THE INVARIANT OF IMMERSIONS UNDER ISOTWIST FOLDING

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ABSTRACT. In this paper we will introduce all types of the isotwist foldings of a manifold M into itself. The limits of the isotwist foldings of a manifold are obtained. Also the relations between conditional retraction and this type of the folding are achieved. Finally the variant and invariant of the immersion under the type of folding are deduced.

1. Introduction

The notion of isometric folding was introduced by S. A. Robertson who studied the stratification determined by the folds or singularities [14]. Then the theory of isometric foldings has been pushed and also different types of foldings are introduced by E. El-Kholy and others [8, 9, 10]. The conditional foldings of manifolds are defined by M. El-Ghoul in [2, 3, 4, 6, 7]. Some applications on the folding of a manifold into it self was introduced by P. Di. Francesco in [11].

Let $\alpha : (a, b) \to E^3$ be a regular curve and let $t_0 \in (a, b)$. Set $L(t) = \int_{t_0}^{t} \left| \frac{d\alpha}{dt} \right| dt$. $s = L(t)$ is called arc length along α. If the curve α parametrized by the arc length s then $\left| \frac{d\alpha}{ds} \right| = 1$. So let $\alpha : (a, b) \to E^3$ be unit speed curve the principal normal vector field is (unit) vector field $N(s) = T^{\backslash}(s)/\kappa(s)$. The binormal vector filed to $\alpha(s)$ is $B(s) =$

$T(s) \times N(s)$. The torsion of α is real -valued function

$$\tau = \left\langle B^{\backslash}(s), N(s) \right\rangle.$$

where $T(s)$ is unit tangent to $\alpha(s)$ and $\kappa(s)$ is the curvature of $\alpha(s)$ [13].

DEFINITION 1.1. Let α, β are two immersed curves in E^2 we say that α, β are called athwart immersions if they have not common tangent space at any points of α, β [1].

DEFINITION 1.2. Let α, β are two immersed curves in E^2 we say that α, β are called not athwart immersions if they have common tangent space at some points of α, β [1].

DEFINITION 1.3. Let M be n-manifold, $A \subset M$ subset. A is called a retract of M if there is a retraction $r : M \to A$, i.e., a continuous map with $r|_A = Id_A$ [12].

2. Isotwist folding

In this section we will give the definitions of all type of the isotwist folding of a manifold M into itself. The limits of the isotwist foldings of a manifold are obtained. Also the relations between conditional retraction and this type of the folding achieved.

DEFINITION 2.1. Let M be an n-dimensional Riemannian manifold. A topological folding $f_\tau : M \to M$ from M into itself is said to be isotwist folding if and only if f_τ preserve to the sign of the twist (i.e., f_τ maps M_1 into M_2 where $M_1, M_2 \subset M$ such that $sign\ \tau(M_1) = sign\ \tau(M_2)$.

DEFINITION 2.2. Let M be an n-dimensional Riemannian manifold. A topological folding $f_\tau : M \to M$ from M into itself is said to be isotwist folding if and only if f_τ preserve to the twist at corresponding points (i.e., f_τ maps M_1 into M_2 where $M_1, M_2 \subset M$ such that $\tau(M_1) = \tau(M_2)$ at the corresponding points.

In the general case the limit of folding of an n-dimensional Riemannian manifold into itself is an Riemannian manifold manifold of dimension $n-1$, i.e., $\lim_{r \to \infty} f_r(M^n) = M^{n-1}$ [5]. Now let α be unit speed curve in E^3 and τ is the torsion of α. Also $\bar{\alpha} \subset \alpha$ where $\tau > 0$ at each point $\bar{t} \in \bar{\alpha}$ and $\underline{\alpha} \subset \alpha$ where $\tau < 0$ at each point $\underline{t} \in \underline{\alpha}$

THEOREM 2.1. *In the folding of a manifold into itself which preserve the limit of sequences of foldings must be a manifold of the same dimension.*

Proof. Let α be unit speed curve in E^3 and $f_\tau : \alpha \to \alpha$ is isotorsion folding which preserve the sign of the torsion $f_\tau(\alpha_1) = \alpha_2$ such that $\tau(\alpha_1), \tau(\alpha_2) > 0$ $\alpha_1, \alpha_2 \subset \alpha$. Also let α be a curve with $\tau = 0$ at every point $p \in \alpha$ and $d : \alpha \to \alpha$ be a deformation such that $\tau \mid_p = 0$, $\tau < 0$ in the lower of p and $\tau > 0$ in the upper of p. Consider a sequence of foldings and $f_{\tau_1} : \bar{\alpha}_d \to \bar{\alpha}_d$ such that $f_1(\bar{\alpha}_{1,\tau>0}) = \bar{\alpha}_{2,\tau>0}$, $f_{\tau_1}(\underline{\alpha}_{-1,\tau<0}) = \underline{\alpha}_{-2,\tau<0}$ and $f_{\tau_2} : f_1(\alpha_d) \to f_1(\alpha_d)$ such that $f_2(\bar{\alpha}_{2,\tau>0}) = \bar{\alpha}_{3,\tau>0}$, $f_2(\underline{\alpha}_{-2,\tau<0}) = \underline{\alpha}_{-3,\tau<0}$, $\cdots$, $f_{\tau_r} : f_{\tau_{r-1}}(\alpha_d) \to f_{\tau_{r-1}}(\alpha_d)$, $f_{\tau_r}(\bar{\alpha}_{r,\tau>0}) = \bar{\alpha}_{r+1,\tau>0}$, $f_{\tau_r}(\underline{\alpha}_{-r,\tau<0}) = \underline{\alpha}_{-r+1,\tau<0}$. Thus $\lim f_{\tau_r} = g_1$ at $\tau > 0$ and $\lim f_{\tau_r} = g_2$ at $\tau < 0$ such that g_1 is an upper geodesic, g_2 is an lower geodesic. So the α after the sequence of the folding is piecewise geodesics of the same dimension of α. □

Now we will give the definition of the tiling of the curves and the classification of this tiling.

DEFINITION 2.3. A tiling of unit speed curve α is a collection $\Im = \{\alpha_i, \ i \in I = \{1, 2, 3, \cdots\}\}$ of segment curve (tile) which cover the manifold.

Classification of the tiling

In the general case the arclength of tiles (segment curve) are different. In the following we give two types of the tiling.

(1) Equitiling in this case all tiles have the same arclength, i.e., $L(\alpha_i) = const$ for all $i \in I$.

(2) Uniform tiling in this case $L(\alpha_i) = L(\alpha_{i+2})$ for all $i \in I$.

DEFINITION 2.4. Let α be equipped by tiling $\Im(\alpha)$. Then the tiling folding of α is the folding $g : \alpha \to \alpha$ such that $g(\alpha_i) = \alpha_j$, where $\alpha_i, \alpha_j \in \Im(\alpha)$

LEMMA 2.2. Let M equipped by tiling $\Im$ then the Min tiling folding of M is either tile with $\tau \geq 0$, $\tau \leq 0$, tile with different sign torsion or two tiles with different sign torsion.

EXAMPLE 2.3. In the example we introduce the different types of Min tiling folding. Let α be unit speed curve in E^3 equipped by tiling. and if p is point in α thus that $\tau_p(\alpha) = 0$ and let $g : \alpha \to \alpha$ be a tiling folding of α into itself. Then the Min tiling folding of α depends on the position of p.

(1) In the first case let the point p lies in the boundary of tiles $\bar{\alpha}_1$ where $\tau > 0$ and $\underset{-1}{\alpha}$ where $\tau < 0$. Then the Min tiling folding of α is

$$\lim_{r \to \infty} g_r(\alpha) = \begin{cases} \bar{\alpha}_1 \text{ where } \tau(\bar{\alpha}_1) \geq 0 \text{ for all } t \in \bar{\alpha}_1 \\ \underset{-1}{\alpha} \text{ where } \tau(\underset{-1}{\alpha}) \leq 0 \text{ for all } t \in \underset{-1}{\alpha} \end{cases} . \text{ See Fig. 1.}$$
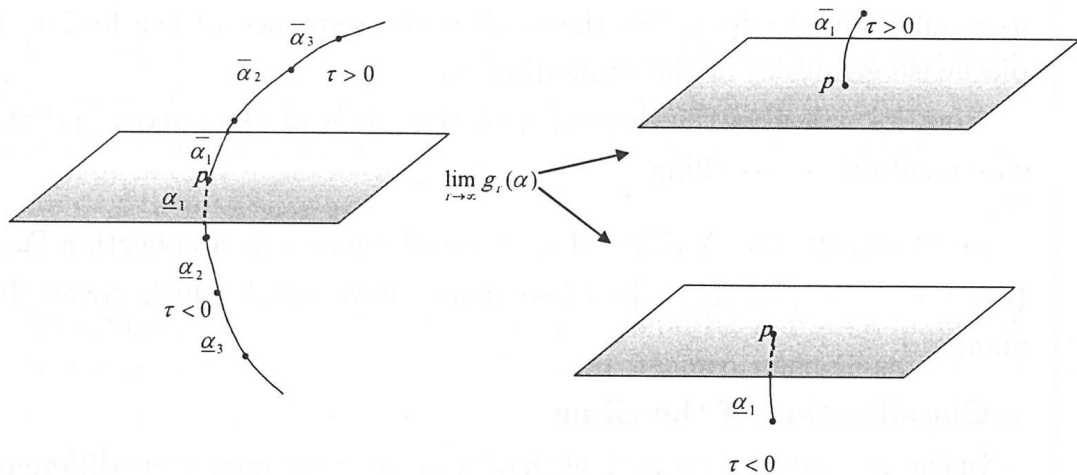

Fig 1

(2) In the second case let α_0 be the tile that the point p lies inside it. Then the Min tiling folding $\lim\limits_{r \to \infty} g_r(\alpha) = \alpha_0$ where the the torsion of α_0 has different sign. See Fig. 2.
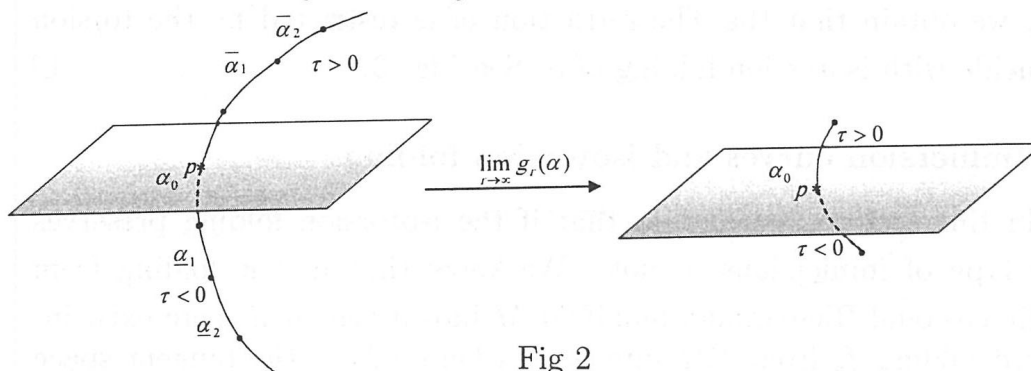

Fig 2

Now in the following theorem we will discuss the relation between the retraction and the isotorsion folding to the curves in E^3. In the general case the retraction of the manifold M does not coincide with the folding of M, but under some restrictions we obtain the following.

THEOREM 2.4. *Let α be unit speed curve The retraction of α restricted by the torsion coincide with isotorsion folding of α.*

Proof. Let α be unit speed curve such that the torsion τ at some points $p \in \alpha$ equal 0, i.e., $\tau_p = 0$ otherwise the torsion is $\tau > 0$ or $\tau < 0$ So we can equipped α by tiling $\mathfrak{I}$ such that the point p lies between two tiles. Let $r : \alpha \to A$ be the retraction of α restricted by the torsion, where $A \subset \alpha$, then $r_1 \mid_{\tau>0} (\bar{\alpha}) = \bar{A}_1$, $r_1 \mid_{\tau<0} (\underline{\alpha}) = A_{-1}$ and $r_2 \mid_{\tau>0} (\bar{A}_1) = \bar{A}_2$, $r_2 \mid_{\tau<0} (\bar{A}_1) = A_{-2}, \cdots, r_n \mid_{\tau>0} (\bar{A}_{n-1}) = \bar{A}_n$, $r_n \mid_{\tau<0} (\bar{A}_{n-1}) = A_{-n}$.
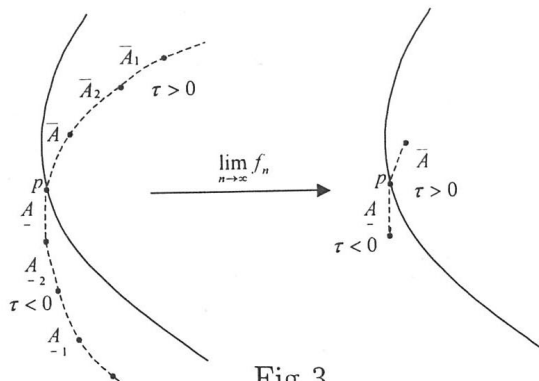

Fig 3

Then $\lim r_n |_{\tau > 0} (\bar{A}_{n-1}) = \bar{A}$ and $\lim r_n |_{\tau < 0} (\bar{A}_{n-1}) = \underline{A}$, where $\bar{A}$, $\underline{A}$ are two tiles about the point p, $\tau(\bar{A}) > 0, \tau(\underline{A}) < 0$. By Theorem 2.1, we obtain that the The retraction of α restricted by the torsion coincide with isotorsion folding of α. See Fig. 3. $\square$

3. Immersion curves and isotorsion folding

In this section, we discuss that if the isotorsion folding preserves the type of immersions or not. We know that if f is folding from n-dimensional Riemannian manifold M into it self then there exist induced folding f_* from TM into itself where TM is the tangent space of M.

THEOREM 3.1. *Let α, β are not athwart immersed curves in E^2 then the isotorsion folding f_τ of α into itself, β into itself preserve the immersion if $f_*(t_{com}) = t_{com}$ where t_{com} is common tangent at some points.*

Proof. Let $f_\tau : \alpha \to \alpha$, $\tau(\alpha) = \tau(f_\tau(\alpha))$ and $f_\tau : \beta \to \beta$, $\tau(\beta) = \tau(f_\tau(\beta))$ and let the two immersed manifolds have common tangent at some points. So there exist induced folding $f_* : T\alpha \to T\alpha$ such that $f_*(t_{com}) = t_{com}$. See Fig. 4.

Fig 4
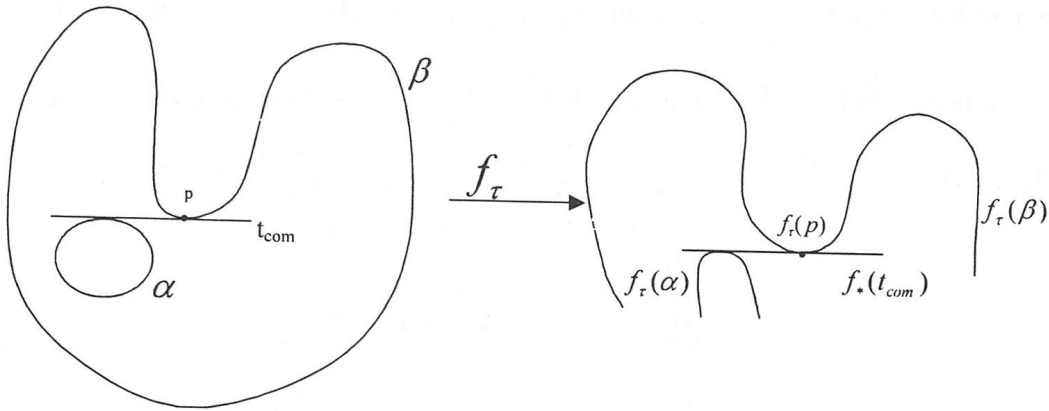
Thus $f_\tau(\alpha)$, $f_\tau(\beta)$ are not remain athwart immersions, but if $\bar{\bar{f}}_\tau$ is the type of isotorsion folding which $\bar{f}_*(t_{com}) \neq t_{com}$. See Fig. 5.
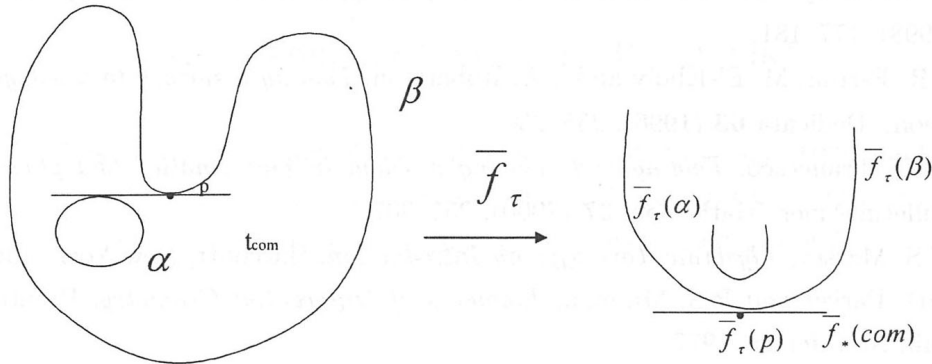

Fig 5

Then $\bar{f}_\tau(\alpha)$, $\bar{f}_\tau(\beta)$ are athwart immersions. □

Above theorems can be generalized to n- manifold but before this we must define the twist of n dimensional manifold as the deformation in the direction of normal vector on the least space immersed in it.

REFERENCES

1. F.J.C. de Carvalho and S.A. Robertson, *Athwart immersion in Euclidean space,* Proc. Roy. Soc. Edingburg Sect. A (to appear).

2. M. El-Ghoul, *Unfolding of Riemannian manifolds,* Commu. Fac. Sci. Univ. Ankara **A37** (1988), 1–4.

3. M. El-Ghoul, *The deformation retract of the complex projective space and its folding,* J. Material Sci. **30** (1995), 4145–4148.

4. M. El-Ghoul, *Folding of fuzzy torus and fuzzy graphs,* Fuzzy Sets and Systems **80** (1996), 389–396.

5. M. El-Ghoul, *The limit of folding of a manifold and its deformation retract,* J. Egyptian Math. Soc. **5** (1997), 133–140.

6. M. El-Ghoul, A.E. Ahmady and H. Rafat, *Folding-retraction of chaotic dynamical manifold and VAK of vacuum fluctuation,* Chaos, Solitons & Fractals **20** (2004), 209–217.

7. M. El-Ghoul and M.H. Mostafa, *Homology and deformation retract of some real manifolds and their folding,* Bull. Col. Soc. **89** (1997), 355–362.

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8. M. El-Kholy and M. El-Ghoul, *Simplicial foldings,* J. Faculty Education **18** (1993), 443–455.

9. M. El-Kholy and R.M. Shahin, *Cellular folding,* J. Inst. Math. Comp. Sci. **3** (1998), 177–181.

10. H.R. Farran, M. El-Kholy and S.A. Robertson, *Folding a surface to a polygon,* Geom. Dedicata **63** (1996), 255–266.

11. P. Di. Francesco, *Folding and coloring problem in mathematics and physics,* Bulletin Amer. Math. Soc. **37** (2000), 251–307.

12. W.S. Massey, *Algebraic Topology: an Introduction,* Harcourt, New York, 1967.

13. G.D. Parker and R.S. Millman, *Elements of Differential Geometry,* Prentice-Hall, New Jersey, 1977.

14. S.A. Robertson, *Isometric folding of Riemannian manifolds,* Proc. Roy. Soc. Edingburg Sect. A **79** (1977), 275–284.

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