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BILINEAR SYSTEMS CONTROLLER DESIGN WITH APPROXIMATION TECHNIQUES

Sang-Hyuk Lee* and Keonhee Lee**

Abstract. Using the iterative method, we derive an controller realization of the bilinear system, which is resulted from the system reformulation. We utilize Banach Fixed Point Theorem to support proposed controller, and the simulation results are also illustrated to verify usefulness of this technique.

1. Introduction

To consider the optimal controller structure for the bilinear system with quadratic cost, iterative design is stated and the canonical equations of the problem are given with the presentation closely related to the Riccati approach in linear quadratic optimization. Using the properties of the Banach fixed point theorem, we prove the convergence of the iteration procedure. Bilinear systems are special class of nonlinear systems possessing many of the properties of linear systems. Hence, there is a need for theoretical results sufficiently close to those already applied for linear systems. The design problem of controllers for bilinear system has been studied by numerous authors ([2], [3]). Most of the obtained results rely on optimization theory, either using quadratic cost or criteria linear in control, specially through the application of Pontryagin's principle leading to bang-bang controls or minimizing controls or minimizing control time. Obtaining the optimal solution was not easy because of the nonlinearity in bilinear

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system. For the purpose of obtaining the optimal solution for bilinear systems, it is required to use closely related Riccati approach in linear quadratic optimization.

2. Riccati approach for the solution of linear system

In this section, we introduce the useful definitions and theorem to guarantee convergence of variables. Furthermore simple controller realization procedure for linear systems has been illustrated, and we also formulate and provide iterative procedure of the time varying system.

2.1. Statement of the problem.

We consider dynamical systems generated by ordinary differential equations in $\mathbf{R}^p$. Next, $x(t) \in\mathbf{R}^p$ denotes a vector valued function of $t \in\mathbf{R}$.

Consider the following problem

$$\dot{x}(t) = f(t), \quad x(0) = x_0 \in \mathbf{R}^p,$$

where $f \in C(\mathbf{R}^p, \mathbf{R}^p)$.

DEFINITION 2.1. (Invariant Set) A subset D of a Banach space $(B, ||\cdot||)$ is invariant under operator T if $T(D) \subseteq D$ is satisfied.

DEFINITION 2.2. (Contraction Mapping Theorem) Suppose that $F : D \to D$ where D is a closed subset of a Banach space $(B, ||\cdot||)$ and that $F(\cdot)$ is a contraction on D with constant μ,

$$\exists \mu < 1, \quad ||F(x) - F(y)|| \leq \mu ||x - y||, \quad \forall x, y \in D.$$

Then there exactly one point $x^* \in D$ such that $F(x^*) = x^*$. The fixed point x^* denotes the Banach fixed point.

Control solution to the linear systems

Now, we briefly introduce the one of controller design procedures for linear systems. Consider the finite dimensional linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t),$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}$ and $y(t) \in \mathbf{R}$ are state, input and output vectors, respectively, and the matrices $A \in \mathbf{R}^{n\times m}$, $B \in \mathbf{R}^{n\times 1}$ and $C \in \mathbf{R}^{1\times m}$ are satisfied. For the final time t^f, cost function is

$$\begin{aligned} J = &\frac{1}{2}(Cx(t^f) - r(t^f))^T\overline{P}(Cx(t^f) - r(t^f)) \qquad (1) \\ &+ \frac{1}{2}\int_0^{t^f} \{(Cx(t) - r(t))^T\overline{Q}(Cx(t) - r(t)) + u(t)^T\overline{R}u(t)\}dt, \end{aligned}$$

where $\overline{P} = \overline{P}^T \geq 0$, $\overline{Q} = \overline{Q}^T \geq 0$ and $\overline{R} > 0$.

With the Hamiltonian of the problem

$$\begin{aligned} H(x,u,p) = &\frac{1}{2}\{(Cx(t) - r(t))^T\overline{Q}(Cx(t) - r(t)) + u(t)^T\overline{R}u(t)\} \\ &+ p(t)^T\{Ax(t) + Bu(t)\}, \qquad (2) \end{aligned}$$

the optimal control $u^*(t) = -\overline{P}^{-1}B^Tp(t)$ is determined by $\frac{\partial H}{\partial u} = 0$ ([6]). And the costate equation is represented by $\frac{\partial H}{\partial x} = -\dot{p}$. For the tracking problem, costate $p(t) = Sx(t) - v(t)$ is obtained from the following relations

$$\begin{aligned} 0 &= A^TS + SA - SB\overline{R}^{-1}B^TS + \overline{Q}, \\ -\dot{v}(t) &= (A - B\overline{R}^{-1}B^TS)^Tv(t) + C^T\overline{Q}r(t), \quad v(t^f) = C^T\overline{P}r(t^f). \end{aligned}$$

S is obtained off-line, and $v(t)$ is also precomputed and stored if the reference track $r(t)$ is known a priori.

Control solution to the bilinear systems

Consider a finite dimensional bilinear systems

$$\dot{x}(t) = Ax(t) + Bu(t) + < x(t)N > u(t),$$
$$y(t) = Cx(t),$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}$ and $y(t) \in \mathbf{R}^r$ are state, input, output vectors, respectively; $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times 1}$, $C \in \mathbf{R}^{1\times n}$, $< x(t)N > = \sum_{j=1}^{n} x_j(t)N_j$, $N_j \in \mathbf{R}^{n\times 1}$, $j = 1, \cdots, n$.

At this time, we hold same cost function (1) to bilinear system, then the Hamiltonian of the problem has the following structure

$$\begin{aligned} H(x,u,p) = &\frac{1}{2}\{(Cx(t) - r(t))^T\overline{Q}(Cx(t) - r(t)) + u(t)^T\overline{R}u(t)\} \\ &+ p(t)^T\{Ax(t) + < x(t)N > u(t) + Bu(t)\}. \end{aligned} \tag{3}$$

By the same procedure, optimal control u^*

$$u^*(t) = -\overline{R}^{-1}(B + < x(t)N >)^T p(t)$$

is obtained from the necessary optimality condition $\frac{\partial H}{\partial u} = 0$. Furthermore costate variable $p(t)$ satisfies $\frac{\partial H}{\partial x} = -\dot{p}$. Then, we can rewrite state and costate equations as follows

$$\dot{x} = \tilde{A}x(t) - \tilde{B}\overline{R}^{-1}\tilde{B}^T p(t), \quad x(0) = x^0, \tag{4}$$

$$\dot{p} = -\tilde{Q}x(t) - \tilde{A}^T p(t) + C^T Q r(t), \tag{5}$$
$$p(t^f) = C^T\overline{P}Cx(t^f) - C^T\overline{P}r(t^f).$$

The time-varying matrices $\tilde{A}$, $\tilde{Q}$ and $\tilde{B}\overline{R}^{-1}\tilde{B}^T$ denote

$$\tilde{A} = [\tilde{a_{ij}}] = a_{ij} - \frac{1}{2}[(N_j\overline{R}^{-1}B^T + B\overline{R}^{-1}N_j^T)p(t)]_i,$$

$\tilde{a_{ij}}$ is an i-th row and j-th column element of matrix $\tilde{A}$,

$$\tilde{Q} = [\tilde{q_{ij}}] = [C^T\overline{Q}C]_{ij} - \frac{1}{2}p^T(t)(N_i\overline{R}^{-1}N_j^T + N_j\overline{R}^{-1}N_i^T)p(t),$$

$\tilde{q_{ij}}$ is an i-th row and j-th column element of matrix $\tilde{Q}$,

$$\begin{aligned}\tilde{B}\overline{R}^{-1}\tilde{B}^T &= (B+ < xN >)\overline{R}^{-1}(B+ < xN >)^T \\ &\quad - \frac{1}{2}(< xN > \overline{R}^{-1}B^T + B\overline{R}^{-1} < xN >^T).\end{aligned}$$

The above problems are called two-point boundary-value problems, and they are sometimes rather difficult to solve, even with a high speed computer. Notice that the difference equations of state and costate equations are coupled and time varying, optimal control scheme is different from linear quadratic controller. This version of control law cannot be implemented in practice, since the boundary conditions are split between times $t = 0$ and $t = t^f$. Let us find a more useful version. The optimal control is linear costate feedback, but unfortunately, because of the forcing term in the costate equation and boundary condition $p(t^f)$, it is no longer possible to express it as a linear state feedback as we did for the linear quadratic optimal controller. However, we can express $u(t)$ as a combination of a linear state variable feedback plus a term depending on $r(t)$. From the looks of boundary condition $p(t^f)$, it seems reasonable to assume that for all $t \leq t^f$, we can write

$$p(t) = S(t)x(t) - v(t)$$

for some as yet unknown auxiliary sequences $S(t)$ and $v(t)$. Note that $S(t)$ is an $n \times n$ matrix, whereas $v(t)$ is an n vector.

State equation becomes

$$\dot{x}(t) = \tilde{A}x(t) - \tilde{B}\overline{R}^{-1}\tilde{B}^T(S(t)x(t) - v(t)).$$

Differentiate costate definition to find the intermediate function $S(t)$

$$\dot{p}(t) = \dot{S}(t)x(t) + S(t)\dot{x(t)} - \dot{v}(t). \tag{6}$$

Now, taking into account the costate equation (5), we have obtained following results by equating Eqs. (5) and (6)

$$\begin{aligned} -\dot{S}(t) &= S(t)\tilde{A} + \tilde{A}^T S(t) - S(t)\tilde{B}\overline{R}^{-1}\tilde{B}^T S(t) + \tilde{Q}, \\ -\dot{v}(t) &= \tilde{A}^T v(t) - S(t)\tilde{B}\overline{R}^{-1}\tilde{B}^T v(t) + C^T\overline{Q}r(t). \end{aligned}$$

Since the matrix sequence $S(t)$ is independent of the state trajectory, so the Riccati equation can be solved off-line, and $S(t)$ can be stored. If the reference track $r(t)$ is known a priori, the auxiliary function $v(t)$ can also be precomputed and stored. However, that $v(t)$ has been determined by integrating backward the closed-loop adjoint system with $v(t^f) = C^T\overline{P}r(t^f)$.

Then, the optimal tracking control becomes

$$u(t) = -\overline{R}^{-1}(B + < x(t)N >)^T S(t)x(t) + \overline{R}^{-1}(B + < x(t)N >)^T v(t).$$

2.2. Iterative procedure.

In the previous subsection, we note that $\tilde{A}(t)$, $\tilde{Q}(t)$ and $\tilde{B}\overline{R}^{-1}\tilde{B}^T(t)$ are functions of the costate $p(t)$ and state $x(t)$. And superscript (j) denotes iteration index $j = 0, 1, \cdots$. For the brevity of notation, iteration sequences of $\tilde{A}(p^{(j)}(t))$, $\tilde{Q}(p^{(j)}(t))$ and $\tilde{B}(x^{(j)}(t))\overline{R}^{-1}\tilde{B}^T(x^{(j)}(t))$ are simplified by

$$\begin{aligned} \tilde{A}^{(j)}(t) &= \tilde{A}(p^{(j)}(t)) = A^{(j)}, \\ \tilde{Q}^{(j)}(t) &= \tilde{Q}(p^{(j)}(t)) \end{aligned}$$

and $\tilde{B}^{(j)}(x^{(j)})\overline{R}^{-1}\tilde{B}^{T(j)}(t) = \tilde{B}(x^{(j)})\overline{R}^{-1}\tilde{B}^T(x^{(j)}(t)) = B^{(j)}$.

With the definition of $p(t) = S(t)x(t) - v(t)$, the iterative solutions of the state and costate equations are obtained from the following equations

$$\begin{aligned} -\dot{S}^{(j+1)}(t) &= A^{(j)T}S^{(j+1)}(t) + S^{(j+1)}(t)A^{(j)} \\ &\quad - S^{(j+1)}(t)B^{(j)}S^{(j+1)}(t) + \tilde{Q}^{(j)}, \\ -\dot{v}^{(j+1)}(t) &= (A^{(j)} - B^{(j)}S^{(j+1)}(t))^T v^{(j+1)}(t) + C^T\tilde{Q}^{(j)}r(t), \\ \dot{x}^{(j+1)}(t) &= A^{(j)}x^{(j+1)}(t) - B^{(j)}S^{(j+1)}x^{(j+1)}(t) + B^{(j)}v^{(j+1)}(t) \end{aligned}$$

and

$$\dot{p}^{(j+1)}(t) = -\tilde{Q}^{(j)}x^{(j+1)}(t) - A^{(j)T}p^{(j+1)}(t) + C^T\tilde{Q}r(t),$$

where the boundary conditions denote $S^{(j+1)}(t^f) = C^T\overline{P}C$, $v^{(j+1)}(t^f) = C^T\overline{P}r(t^f)$ and $x^{(0)}(0) = x^0$.

For each iteration, the feedback controller is given by

$$\begin{aligned} &u^{(j+1)}(t) \\ &= -\overline{R}^{-1}(B+ < x^{(j+1)}(t)N >)^T(S^{(j+1)}(t)x^{(j+1)}(t) - v^{(j+1)}(t)), \end{aligned}$$

where the matrix $S^{(j+1)}(t)$ is calculated from the Riccati equation. Hence we confirm that in the bilinear quadratic case the Riccati matrix depends on the initial state x^0 in contrast to the linear quadratic problem. This point seems to be natural because of the nonlinearity of the bilinear system.

3. Analysis on the proof of convergence

In this section, we show the convergence of iterative sequences $x^{(j)}$, $v^{(j)}$ and $S^{(j)}$. Next, we define some definitions ([7]).

DEFINITION 3.1. Let $B_1 = C([0, t^f], R^n)$, $B_2 = C([0, t^f], R^{n\times n})$ be the two Banach spaces with the norms

$$||x||_\epsilon = \sup_{t\in[0,t^f]}[||x(t)||e^{-\epsilon t}], \quad x \in B_1,$$
$$||v||_\epsilon = \sup_{t\in[0,t^f]}[||v(t)||e^{-\epsilon(t^f-t)}], \quad v \in B_1,$$
$$||S||_\epsilon = \sup_{t\in[0,t^f]}[||S(t)||e^{-\epsilon(t^f-t)}], \quad S \in B_2,$$

where $||x|| = [\Sigma_{i=1}^n x_i^2]^{1/2}$, $||v|| = [\Sigma_{i=1}^n v_i^2]^{1/2}$ and $||S|| = [\Sigma_{i,j=1}^n S_{ij}^2]^{1/2}$.

In Definition 2.1, B_1 and B_2 are the spaces of all real-valued, continuous, smooth n-vector functions and $n \times n$ matrix defined on $[0, t^f]$, respectively.

DEFINITION 3.2. The three operators T_1, T_2 and T_2 are defined as follows

$$T_1[x, v, S] = \bar{x}, \quad \bar{x} \in B_1,$$
$$T_2[x, v, S] = \bar{v}, \quad \bar{v} \in B_1,$$
$$T_3[x, v, S] = \bar{S}, \quad \bar{S} \in B_2.$$

$B_1 \times B_1 \times B_2$ denotes the ordered triplets (x, v, S), where $x \in B_1$, $v \in B_1$ and $S \in B_2$.

DEFINITION 3.3. Subsets $D_1 \subseteq B_1$, $D_2 \subseteq B_1$ and $D_3 \subseteq B_2$ are invariant under T_1, T_2 and T_3 if $T_1[D_1, D_2, D_3] \subseteq D_1$, $T_2[D_1, D_2, D_3] \subseteq D_2$ and $T_3[D_1, D_2, D_3] \subseteq D_3$ are satisfied.

DEFINITION 3.4. Operators T_1, T_2 and T_3 are called contractive in the space $D_1 \times D_2 \times D_3$ if a 3×3 matrix M exists with all eigenvalues in the unit circle such that, for all $x^{(j)}, x^{(j-1)} \in D_1$, $v^{(j)}, v^{(j-1)} \in D_2$

and $S^{(j)}, S^{(j-1)} \in D_3$, the following inequality

$$\begin{pmatrix} ||T_1[x^{(j)}, v^{(j)}, S^{(j)}] - T_1[x^{(j-1)}, v^{(j-1)}, S^{(j-1)}]||_\epsilon \\ ||T_2[x^{(j)}, v^{(j)}, S^{(j)}] - T_2[x^{(j-1)}, v^{(j-1)}, S^{(j-1)}]||_\epsilon \\ ||T_3[x^{(j)}, v^{(j)}, S^{(j)}] - T_3[x^{(j-1)}, v^{(j-1)}, S^{(j-1)}]||_\epsilon \end{pmatrix} \leq M \begin{pmatrix} ||x^{(j)} - x^{(j-1)}||_\epsilon \\ ||v^{(j)} - v^{(j-1)}||_\epsilon \\ ||S^{(j)} - S^{(j-1)}||_\epsilon \end{pmatrix}$$

holds componentwise.

THEOREM 3.1. *Assuming that the operators T_1, T_2 and T_3 have the invariant sets D_1, D_2 and D_3 and the property of contraction in $D_1 \times D_2 \times D_3$. Then the iteration procedures*

$$x^{(j+1)} = T_1[x^{(j)}, v^{(j)}, S^{(j)}],$$
$$v^{(j+1)} = T_2[x^{(j)}, v^{(j)}, S^{(j)}],$$
$$S^{(j+1)} = T_3[x^{(j)}, v^{(j)}, S^{(j)}],$$

with $x^{(0)} \in D_1$, $v^{(0)} \in D_2$ and $S^{(0)} \in D_3$ converge to the unique fixed points x^*, v^* and S^*. The equivalent mathematical formulation of the theorem is

$$\lim_{j\to\infty} ||x^{(j)} - x^*|| = 0, \quad \lim_{j\to\infty} ||v^{(j)} - v^*|| = 0, \quad \lim_{j\to\infty} ||S^{(j)} - S^*|| = 0.$$

Proof. D_1, D_2 and D_3 denote the invariant sets since it is assumed at first, hence iterative sequences are satisfied by $x^{(j)} \in D_1$, $v^{(j)} \in D_2$ and $S^{(j)} \in D_3$ with initial conditions $x^{(0)} \in D_1$, $v^{(0)} \in D_2$ and $S^{(0)} \in D_3$. By Definition 3.2, the following expression

$$\begin{pmatrix} ||x^{(p)} - x^{(q)}||_\epsilon \\ ||v^{(p)} - v^{(q)}||_\epsilon \\ ||S^{(p)} - S^{(q)}||_\epsilon \end{pmatrix} = \begin{pmatrix} ||T_1[x^{(p-1)}, v^{(p-1)}, S^{(p-1)}] - T_1[x^{(q-1)}, v^{(q-1)}, S^{(q-1)}]||_\epsilon \\ ||T_2[x^{(p-1)}, v^{(p-1)}, S^{(p-1)}] - T_2[x^{(q-1)}, v^{(q-1)}, S^{(q-1)}]||_\epsilon \\ ||T_3[x^{(p-1)}, v^{(p-1)}, S^{(p-1)}] - T_3[x^{(q-1)}, v^{(q-1)}, S^{(q-1)}]||_\epsilon \end{pmatrix}$$

is derived. Assuming $p > q$, then we can derive

$$\begin{pmatrix} ||x^{(p)} - x^{(q)}||_\epsilon \\ ||v^{(p)} - v^{(q)}||_\epsilon \\ ||S^{(p)} - S^{(q)}||_\epsilon \end{pmatrix}$$
$$= \begin{pmatrix} ||x^{(p)} - x^{(p-1)} + x^{(p-1)} - x^{(p-2)} + x^{(p-2)} + \cdots + x^{(q+1)} - x^{(q)}||_\epsilon \\ ||v^{(p)} - v^{(p-1)} + v^{(p-1)} - v^{(p-2)} + v^{(p-2)} + \cdots + v^{(q+1)} - v^{(q)}||_\epsilon \\ ||S^{(p)} - S^{(p-1)} + S^{(p-1)} - S^{(p-2)} + S^{(p-2)} + \cdots + S^{(q+1)} - S^{(q)}||_\epsilon \end{pmatrix}$$
$$\leq \sum_{j=q+1}^{p} \begin{pmatrix} ||x^{(j)} - x^{(j-1)}||_\epsilon \\ ||v^{(j)} - v^{(j-1)}||_\epsilon \\ ||S^{(j)} - S^{(j-1)}||_\epsilon \end{pmatrix}.$$

From Definition 3.4, we obtain the following inequality

$$\begin{pmatrix} ||x^{(p)} - x^{(q)}||_\epsilon \\ ||v^{(p)} - v^{(q)}||_\epsilon \\ ||S^{(p)} - S^{(q)}||_\epsilon \end{pmatrix} \leq \sum_{j=q+1}^{p} M^{j-1} \begin{pmatrix} ||x^{(1)} - x^{(0)}||_\epsilon \\ ||v^{(1)} - v^{(0)}||_\epsilon \\ ||S^{(1)} - S^{(0)}||_\epsilon \end{pmatrix},$$

and with the relation

$$(M^q + M^{q+1} + \cdots + M^{p-1})(I - M) = M^q - M^p,$$

the following inequality

$$\begin{pmatrix} ||x^{(p)} - x^{(q)}||_\epsilon \\ ||v^{(p)} - v^{(q)}||_\epsilon \\ ||S^{(p)} - S^{(q)}||_\epsilon \end{pmatrix}$$
$$\leq M^q(I - M^{p-q})(I - M)^{-1} \begin{pmatrix} ||x^{(1)} - x^{(0)}||_\epsilon \\ ||v^{(1)} - v^{(0)}||_\epsilon \\ ||S^{(1)} - S^{(0)}||_\epsilon \end{pmatrix} \tag{7}$$

is carried out.

By the contractive property of the operator T_1, T_2 and T_3, all eigenvalues of the matrix M are in the unit circle. If $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the eigenvalues of M, then $\lambda_1^k, \lambda_2^k, \cdots, \lambda_n^k$ are the eigenvalues of M^k. We can check the fact that $\rho(M)^k = \rho(M^k)$ ([4]), where $\rho(M)$ is the eigenvalue of M. Hence the eigenvalues of $M^q + M^{q+1} + \cdots + M^{p-1}$ can be made arbitrarily small by choosing q sufficiently large. Thus, for every $\epsilon_1 > 0$, $\epsilon_2 > 0$ and $\epsilon_3 > 0$, there is an integer N such that

$$||x^{(p)} - x^{(q)}||_\epsilon \leq \epsilon_1, \quad ||v^{(p)} - v^{(q)}||_\epsilon \leq \epsilon_2, \quad ||S^{(p)} - S^{(q)}||_\epsilon \leq \epsilon_3$$

if $p \geq N$ and $q \geq N$ hold. Then by (7), $||x^{(p)} - x^{(q)}||_\epsilon \to 0$, $||v^{(p)} - v^{(q)}||_\epsilon \to 0$ and $||S^{(p)} - S^{(q)}||_\epsilon \to 0$ as $p, q \to \infty$. Hence the sequences x, v and S are Cauchy sequence, and the sequences x, v and S converge. We now assert that x^*, v^* and S^* are fixed points of T_1, T_2 and T_3. Since T_1, T_2 and T_3 are continuous, we know

$$\lim_{q\to\infty} T_1[x^{(q)}, v^{(q)}, S^{(q)}] = x^*,$$
$$\lim_{q\to\infty} T_2[x^{(q)}, v^{(q)}, S^{(q)}] = v^*,$$
$$\lim_{q\to\infty} T_3[x^{(q)}, v^{(q)}, S^{(q)}] = S^*,$$

as desired. □

If iterative sequences $x^{(j)}$, $v^{(j)}$ and $S^{(j)}$ converge to the unique fixed point x^*, v^* and S^*, then the convergence of input magnitude $|u|$ is derived with the iteration procedure. In order to apply Theorem 3.1, the existence of invariant subsets D_1, D_2 and D_3 the contractiveness of the operators T_1, T_2 and T_3 in $D_1 \times D_2 \times D_3$ have to be required.

3.1. Approximation techniques.

When Theorem 3.1 is applied to the proof of convergence of iterative scheme, we give another approach that the differential equations are replaced by

$$\dot{x}^{(j+1)}(t) := \overline{A}^{(j)} x^{(j+1)}(t) + B^{(j)} v^{(j+1)}(t),$$
$$\dot{v}^{(j+1)}(t) := \overline{A}^{(j)T} v^{(j+1)}(t) + C^T \overline{Q} r(t),$$
$$\dot{P}^{(j+1)}(t) + \overline{A}^{T(j)} P^{(j+1)}(t) + P^{(j+1)}(t) \overline{A}^{(j)} + \overline{Q}^{(j)} = 0,$$

where $\overline{A}^{(j)} = A^{(j)} - B^{(j)}P^{(j)}$ and $\overline{Q}^{(j)} = \tilde{Q}^{(j)} + P^{(j)}B^{(j)}P^{(j)}$ ([1]).

Then the norm of differences $P^{(j+1)}(t) - P^{(j)}(t)$, $v^{(j+1)}(t) - v^{(j)}(t)$ and $x^{(j+1)}(t) - x^{(j)}(t)$ are satisfied as follows

$$\begin{aligned} ||P^{(j+1)}(t) - P^{(j)}(t)||_\epsilon \le v_7||x^{(j)}(t) - x^{(j-1)}(t)||_\epsilon \\ + v_8||v^{(j)}(t) - v^{(j-1)}(t)||_\epsilon + v_9||P^{(j)}(t) - P^{(j-1)}(t)||_\epsilon, \\ ||v^{(j+1)}(t) - v^{(j)}(t)||_\epsilon \le v_4||x^{(j)}(t) - x^{(j-1)}(t)||_\epsilon \\ + v_5||v^{(j)}(t) - v^{(j-1)}(t)||_\epsilon + v_6||P^{(j)}(t) - P^{(j-1)}(t)||_\epsilon, \\ ||x^{(j+1)}(t) - x^{(j)}(t)||_\epsilon \le v_1||x^{(j)}(t) - x^{(j-1)}(t)||_\epsilon \\ + v_2||v^{(j)}(t) - v^{(j-1)}(t)||_\epsilon + v_3||P^{(j)}(t) - P^{(j-1)}(t)||_\epsilon, \end{aligned}$$

where the values of $v_1, v_2, \cdots, v_9$ are determined by

$$\begin{aligned} v_1 &= (N_1 + \beta_4 v_4 + \beta_3 v_7)|t^f|e^{\epsilon(t^f - t)}, \\ v_2 &= (N_2 + \beta_4 v_5 + \beta_3 v_8)|t^f|e^{\epsilon(t^f - t)}, \\ v_3 &= (N_3 + \beta_4 v_6 + \beta_3 v_9)|t^f|e^{\epsilon(t^f - t)}, v_4 = (N_4 + \delta_3 v_7)|t^f|e^{\epsilon t}, \\ v_5 &= (N_5 + \delta_3 v_8)|t^f|e^{\epsilon t}, v_6 = (N_6 + \delta_3 v_9)|t^f|e^{\epsilon t}, \\ v_7 &= N_7|t^f|e^{\epsilon t}, v_8 = N_8|t^f|e^{\epsilon t}, v_9 = N_9|t^f|e^{\epsilon t}. \end{aligned}$$

The above values of $v_1, v_2, \cdots, v_9$ can be founded in [7].

In the above equations, we show that $||x^{(j+1)} - x^{(j)}||_\epsilon$, $||v^{(j+1)} - v^{(j)}||_\epsilon$ and $||P^{(j+1)} - P^{(j)}||_\epsilon$ are bounded with the combination of the norms of $||x^{(j)} - x^{(j-1)}||_\epsilon$, $||v^{(j)} - v^{(j-1)}||_\epsilon$ and $||P^{(j)} - P^{(j-1)}||_\epsilon$.

Arranging the above relations,

$$\begin{aligned} &||T_1[x^{(j)}, v^{(j)}, P^{(j)}] - T_1[x^{(j-1)}, v^{(j-1)}, P^{(j-1)}]||_\epsilon \\ &\quad \le v_1||x^{(j)} - x^{(j-1)}||_\epsilon + v_2||v^{(j)} - v^{(j-1)}||_\epsilon + v_3||P^{(j)} - P^{(j-1)}||_\epsilon, \end{aligned}$$

$$\begin{aligned} &||T_2[x^{(j)}, v^{(j)}, P^{(j)}] - T_2[x^{(j-1)}, v^{(j-1)}, P^{(j-1)}]||_\epsilon \\ &\quad \le v_4||x^{(j)} - x^{(j-1)}||_\epsilon + v_5||v^{(j)} - v^{(j-1)}||_\epsilon + v_6||P^{(j)} - P^{(j-1)}||_\epsilon, \end{aligned}$$

$$||T_3[x^{(j)}, v^{(j)}, P^{(j)}] - T_3[x^{(j-1)}, v^{(j-1)}, P^{(j-1)}]||_\epsilon$$
$$\leq v_7||x^{(j)} - x^{(j-1)}||_\epsilon + v_8||v^{(j)} - v^{(j-1)}||_\epsilon + v_9||P^{(j)} - P^{(j-1)}||_\epsilon,$$

the equivalent matrix notation is satisfied as follows

$$\begin{pmatrix} ||T_1[x^{(j)}, v^{(j)}, P^{(j)}] - T_1[x^{(j-1)}, v^{(j-1)}, P^{(j-1)}]||_\epsilon \\ ||T_2[x^{(j)}, v^{(j)}, P^{(j)}] - T_2[x^{(j-1)}, v^{(j-1)}, P^{(j-1)}]||_\epsilon \\ ||T_3[x^{(j)}, v^{(j)}, P^{(j)}] - T_3[x^{(j-1)}, v^{(j-1)}, P^{(j-1)}]||_\epsilon \end{pmatrix}$$
$$\leq M \begin{pmatrix} ||x^{(j)} - x^{(j-1)}||_\epsilon \\ ||v^{(j)} - v^{(j-1)}||_\epsilon \\ ||P^{(j)} - P^{(j-1)}||_\epsilon \end{pmatrix},$$

where $M = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_4 & v_5 & v_6 \\ v_7 & v_8 & v_9 \end{pmatrix}$.

As mentioned above, values of $v_1, v_2, \cdots, v_9$ contain the inverse of the design matrix $\overline{R}$ as a multiplicative element and $x^{(j)}$, $x^{(j-1)}$, $v^{(j)}$, $v^{(j-1)}$, $P^{(j)}$ and $P^{(j-1)}$. Hence we can make three operators T_1, T_2 and T_3 are contractive, namely, elements of matrix M can be made arbitrary small by choosing $\overline{R}$ large enough.

3.2. Invariant set.

Now we construct the invariant sets D_1, D_2 and D_3, in which the operators T_1, T_2 and T_3 are contractive. The invariant sets D_1, D_2 and D_3 are constructed by

$$D_1 = \{x \in B_1, ||x - x^{(0)}||_\epsilon \leq l_1\}, \quad l_1 > 0,$$
$$D_2 = \{v \in B_1, ||v - v^{(0)}||_\epsilon \leq l_2\}, \quad l_2 > 0,$$
$$D_3 = \{P \in B_2, ||P - P^{(0)}||_\epsilon \leq l_3\}, \quad l_3 > 0.$$

Then values of l_1, l_2 and l_3 can be solved as follows. For the natural number N,

$$l_1 = \sup_{p \in N}(\Pi_{i=q+1}^{p}\alpha_i + \Pi_{i=q+1}^{p-1}\alpha_i + \cdots + \alpha_{q+2}\alpha_{q+1} + \alpha_{q+1})||x^{(1)} - x^{(0)}||_\epsilon,$$
$$l_2 = \sup_{p \in N}(\Pi_{i=q+1}^{p}\beta_i + \Pi_{i=q+1}^{p-1}\beta_i + \cdots + \beta_{q+2}\beta_{q+1} + \beta_{q+1})||v^{(1)} - v^{(0)}||_\epsilon,$$
$$l_3 = \sup_{p \in N}(\Pi_{i=q+1}^{p}\gamma_i + \Pi_{i=q+1}^{p-1}\gamma_i + \cdots + \gamma_{q+2}\gamma_{q+1} + \gamma_{q+1})||S^{(1)} - S^{(0)}||_\epsilon,$$

where $0 \le \alpha_i, \beta_i, \gamma_i < 1$.

Proof. See also [5]. □

4. Computer simulation

Consider Continuous-Stirred Tank Reactor (CSTR) model, which has the following equation

$$\dot{x}(t) = Ax(t) + Bu(t) + < x(t)N > u(t),$$
$$y(t) = Cx(t),$$

where

$$A = \begin{pmatrix} 13/6 & 5/12 \\ -50/3 & -8/3 \end{pmatrix}, B = \begin{pmatrix} -1/8 \\ 0 \end{pmatrix}, N_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, N_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Cost function is considered by

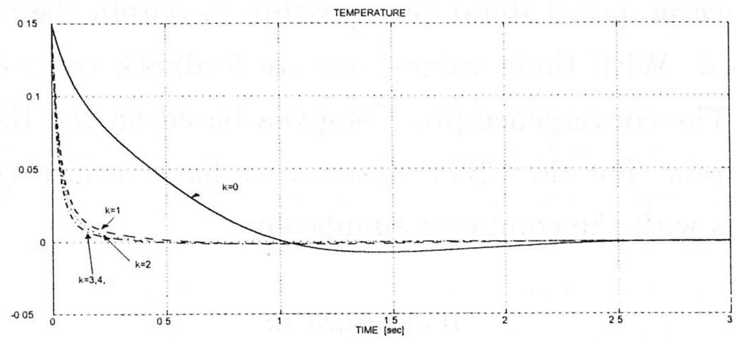
$$J = \frac{1}{2}(Cx(t^f) - r(t^f))^T \overline{P}(Cx(t^f) - r(t^f)) + \frac{1}{2}\int_0^{t^f} \{(Cx(t) - r(t))^T \overline{Q}(Cx(t) - r(t)) + u(t)^T \overline{R} u(t)\} dt,$$

where the weighting matrices are chosen by $\overline{P} = \begin{pmatrix} 1000 & 0 \\ 0 & 1000 \end{pmatrix}$, $\overline{Q} = \begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}$ and $\overline{R} = 1$.

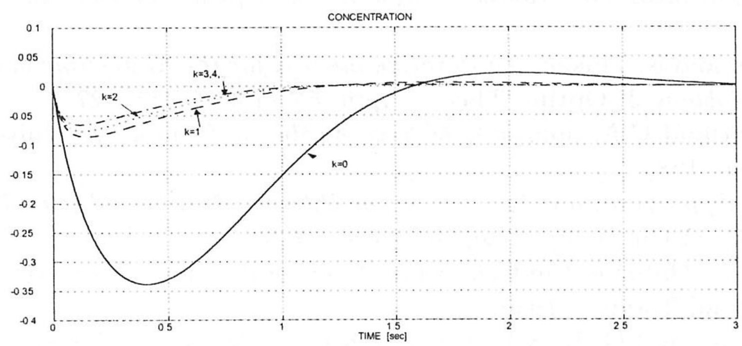
Final time t^f is 3(s), sampling time is 0.001(s), initial value is $x_0 \in ((\,0.15 \quad 0\,))^T$. Figure 1 represents the CSTR temperature and concentration. k represents the iteration number, hence we can find that the response converges as iteration increases. In Figure 2, control variable is illustrated. It can be noticed that as iteration processed, control profile is also converged.

5. Conclusions

We have performed the optimal controller structure for the bilinear system with quadratic cost. We have also generalized the iterative



(a) Temperature



(b) Concentration

Fig. 1 Profiles of temperature, concentration

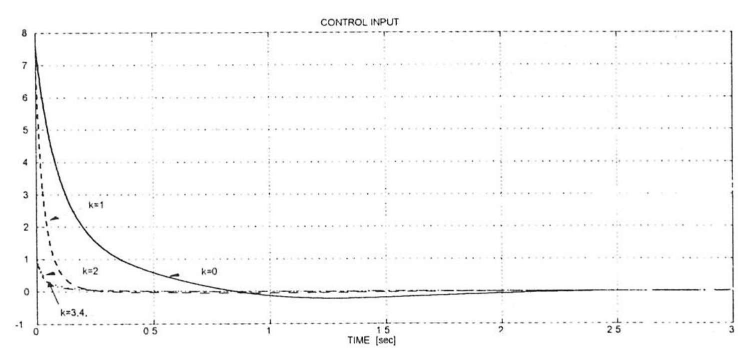


Fig. 2 Profiles of control input

methods for the tracking case. Bilinear system variables are redefined, and we have proposed iterative algorithm to obtain the convergence of variables. With these values, a state feedback controller is constructed. The convergence proof is given based on the Banach fixed point theorem. For the CSTR system, we have verified convergence of variables with the computer simulation.

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Department of Electrical Engineering
Pusan National University
Pusan 609–735, Korea

E-mail: leehyuk@pusan.ac.kr

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Department of Mathematics
Chungnam National University
Daejeon 305–764, Korea

E-mail: khlee@math.cnu.ac.kr