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# ON M–INJECTIVE MODULES AND M–IDEALS

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Abstract. For a left R-module M, we identify certain submodules of M that play a role analogous to that of ideals in the ring R. We investigate some properties of M-ideals in the submodules of M and also study Jacobson radicals of a submodule of M.

We assume throughout the paper that $R$ is an associative ring with identity and a left $R$-module is a unitary left $R$-module.

The module ${}_RX$ is called $M$-injective if each $R$-homomorphism $f : K \to X$ defined on a submodule $K$ of $M$ can be extended to an $R$-homomorphism $\hat{f} : M \to X$ with $f = \hat{f}i$, where $i : K \to M$ is the natural inclusion mapping [1].

The category $\sigma[M]$ is defined to be the full subcategory of $R$-mod that contains all modules ${}_RX$ such that $X$ is isomorphic to a submodule of an $M$-generated module[4].

Two-sided ideals of the ring $R$ correspond to the annihilator of left $R$-modules. Furthermore, for a left $R$-module $X$, we have

$$\begin{aligned} Ann_R(X) &= \{r \in R \mid rx = 0 \text{ for all } x \in X\} \\ &= \bigcap_{f \in \mathrm{Hom}_R(R,X)} \ker(f). \end{aligned}$$

More generally, the annihilator of any class of modules is two-sided ideal of $R$ and this motivates the following definition.

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Definition 1. ([3]) *Let $M$ be any left $R$-module, and let $\mathcal{C}$ be a class of modules in $R$-mod, and let $\Omega$ be the set of kernel of $R$-homomorphisms from $M$ into $\mathcal{C}$. That is*

$$\Omega = \{K \subset M \mid \exists\, w \in \mathcal{C} \text{ and } f \in Hom_R(M, W) \\ \text{with } K = ker(f)\}.$$

*We define the annihilator of $\mathcal{C}$ in $M$ to be $ann_M(\mathcal{C}) = \cap_{K\in\Omega} K$.*

In [1]. what we have called the annihilator of $\mathcal{C}$ in $M$ is called the reject of $\mathcal{C}$ in $M$, and is denoted by $Rej_M(\mathcal{C})$.

Definition 2. ([2]) *The submodule $N$ of $M$ is called an $M$-ideal if there is a class $\mathcal{C}$ of modules in $\sigma[M]$ such that $N = Ann_M(\mathcal{C})$.*

*Note that although the definition of an $M$-ideal is given relative to the subcategory $\sigma[M]$, it is easy to check that $N$ is an $M$-ideal if and only if $N = Ann_M(\mathcal{C})$ for some class $\mathcal{C}$ in $R$-mod. A subfunctor $\rho$ of the identity of $R$-mod is called a radical if $\rho(X/\rho(X)) = O$, for all modules ${}_RX$. For a class $\mathcal{C}$ of $R$-modules, the radical of $R$-mod cogenerated by $\mathcal{C}$ is defined by setting $rad_{\mathcal{C}}(X) = Ann_X(\mathcal{C})$ for al module ${}_RX$.*

Proposition 1. ([3]) *The following conditions are equivalent for a submodule $N \subset M$.*

*(1) $N$ is an $M$-ideal;*

*(2) there exists a radical $\rho$ of $R$-mod such that $N = \rho(M)$;*

*(3) $g(N) = O$ for all $g \in Hom_R(M, M/N)$;*

*(4) $N = Ann_M(M/N)$.*

*Proof.* (1) $\Rightarrow$ (2): Assume that $N = Ann_M(M/N)$ for the class $\mathcal{C}$ of modules in $\sigma[M]$. Then $N = rad_{\mathcal{C}}(M)$ for the radical $\rho = rad_{\mathcal{C}}$ cogenerated by the class $\mathcal{C}$.

(2) $\Rightarrow$ (3): Assume that $\rho$ is a radical of $R$-mod, with $N = \rho(M)$. Since $\rho$ is a radical, we have $\rho(M/N) = \rho(M/\rho(M)) = O$. If $g \in \mathrm{Hom}_R(M, M/N)$, then $g(N) = g(\rho(M)) \subseteq \rho(M/N) = O$.

(3) $\Rightarrow$ (4): It is always true that $Ann_M(M/N) \subseteq N$. Condition (3) implies that $N \subseteq \cap_{f:M\to M/N}\ker(f)$, so we have equality in this case.

(4) $\Rightarrow$ (1): This follows from the definition of $M$-ideal. □

Let $N \subset K$ be submodules of $M$. It is not true that if $N$ is an $M$-ideal, then $N$ is a $K$-ideal.

PROPOSITION 2. *Let $N \subset K$ be submodules of $M$. If $N$ is an $M$-ideal and $K/N$ is $M$-injective, then $N$ is a $K$-ideal.*

*Proof.* Let $g : K \to K/N$ be a homomorphism. Since $K/N$ is $M$-injective, there exists $\bar{g} : M \to K/N$ such that $\bar{g} \circ i = g$ where $i : K \to M$ is an inclusion homomorphism. Since $N$ is an $M$-ideal, $\bar{g}(N) = O$.

$$g(N) = \bar{g}i(N) = \bar{g}(N) = O.$$

Then $N$ is a $K$-ideal. □

PROPOSITION 3. *Let $N \subset K$ be submodules of $M$. If $K$ is a direct summand of $M$ and $N$ is a $M$-ideal, then $N$ is a $K$-ideal.*

*Proof.* Let $f : K \to K/N$ be a homomorphism. There exists $g : M \to K/N$ such that $g \circ i = g$ where $i : K \to M$ is the inclusion homomorphism. Since $g(N) = O$ and $f(N) = O$, $N$ is a $K$-ideal. □

COROLLARY 4. *Let $R$ be a semisimple ring and $M$ be an $R$-module. Let $N \subseteq K$ be submodules of $M$. If $N$ is a $M$-ideal, then $N$ is a $K$-ideal.*

PROPOSITION 5. *Let $f : M \to X$ be an epimorphism of $R$-modules. If $N$ is a $M$-ideal, then $f(N)$ is a $X$-ideal.*

*Proof.* There exists a radical $\rho$ of $R$-mod such that $N = \rho(M)$.

$$f(N) = f(\rho(M)) = \rho(f(M)) = \rho(X).$$

Then $f(N)$ is a $X$-ideal. □

PROPOSITION 6. *Let $N \subset K$ be submodules of $M$. $N$ is a $M$-ideal and $K/N$ is a $M/N$-ideal. If for $f \in Hom_R(M, M/K)$, there exists $\bar{f} \in Hom_R(M, M/N)$ such that $\pi \circ \bar{f} = f$ where $\pi$ is the natural homomorphism of $M/N$ onto $M/K$, then $K$ is an $M$-ideal.*

*Proof.* Let $\pi : M/N \to M/K$ be the natural homomorphism. Let $f \in \mathrm{Hom}_R(M, M/K)$. By the assumption, there exists $\bar{f} \in \mathrm{Hom}_R(M, M/N)$ such that $\pi \circ \bar{f} = f$. Since $N$ is a $M$-ideal, $\bar{f}(N) = O$. Since $K/N$ is a $M/N$-ideal, $\pi\bar{f}(K) = O$. This implies that $f(K) = O$. Thus $K$ is an $M$-ideal. □

PROPOSITION 7. *Let $N \subset K$ be submodules of $M$. If $N$ is a $K$-ideal (in $K$) and for every $f \in Hom_R(M, M/N)$, $f(K) \subseteq K/N$, then $N$ is an $M$-ideal.*

*Proof.* If $f \in \mathrm{Hom}_R(M, M/N)$, then $f|_K : K \to K/N$. Since $N$ is a $K$-ideal (in $K$), $f(N) = O$. Thus $N$ is an $M$-ideal. □

The next step is to define the product of two $M$-ideal. We give the definition more generally, constructing a product $N \cdot X$ for any submodule $N$ of $M$ and any module ${}_RX$.

DEFINITION 3. ([2]) *Let $N$ be a submodule of $M$. For each module ${}_RX$, we define $N \cdot X = Ann_X(\mathcal{C})$, where $\mathcal{C}$ is the class of modules ${}_RW$ such that $f(N) = O$ for all $f \in Hom_R(M, W)$.*

It follows from definition that $N \cdot X = O$ if and only if $f(N) = O$ for all $f \in \mathrm{Hom}_R(M, X)$.

PROPOSITION 8. *Let $N \subset K$ be submodules of $M$ and $X$ be any $R$-module. For any $f \in Hom_R(M,X)$, $f$ can be extended to $\bar{f} : M \to X$. Then $N \cdot X = O$ in $K$ if and only if $f(N) = O$ in $M$.*

*Proof.* $N \cdot X = O$ if and only if $f(N) = O$ for all $f \in \mathrm{Hom}_R(M,X)$. Assume that $N \cdot X = O$ in $M$. Let $f \in \mathrm{Hom}_R(K,X)$. Then there exists $\bar{f} : M \to X$ such that $\bar{f}$ is an extension of $f$. $\bar{f}(N) = O$ implies $f(N) = O$. Thus $N \cdot X = O$ in $K$.

Conversely, assume that $N \cdot X = O$ in $K$. Let $f \in \mathrm{Hom}_R(K,X)$ and $i : K \to M$ be an inclusion. $f \circ i = O$ implies $f(N) = O$ and $N \cdot X = O$ in $M$. □

COROLLARY 9. *Let $K$ be a direct summand of $M$ and $N \subseteq K$ be a submodule of $M$. Then $N \cdot X = O$ in $K$ if and only if $N \cdot X = O$ in $M$.*

*Proof.* Every homomorphism $f : K \to X$ can be extended to a homomorphism $\bar{f} : M \to X$. □

PROPOSITION 10. *Let $N \subset K$ be submodules of $M$. If $K$ is a direct summand of $M$, then $N \cdot X$ in $K$ equals to $N \cdot X$ in $M$ for any module $X$.*

*Proof.* Let $\mathcal{C}$ be the class of modules ${}_RW$ such that $f(N) = O$ for all $f \in \mathrm{Hom}_R(M,W)$. $\mathcal{C}$ is also the class of modules ${}_RW$ such that $f(N) = O$ for all $f \in \mathrm{Hom}_R(K,W)$. Thus $N \cdot X = Ann_X(\mathcal{C})$ in $K$ equals $N \cdot X = Ann_X(\mathcal{C})$ in $M$. □

It is not true that $N \cdot X$ in $K = N \cdot X$ in $M$.

DEFINITION 4. ([2]) *The module ${}_RX$ is said to be $M$-prime if $Hom_R(M,X) \neq O$, and $Ann_M(Y = Ann_M(X)$ for all submodules $Y \subseteq X$ such that $Hom_R(M,Y) \neq O$.*

Proposition 11. *Let $M$ be a nonzero $M$-prime module. If Hom $_R(K,Y) \neq O$ implies Hom$_R(M,Y) \neq O$ for any nonzero $Y \leq K \leq M$, then $K$ is a $K$-prime module.*

*Proof.* Let $Y$ be a nonzero submodule of $K$ such that $\text{Hom}_R(K,Y) \neq O$. $\text{Hom}_R(M,Y) \neq O$ by assumption. $Ann_K(Y) = Ann_M(M) = O$ since $M$ is $M$-prime. $Ann_K(Y) \subseteq Ann_M(Y) \cap K = O$. This implies that $Ann_K(Y) = O$ and $K$ is a $K$-prime module. □

Proposition 12. *Let $X$ be a $M$-prime module and $K$ a submodule of $M$. Hom$_R(K,Y) \neq O$. Let $Y$ be a submodule of $X$. Every homomorphism $f \in$ Hom$_R(K,Y)$ can be extended to $\bar{f} \in$ Hom$_R(M,Y)$. Then $X$ is an $K$-prime module.*

*Proof.* Let $Y$ be a nonzero submodule of $K$ such that $\text{Hom}_R(K,Y) \neq O$. $Ann_K(Y) = K \cap Ann_M(Y) = K \cap Ann_M(X) = Ann_K(X)$. This proves that $X$ is a $K$-prime module. □

Corollary 13. *Let $X$ be a $M$-prime module and $K$ is a direct summand of $M$. Hom$_R(K,X) \neq O$. Then $X$ is a $K$-prime module.*

The Jacobian radical of the ring $R$ is generally defined to be the intersection of maximal left ideals of $R$. The definition is extended to modules, by defining the Jacobson radical $J(X)$ of a module $_RX$ to be the intersection of all maximal submodules of $X$. Equivalently, $J(X) = Ann_X(\mathcal{C})$, where $\mathcal{C}$ is the class of simple left $R$-modules.

Proposition 14. *Let $A$ be a submodule a module $M$. If $A$ is direct summand of $M$, then $J(A) = J(M) \cap A$.*

*Proof.* $J(A) = Ann_A(\mathcal{C})$ where $\mathcal{C}$ is the class of simple left $R$-modules. $Ann_A(\mathcal{C}) = A \cap Ann_M(\mathcal{C}) = A \cap J(M)$. □

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