu+ud(v)u+f(v)u^2+vd(u)u\\
&+uvd(u)+vud(u) \quad \text{for all } u,v\in U.
\end{aligned}
$$

Comparing (1.3) and (1.4), using $charR\neq 2$, we get the required result.

iii) Linearizing (ii) on u, we get

(1.5)

$$
\begin{aligned}
&f((u+w)v(u+w)) = f(uvu+uvw+wvu+wvw)\\
&= f(uvu)+f(uvw+wvu)+f(wvw) = f(u)vu+ud(v)u+uvd(u)\\
&+f(uvw+wvu)+f(w)vw+wd(v)w+wvd(w) \text{ for all } u,v,w\in U.
\end{aligned}
$$

Now compute $f((u+w)v(u+w))$ in other way, we get

(1.6)

$$
\begin{aligned}
&f((u+w)v(u+w)) = f(u+w)(vu+vw)+(u+w)d(v)(u+w)\\
&+(uv+wv)d(u+w) = f(u)vu+f(u)vw+f(w)vu+f(w)vw\\
&+ud(v)u+ud(v)w+wd(v)u+wd(v)w\\
&+uvd(u)+uvd(w)+wvd(u)+wvd(w) \text{ for all } u,v,w\in U.
\end{aligned}
$$

Comparing (1.5) and (1.6), we have

$$
\begin{aligned}
&f(uvw+wvu) = f(u)vw+ud(v)w+uvd(w)+f(w)vu\\
&+wd(v)u+wvd(u) \quad \text{for all } u,v,w\in U.
\end{aligned}
$$

□

REMARK 1. *We introduce abbreviation*

$u^v = f(uv) - f(u)v - ud(v)$ *for all* $u, v \in U$.
Observe also by Lemma 1 (i), we have
$f(uv + vu) = f(u)v + ud(v) + f(v)u + vd(u)$
and so,
$f(uv) - f(u)v - ud(v) = -(f(vu) - f(v)u - vd(u))$.
That is,

$$u^v = -v^u \text{ for all } u, v \in U. \tag{1.7}$$

LEMMA 2. *For all* $u, v \in U$, $u^v[u, v] = 0$.

Proof. Replace w by uv in Lemma 1 (iii) and using the fact that $char R \neq 2$, we get

$$\begin{aligned} f((uv)^2 + uv^2u) &= f(uv)uv + uvd(uv) \\ &+ f(u)v^2u + ud(v^2)u + uv^2d(u) \\ &= f(uv)uv + uvd(u)v + uvud(v) + f(u)v^2u \\ &+ ud(v)vu + uvd(v)u + uv^2d(u) \text{ for all } u, v \in U. \end{aligned} \tag{1.8}$$

On the other hand, we get

$$\begin{aligned} f(uv(uv) + (uv)vu) &= f(u)vuv + ud(v)uv + uvd(uv) \\ &+ f(uv)vu + uvd(v)u + uv^2d(u) \\ &= f(u)vuv + ud(v)uv + uvd(u)v + uvud(v) \\ &+ f(uv)vu + uvd(v)u + uv^2d(u) \text{ for all } u, v \in U. \end{aligned} \tag{1.9}$$

Comparing this two equations, we have

$$\begin{aligned} &f(uv)uv + f(u)v^2u + ud(v)vu \\ &= f(u)vuv + ud(v)uv + f(uv)vu. \end{aligned}$$

That is,

$$u^v[u, v] = 0 \text{ for all } u, v \in U.$$

□

THEOREM 1. *Let* R *be a non-commutative prime ring with characteristic not two,* U *a noncentral Lie ideal of* R *such that* $u^2 \in U$ *for all* $u \in U$. *If* f *be a generalized Jordan derivation on* U *then* f *is a generalized derivation on* U.

Proof. From Lemma 1 (iii), we have

$$f(uwv+vwu)=f(u)wv+ud(w)v+uwd(v)$$
$$+f(v)wu+vd(w)u+vwd(u) \text{ for all } u,v,w\in U. \tag{1.10}$$

Replacing u by uv and v by vu in (1.10), we get

(1.11)
$$\begin{aligned} f((uv)w(vu)+(vu)w(uv)) &= f(uv)wvu+uvd(w)vu+uvwd(vu) \\ &\quad +f(vu)wuv+vud(w)uv+vuwd(uv) \\ &= f(uv)wvu+uvd(w)vu+uvwd(v)u+uvwvd(u) \\ &\quad +f(vu)wuv+vud(w)uv+vuwd(u)v \\ &\quad +vuwud(v) \text{ for all } u,v,w\in U. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} f((uv)w(vu)+(vu)w(uv)) &= f(u(vwv)u)+f(v(uwu)v) \\ &= f(u)vwvu+ud(vwv)u+uvwvd(u) \\ &\quad +f(v)uwuv+vd(uwu)v+vuwud(v) \\ &= f(u)vwvu+ud(v)wvu+uvd(w)vu \\ &\quad +uvwd(v)u+uvwvd(u)+f(v)uwuv \\ &\quad +vd(u)wuv+vud(w)uv \\ &\quad +vuwd(u)v+vuwud(v) \text{ for all } u,v,w\in U. \end{aligned} \tag{1.12}$$

Comparing equations (1.11) and (1.12), we obtain

$$\{f(vu)-f(v)u-vd(u)\}wuv$$
$$+\{f(uv)-f(u)v-ud(v)\}wvu=0$$

and hence

$$v^u wuv+u^v wvu=0 \text{ for all } u,v,w\in U.$$

Using the $u^v=-v^u$, we get

$$u^v U[v,u]=0 \ \text{ for all } u,v\in U.$$

Since $U\not\subseteq Z$, by [3, Lemma 4] we obtain for each pair $u,v\in U$ either $u^v=0$ or $[u,v]=0$. Notice that the mappings $(u,v)\to u^v$ and $(u,v)\to[u,v]$ satisfy the requirements of the [5, Lemma 4]. Hence $u^v=0$ for all $u,v\in U$ or $[u,v]^2=0$ for all $u,v\in U$. If $[u,v]^2=0$ for all $u, v \in U$ then for each $u \in U$, $I_u(v)^2 = 0$, for all $v \in U$, where I_u is the inner derivation. Hence we get $I_u(U) = 0$ by [3, Theorem 1]. This yields that $U \subset Z$, a contradiction. Thus, we have $u^v = 0$ for all $u, v \in U$. This complates the proof. $\square$

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