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A CONTINUED FRACTION FROM UNORGANIZED PORTIONS OF RAMANUJAN'S NOTEBOOKS AND PARTITIONS

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Abstract. In this paper, we consider the finite form of the continued fraction found in the unorganized material, Entry 9 [2], and give the sum of n terms. We give the n^{th} convergent series in two ways-one by simple expansion and the other by using partition theory.

1. Introduction

In the second notebook of Ramanujan, there are organized material in the 21 chapters. Chapter 12 [3] is devoted entirely to continued fractions. Some continued fractions are also found in Chapter 16 [1]. After these 21 chapters there are 100 pages containing material which is unorganized and there are only 33 pages in the third notebook, containing material, which is also unorganized. Thus there are 133 pages, the contents of which are unorganized. Andrews, Berndt, Jacobson and Lamphere [2] have proved all the continued fractions found in these 133 pages.

In this paper, we consider the finite form of the continued fraction found in the unorganized material, Entry 9 [2],

$$(1) \qquad 1-\cfrac{xq}{(1+q)+\cfrac{xq^3}{(1+q^2)-\cfrac{xq^2}{(1+q^3)+.}}}=\frac{\sum_{n=0}^{\infty}\frac{q^{\frac{n^2+n}{2}}(-x)^n}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{q^{\frac{n^2+3n}{2}}(-x)^n}{(q^2;q^2)_n}}=\frac{P(x)}{Q(x)},$$

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and give the sum of n terms. We give the n^{th} convergent series in two ways-one by simple expansion and the other by using partition theory. We choose this continued fraction as Andrews et. al., say, the most interesting results are Entry 7 and Entry 9 involving modest generalizations of the Rogers-Ramanujan continued fraction. Andrews et. al. did not consider its finite form. They proved the result of Ramanujan by first proving two three term relations. We give a simple proof.

We shall use the following usual basic hypergeometric notations: For $|q| < 1$,

$(a)_0 = 1,$

$(a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $1 \leq n < \infty,$

$(a)_\infty = \Pi_{r=0}^{\infty}(1-aq^r),$

$\left[{n \atop r}\right] = \frac{(q)_n}{(q)_r(q)_{n-r}},$

$\sum_k p(k,n,r)x^k = \frac{(q)_{r+n}}{(q)_r(q)_n}.$

2. Expression for P_n and Q_n

Theorem 2.1. *For $n = 2m$, the n^{th} term $= \frac{xq^{3m}}{1+q^{2m}}$ and for $n = 2m-1$, the n^{th} term $= \frac{-xq^m}{1+q^{2m-1}}$,*

$$1 - \cfrac{xq}{(1+q) + \cfrac{xq^3}{(1+q^2) - \cfrac{xq^2}{(1+q^3) + \cfrac{xq^6}{(1+q^4)\cdots n terms}}}} = \frac{P_n(x)}{Q_n(x)},$$

where

$$(2) \qquad P_n(x) = \sum_{r=0}^{n+1}(-1)^r q^{\frac{r^2-r}{2}} x^r \sum_{s=0}^{n+1-r}(-q)^{n+1-s}\left[{r+s \atop r}\right]\left[{n-s \atop r-1}\right],$$

$$Q_n(x) \ = \ \sum_{r=0}^{n}(-1)^r q^{\frac{r^2+r}{2}} x^r \sum_{s=0}^{n-r}(-q)^{n-s}\left[{r+s \atop r}\right]\left[{n-s-1 \atop r-1}\right]. \tag{3}$$

Proof. Let

$$\begin{aligned} P(x,z) \ &= \ 1+\sum_{n=0}^{\infty} P_n(x) z^{n+1} \\ &= \ \sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{r^2+r}{2}}(xz)^r}{(z)_{r+1}(-zq)_r}, \qquad |xz|<1, \end{aligned} \tag{4}$$

$$Q(x,z) \ = \ \sum_{n=0}^{\infty} Q_n(x) z^n = \sum_{r=0}^{\infty} \frac{(-1)^r x^{\frac{r^2+3r}{2}}(xz)^r}{(z)_{r+1}(-zq)_r}.$$

Applying Abel's Lemma, we get

$$\begin{aligned} P(x) \ &= \ \lim_{n\to\infty} P_n = \lim_{z\to 1^-}(1-z)P(x,z) \\ &= \ \sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{r^2+r}{2}}(x)^r}{(q)_r(-q)_r}, \qquad |x|<1. \end{aligned}$$

Similarly,

$$Q(x) = \sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{r^2+3r}{2}}(x)^r}{(q)_r(-q)_r}, \qquad |x|<1.$$

So we get (1).

Now by (4),

$$\begin{aligned} P(x,z) \ &= \ \sum_{n=0}^{\infty} P_n(x) z^n = \sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{r^2+r}{2}}(xz)^r}{(z)_{r+1}(-zq)_r} \\ &= \ \sum_{r=0}^{\infty}(-1)^r q^{\frac{r^2+r}{2}} x^r z^r \sum_{s=0}^{\infty} z^s \left[{r+s \atop r}\right] \sum_{t=0}^{\infty}(-1)^t q^t z^t \left[{r-1+t \atop r-1}\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} P_n(x) &= \sum_{r+s+t=n+1} (-1)^r q^{\frac{r^2+r}{2}} x^r (-1)^t q^t \left[{r+s \atop r}\right]\left[{r-1+t \atop r-1}\right] \\ &= \sum_{r=0}^{n+1} (-1)^r q^{\frac{r^2+r}{2}} x^r \sum_{s=0}^{n+1-r} (-q)^{n+1-r-s} \left[{r+s \atop r}\right]\left[{n-s \atop r-1}\right], \end{aligned}$$

which proves (2).

Similarly,

$$\begin{aligned} Q(x,z) &= \sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{r^2+3r}{2}} (xz)^r}{(z)_{r+1}(-zq)_r} \\ &= \sum_{r=0}^{\infty} (-1)^r q^{\frac{r^2+3r}{2}} x^r z^r \sum_{s=0}^{\infty} z^s \left[{r+s \atop r}\right] \sum_{t=0}^{\infty} (-1)^t q^t z^t \left[{r-1+t \atop r-1}\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} Q_n(x) &= \sum_{r+s+t=n} (-1)^r q^{\frac{r^2+3r}{2}} x^r (-1)^t q^t \left[{r+s \atop r}\right]\left[{r-1+t \atop r-1}\right] \\ &= \sum_{r=0}^{n} (-1)^r q^{\frac{r^2+3r}{2}} x^r \sum_{s=0}^{n-r} (-q)^{n-r-s} \left[{r+s \atop r}\right]\left[{n-s-1 \atop r-1}\right], \end{aligned}$$

which proves (3). □

Now we shall now use partition theory to give P_n and Q_n.

We have

$$P(z) = 1 + \sum_{n=0}^{\infty} P_n(x) z^{n+1} = \sum_{r=0}^{\infty} \frac{(-1)^r q^{\frac{r^2+r}{2}} (xz)^r}{(z)_{r+1}(-zq)_r}.$$

Hence

$$\begin{aligned} P_n(x) &= \sum_{r=0}^{n+1} (-1)^r q^{\frac{r^2+r}{2}} x^r \sum_{\alpha_0+\alpha_1+\ldots+\alpha_r=s} 1^{\alpha_0} q^{\alpha_1} (q^2)^{\alpha_2} \ldots (q^r)^{\alpha_r} \\ &\qquad \times \sum_{\beta_1+\beta_2+\cdots+\beta_r=n+1-s-r} q^{\beta_1} (q^2)^{\beta_2} \cdots (q^r)^{\beta_r} \end{aligned}$$

$$= \sum_{r=0}^{n+1}(-1)^r q^{\frac{r^2+r}{2}} x^r \sum_{\alpha_1+\alpha_2+\cdots+\alpha_r\le s} q^{\alpha_1+2\alpha_2+\cdots+r\alpha_r}$$

$$\times \sum_{\beta_1+\cdots+\beta_r=n+1-s-r} q^{\beta_1+2\beta_2+\cdots+r\beta_r}$$

$$= \sum_{r=0}^{n+1}(-1)^r q^{\frac{r^2+r}{2}} x^r \sum_k p(k,r,s)x^k \sum_k p(k,r,n+1-r-s)x^k$$

$$= \sum_{r=0}^{n+1}(-1)^r q^{\frac{r^2+r}{2}} x^r \frac{(q)_{r+s}}{(q)_r(q)_s}\frac{(q)_{n+1-s}}{(q)_r(q)_{n+1-r-s}}.$$

Similarly,

$$Q_n(x) = \sum_{r=0}^{n+1}(-1)^r q^{\frac{3r^2+r}{2}} x^r \frac{(q)_{r+s}}{(q)_r(q)_s}\frac{(q)_{n-s}}{(q)_r(q)_{n-r-s}},$$

where

$$\sum_k p(k,r,n)x^k = \frac{(x)_{r+n}}{(x)_r(x)_n}$$

and $p(k,r,n)$ is the number of partition of k into at most r parts not exceeding n.

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