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THE CONNECTIVITY OF INSERTED GRAPHS

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Abstract. The aim of the paper is to study the connectivity and the edge-connectivity of inserted graph $I(G)$ of a graph G with the help of connectivity and the edge-connectivity of that graph G.

1. Introduction

We consider ordinary graphs (finite, undirected,with no loops or multiple edges). Let G be a graph with vertex set V_G and edge set E_G. Each member of $V_G \cup E_G$ will be called an element of G. A graph G is called trivial graph if it has a vertex set with single vertex and a null edge set. If e be an edge of a graph G with end vertices x and y, then we denote the edge $e = xy$.

We introduce the notions of box graph $B(G)$ and inserted graph $I(G)$ of a non-trivial graph G in [2]. It is an elementary basic fact that the inserted graph $I(G)$ of a non-trivial connected graph G in connected.

There are two major measures how highly connected a graph can be, namely the connectivity and edge-connectivity.

The connectivity $k(G)$ of a graph G is the least number of vertices whose removal (along with all incident edges) disconnected G or reduces it to the trivial graph; a set of $k(G)$ vertices satisfying this condition is called a minimal separating vertex set of G. Moreover G is n-connected if and only if $k(G) \geq n$. On the other hand, the edge-connectivity $\lambda(G)$ of a graph G is the least number of edges whose removal disconnected

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G or reduces it to the trivial graph; and a set of $\lambda(G)$ edges satisfying this condition is called a minimal separating edge set of G. Moreover G is m-edge-connected if and only if $\lambda(G) \geq m$. Thus a non-trivial graph is connected if and only if it has positive connectivity (and edge-connectivity).

In Section 2, we recall some definitions and results to be used in this paper and also give an example of connectivity and edge-connectivity of a graph G and its inserted graph $I(G)$.

In Section 3, we investigate the relationship between the connectivity and edge-connectivity of a graph and its inserted graph. In particular, if $k(G_1) = n$ and $\lambda(G_2) = m$, then $k(I(G_1)) \geq n$ and $\lambda(I(G_1)) \geq 2n-2$ while $k(I(G_2)) \geq m$ and $\lambda(I(G_2)) \geq 2m-2$.

2. Preliminaries

In this section at first we recall some definitions.

Definition 2.1. ([2]) A graph can be constructed by inserting a new vertex on each edge of G, the resulting graph is called *box graph* of G, denoted by $B(G)$. For an edge e of G, $\overline{e}$ denote the vertex of $B(G)$ corresponding to the edge e.

The graph $B(G)$ has the property that, there always exists a one-one correspondence between the vertices and the elements of G such that any two vertices of $B(G)$ are adjacent if and only if the corresponding elements of G are an edge and an incident vertex. Obviously, $B(G)$ is a bipartite graph whose number of vertices is equal to the number of elements of G. Moreover if $V_G = \{v_1, v_2, \cdots, v_n\}$ and $E_G = \{e_1, e_2, \cdots, e_m\}$ then $V_{B(G)} = \{v_1, v_2, \cdots, v_n, \overline{e}_1, \overline{e}_2, \cdots, \overline{e}_m\}$.

Definition 2.2. ([2]) Let I_G be the set of all inserted vertices in $B(G)$. A graph $I(G)$ with vertex set I_G is called the inserted graph in which any two vertices are adjacent if they are joined by a path of length two in $B(G)$. Therefore, if $E_G = \{e_1, e_2, \cdots, e_m\}$ then $I_G = V_{I(G)} = \{\overline{e}_1, \overline{e}_2, \cdots, \overline{e}_m\}$.

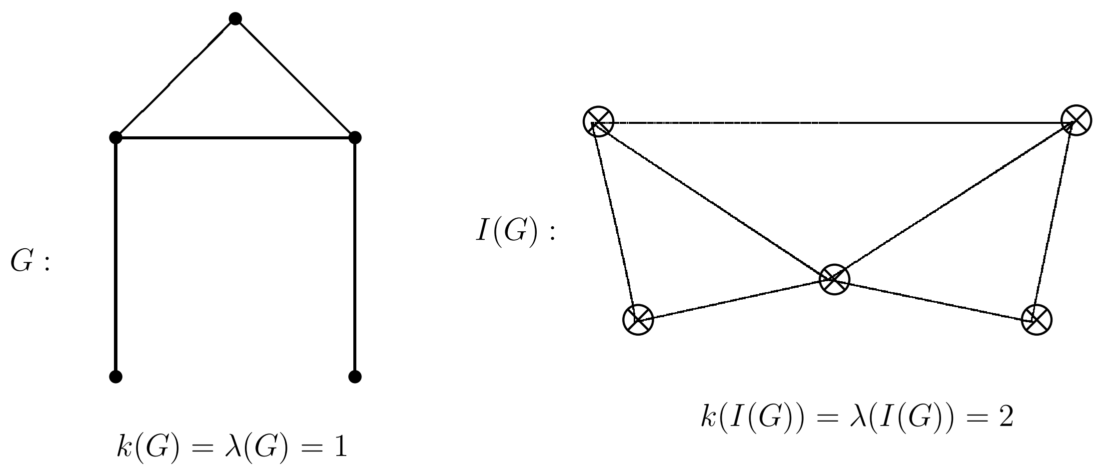

Fig. 1: Connectivity and edge-connectivity of a graph and its inserted graph

These concepts are illustrated for a graph G and its inserted graph $I(G)$ in the Fig. 1. Here $\otimes$ marked vertices are the newly inserted vertices.

Now we review some results related to edge-connectivity and connectivity, to which we shall have occasion to refer in what follows. Characterizations of n-connected graphs and m-edge-connected graphs are presented bellow ([3]).

THEOREM 2.1. *A graph G is n-connected (m-edge-connected) if and only if between every pair of distinct vertices there exist at least n disjoint (m edge-disjoint) paths.*

The following criterion for m-edge-connected graphs will be useful in the proof of one of our results [4].

THEOREM 2.2. *A graph G is m-edge-connected if and only if for every non-empty proper subset A of the vertex set V_G of the graph G, the number of edges joining A and $V_G - A$ is at least m.*

The next observation is due to Whitney [5]. We write min deg G to denote the smallest degree among the vertices of G.

Theorem 2.3. *For any graph* G, $k(G) \leq \lambda(G) \leq$ *min deg* G.

3. Connectivity and edge-connectivity of $I(G)$

In this section we investigate the relationship between the connectivity and edge-connectivity of a graph and its inserted graph.

Theorem 3.1. *If a graph* G *is* m*-edge-connected,* $m \geq 2$*, then* $I(G)$ *is* m*-connected.*

Proof. Let $\overline{x}$ and $\overline{y}$ be two arbitrary distinct vertices of $I(G)$, where G is a m-edge-connected graph with $m \geq 2$ and let $x = uu_1$ and $y = vv_1$ be those edges of G corresponding to the vertices $\overline{x}$ and $\overline{y}$ in $I(G)$. Consider the vertices u and v. Since G is m-edge-connected, by Theorem 2.1 there exist m edge-disjoint paths $P_i, 1 \leq i \leq m$, joining u and v. At most one of the path P_i contains x; however, those paths which fail to contain x have their initial edge adjacent with x. Similarly, at most one P_i contain y, but any such path not containing y has its terminal edge adjacent with y. Corresponding to the paths P_i in G, there are m paths Q_i in $I(G)$ formed by adjoining to $I(P_i)$ the edges uw_{i1} and $w_{ik}v$ (if not already present), where w_{i1} and w_{ik} are the initial and terminal vertices of Q_i. Since the P_i are edge-disjoint, the Q_i are disjoint so that, by Theorem 2.1, $I(G)$ is m-connected. □

The following corollaries are immediate.

Corollary 3.2. *If* G *is a graph for which* $\lambda(G) \geq 2$*, then* $\lambda(G) \leq k(I(G))$.

Corollary 3.3. *If* G *is* n*-connected,* $n \geq 2$*, then* $I(G)$ *is* n*-connected.*

Theorem 3.4. *If a graph* G *is* m*-edge-connected, then* $I(G)$ *is* $(2m-2)$*-edge-connected.*

Proof. Assume that $m \geq 2$ (the result is obvious for $m = 1$). Let Y denote any nonempty proper subset of the edge set E_G of G, thus Y induces a nonempty proper subset $\overline{Y}$ of the vertex set $V_{I(G)}$, and let

$C[Y] = \{\{y_1, y_2\} | y_1 \in Y, y_2 \in E_G - Y, y_1$ is adjacent to y_2 in $G\}$. For each vertex u in G, denote by $\delta(u)$ the number of edges of Y incident with u and by $\delta^{'}(u)$ the number of edges of $E_G - Y$ incident with u. If $W = \{u | \delta(u) > 0, \delta^{'}(u) > 0\}$, then $|C[Y]| = \sum_{w \in W} \delta(w)\delta^{'}(w)$ is the number of edges in $I(G)$ joining vertices of $\overline{Y}$ with vertices of $V_{I(G)} - \overline{Y}$. In order to conclude that $I(G)$ is $(2n-2)$-edge-connected, it sufficient to show, by Theorem 2.2, that $|C[Y]| \geq (2m-2)$ for each $C[Y]$.

Since G is connected and Y is a non-empty proper subset of E_G, it follows that W is non-empty. At this vertex we distinguish two cases.

Case-1: The set W consists of a single vertex, say v. In this case, the removal of the edges of Y incident with v necessarily disconnects G as does the removal of the edges of $E_G - Y$ incident with v. Since G cannot be disconnected by the deletion of fewer than m-edges, $|C[Y]| = \delta(v)\delta^{'}(v) \geq m^2 \geq (2m-2)$.

Case-2: The set W consists of at least two vertices. Here we have $|W| \geq 2$, so that $|C[Y]| \geq |W|\delta(u)\delta^{'}(u)$, where $u \in W$ is so chosen that $\delta(u)\delta^{'}(u)$ is minimum. Since $\delta(u) + \delta^{'}(u) \geq m$, by Theorem 2.3, the minimum value of $\delta(u)\delta^{'}(u)$ is not less than $m-1$; hence $|C[Y]| \geq (2m-2)$. This completes the proof. □

Since $\lambda(G)$ is the largest value of m for which a graph G is m-edge-connected. Now we state the following:

Corollary 3.5. *If G is a graph for which $\lambda(G) = m$, then $\lambda(I(G)) \geq (2m-2)$.*

If $\lambda(G) = m$ and G contains two adjacent vertices, each of degree m, then $I(G)$ contains a vertices of degree $2m-2$, so by Theorem 2.3, $\lambda(I(G)) \leq (2m-2)$. And then the above corollary implies that $\lambda(I(G)) = (2m-2)$.

Conversely, suppose $\lambda(I(G)) = (2m-2)$, where $\lambda(G) = m$. If $m = 1$, G contains a single edge. For $m \geq 2$, a non empty proper subset Y of the edge set E_G of G can be selected such that $W = \{u, v\}$ (if not, $W = \{u\}$ for every non empty proper subset Y of E_G implies $\lambda(I(G)) > (2m-2))$ and $\delta(u)\delta'(u) = \delta(v)\delta'(v) = m-1$ (inasmuch as each product is no less than $m-1$ and their sum is $2m-2$). In particular, this implies the degree of each of u and v is m. Since G is connected, the set Y necessarily induces a connected subgraph of G for otherwise W would contain more than two elements. If, in addition, $m \geq 3$, then u and v are adjacent. To see this, assume the contrary. By Theorem 2.1, there exist at least three edge-disjoint paths joining u and v. Now each such path must be completely contained with in Y or $E_G - Y$, let W contain more than two vertices, so that each of u and v is incident with precisely one edge of Y or precisely one edge of $E_G - Y$. Thus G can be disconnected by the removal of two edges, violating the hypothesis $\lambda(I(G)) \geq 3$. We summarize these observations below.

Corollary 3.6. *If G is a graph for which $\lambda(G) = m \neq 2$, then $\lambda(I(G)) \geq (2m-2)$ if and only if there exist two adjacent vertices in G with degree m.*

Corollary 3.6 cannot be extended to include $m = 2$ as illustrated in Fig. 2.

One might well expect a result for n-connectedness analogous to that obtained for m-edge-connectedness in Theorem 3.4; however the following shows that Corollary 3.3 cannot be improved in general. Let the graph G_n consist of two disjoint copies of complete graph K_{n+1}.

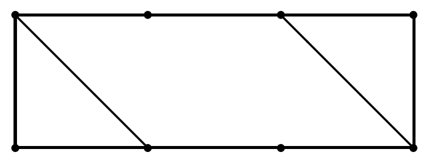

Fig.2 : A graph G for which $\lambda(I(G)) = 2\lambda(G) - 2 = 2$ but containing no adjacent vertices with degree $\lambda(G)$

whose vertices are labelled x_i and $y_i, 0 \leq i \leq n$, where in addition, the edges $e_i = x_i y_i, 1 \leq i \leq n$, are inserted. The graph G_n has connectivity n (and so is n-connected) as does $I(G_n)$ [and so is not $n+1$-connected].

Fig. 3 shows the case when $n = 3$.

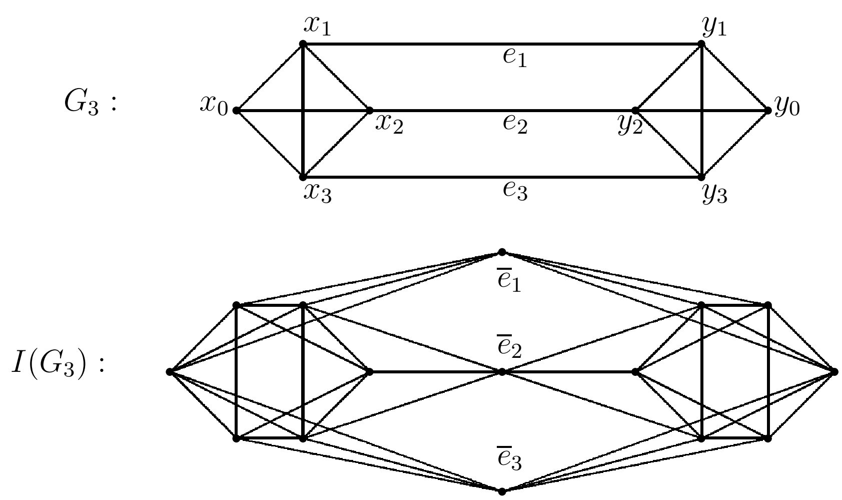

Fig. 3 : Graph G_3 of the class of subgraph G_n for which $k(I(G_n)) = k(G_n) = n$

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