JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **18**, No.1, April 2005

COMMON FIXED POINTS OF A WEAK-COMPATIBLE PAIR OF A SINGLE VALUED AND A MULTIVALUED MAPS IN D-METRIC SPACES

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Abstract. The object of this paper is to prove two unique common fixed point theorems for a pair of a set-valued map and a self map satisfying a general contractive condition using orbital concept and weak-compatibility of the pair. One of these results generalizes substantially, the result of Dhage, Jennifer and Kang [4]. Simultaneously, its implications for two maps and one map improves and generalizes the results of Dhage [3], and Rhoades [11]. All the results of this paper are new.

1. Introduction

The fixed point theory for the set-valued mappings is a major branch of set-valued analysis and at present a very extensive literature is available in this direction. Most of these results are extensions and generalizations of the celebrated fixed point theorem for set-valued maps first established by Nadler [10] in metric spaces. The common fixed point theorems for the pairs of self map and a set-valued map have been studied by Fisher [6, 7], Garegnani and Zanco [8] etc. under weaker versions of the commutativity condition.

Generalizing the notion of metric space, Dhage [2] introduced D-metric space and proved the existence of unique fixed point of a self-map satisfying a contractive condition. Dealing with D-metric space Ahmad

Received by the editors on February 01, 2005.
2000 *Mathematics Subject Classifications* : Primary 54H25, 47H10.
Key words and phrases: D-metric space, weak-compatible maps, orbit, bounded orbit, unique common fixed point.

et. al. [1], Dhage [2, 3], Dhage et. al [5], Rhoades [11], Singh, Jain and Jain [12], and others made a significant contribution in fixed point theory of D-metric spaces while Veerapandi et. al. [13] established some fixed point theorems for set-valued maps in D-metric spaces. Recently Dhage, Jennifer and Kang [4] deal with some results for fixed points of a pair of coincidentally commuting set-valued map and a self map in a D-metric space which is being generalized by our results.

The first result of this paper establishes a unique common fixed point theorem in an unbounded and incomplete D-metric space. The second result of this paper is a unique common fixed point theorem for the pair of a self map and a set-valued map satisfying a general contractive condition under weak-compatibility of them. It uses using orbital concept for the domains of variables x, y and for the completeness and boundedness as well. The results of the said references of D-metric spaces are also generalized significantly in this paper.

2. Preliminaries

DEFINITION 2.1. ([2]) Let X be a non-empty set. A generalized metric (or D-metric) on X is a function from $X \times X \times X \to R^+$ (the set of non-negative real numbers) satisfying:
(D-1) $\rho(x, y, z) = 0$ if and only if $x = y = z$,
(D-2) $\rho(x, y, z) = \rho(y, x, z) = \cdots$,
(D-3) $\rho(x, y, z) \leq \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z), \forall x, y, z, a \in X$.
The pair (X, ρ) is called a D-metric space.

DEFINITION 2.2. ([2]) A sequence $\{x_n\}$ of points in a D-metric space (X, ρ) is said to be D-convergent to a point $x \in X$ if for $\epsilon > 0, \exists n_0 \in N$ such that $\forall m, n \geq n_0$, $\rho(x_m, x_n, x) \leq \epsilon$. This sequence is said to be D-Cauchy sequence if for $\epsilon > 0, \exists n_0 \in N$ such that $\forall m > n, p > m, n \geq n_0, \rho(x_n, x_m, x_p) \leq \epsilon.(X, \rho)$ is said to be complete if every D-Cauchy sequence in it converges to some point of X.

DEFINITION 2.3. Let F be a multivalued map on D-metric space (X, ρ). Let $x_0 \in X$ be arbitrary. A sequence $\{x_n\}$ in X is said to be an orbit of F at x_0 denoted by $O(F, x_0)$ if $x_n \in F^n(x_0), \forall n \in N$. If F is a single- valued self map on X then for $x_0 \in X$, let $x_1 = Fx_0, x_2 = Fx_1 = F^2x_0, \cdots, x_{n-1} = F^{n-1}x_0, \cdots$. Then the sequence $\{x_n\}$ is called the orbit of F at the point x_0 and is denoted by $O(F, x_0)$.

DEFINITION 2.4. Let F be a multivalued map on D-metric space (X, ρ). An orbit $O(F, x_0)$ is said to be complete if every D-Cauchy sequence in it converges to an element of X.

DEFINITION 2.5. A subset A of a D -metric space (X, ρ) is said to be bounded if there exists $M > 0$ such that $\rho(u, v, w) \leq M, \forall u, v, w \in A$ and M is said to be a bound of it.

DEFINITION 2.6. ([13]) Let $CB(X)$ be the collection of all non-empty bounded and closed subsets of a D-metric space (X, ρ) and $A, B, C \in CB(X)$. Let
$\delta(A, B, C) = Sup\{\rho(a, b, c) : a \in A, b \in B, c \in C\}$, Then $(CB(X), \delta)$ is a D-metric space.

DEFINITION 2.7. Let F be a multivalued map on D -metric space (X, ρ). A point $u \in X$ is said to be a fixed point of F if $u \in Fu$. Also for a sequence $\{x_n\} \in X$, if $Lim_{m,n\to\infty}\delta(Fx_m, Fx_n, z) = 0$, then we say $\{Fx_n\} \to z \in X$.

DEFINITION 2.8. ([4]) Let F be a multivalued map and g be a self map on D-metric space (X, ρ). The pair (F, g) is said to be weak-compatible if $Fy = \{gy\}$, for some $y \in X$ implies $Fgy = \{gFy\}$.

Let Φ denote the class of functions $\phi : R^+ \to R^+$ such that ϕ is upper semi-continuous, ϕ is non-decreasing, $\phi(t) < t$, for $t > 0$.

To prove the main results, we require the following proposition and lemma.

Proposition 2.1. *Let g be a self map in a D-metric space (X,ρ) and $F : X \to CB(X)$ such that $F(X) \subseteq g(X)$. For some $x_0 \in X$, define sequences $\{x_n\}$ and $\{y_n\}$ in X by*

$$y_n = gx_n \in Fx_{n-1}, \forall n \in N.$$

Then

$$\{x_o, x_1, x_2, \cdots\} = \{x_n\} \in O(g^{-1}F, x_0),$$
$$\{y_1, y_2, y_3, \cdots\} = \{y_n\} \in O(Fg^{-1}, y_1), \text{ where } y_1 = Fx_0.$$

Proof. Since $gx_1 \in Fx_0$, we have $x_1 \in g^{-1}Fx_0$. Also, $gx_2 \in Fx_1$ gives $x_2 \in g^{-1}Fx_1 = (g^{-1}F)^2x_0$. Similarly, $gx_n \in Fx_{n-1}$ gives $x_n \in g^{-1}Fx_{n-1} = (g^{-1}F)^nx_0$. Again $y_1 = gx_1 \in Fx_0, y_2 = gx_2 \in Fx_1 \in F(g^{-1}Fx_0) = (Fg^{-1})Fx_0, y_3 = gx_3 \in Fx_2 \in F(g^{-1}F)^2x_0 = (Fg^{-1})^2Fx_0$. Similarly, $y_n \in (Fg^{-1})^{n-1}Fx_0$. □

Note that $\{y_n\} = \{y_1, y_2, y_3, \cdots\} = O(Fg^{-1}, y_1)$, where $y_1 = Fx_0$, is said to be an (F/g)-orbit at x_0. It is also written as $O(Fg^{-1}, Fx_0)$.

Lemma 2.2. *Let g be a self-map in a D-metric space (X,ρ) and $F : X \to CB(X)$ be such that $F(X) \subseteq g(X)$. For some $x_0 \in X$, and for some $\phi \in \Phi$, let*

$$\delta(Fx, Fy, Fz) \le \phi \ \ Max \begin{pmatrix} \rho(gx, gy, gz), \delta(Fx, Fy, gz), \\ \delta(gx, Fx, gz), \delta(gy, Fy, gz), \\ \delta(gx, Fy, gz), \delta(gy, Fx, gz), \\ \delta(gx, gy, Fz), \delta(gx, Fx, Fz), \\ \delta(gy, Fy, Fz), \delta(gx, Fy, Fz), \\ \delta(gy, Fx, Fz) \end{pmatrix} \tag{2.1}$$

for all $x, y, z \in O(g^{-1}F, x_0)$.

Let $\{x_n\}$ and $\{y_n\}$ be sequences defined in X as above. Let $\{X_n\}$ be a sequence in $CB(X)$ given by

$$y_n = gx_n \in Fx_{n-1} = X_n, \forall n \in N.$$

If $F(\{x_n\}) = \bigcup_{i \in N} X_i$ is bounded, then

(i) $\{y_n\}$ is a $D-$ Cauchy sequence in $O(Fg^{-1}, Fx_0)$.

(ii) If δ is continuous in one variable, then $gx_n \to u$ implies $Fx_n \to \{u\}$.

Proof. **(i)** Define a positive real sequence $\{\gamma_n\}$ in R^+ by

$$\gamma_i = Sup_{j,k\in N}\delta(X_i, X_{i+j}, X_{i+j+k}), \forall i \in N.$$

Then $\gamma_i \geq 0$ and γ_i is a non-increasing sequence for all i. Each γ_i is finite as $\bigcup_{i\in N} X_i$ is bounded. Hence it tends to a limit, say, γ. In the following, we show that $\gamma = 0$. We have, using (2.1), for $m > n$,

$$\delta(X_n, X_{n+p}, X_m) = \delta(Fx_{n-1}, Fx_{n+p-1}, Fx_{m-1})$$

$$\leq \phi \;\; Max\begin{pmatrix} \rho(y_{n-1}, y_{n+p-1}, y_{m-1}), \delta(X_n, X_{n+p}, y_{m-1}), \\ \delta(y_{n-1}, X_n, y_{m-1}), \delta(y_{n+p-1}, X_{n+p}, y_{m-1}), \\ \delta(y_{n-1}, X_{n+p}, y_{m-1}), \delta(X_n, y_{n+p-1}, y_{m-1}), \\ \delta(y_{n-1}, y_{n+p-1}, X_m), \delta(y_{n-1}, X_n, X_m), \\ \delta(y_{n+p-1}, X_{n+p}, X_m), \delta(y_{n-1}, X_{n+p}, X_m), \\ \delta(X_n, y_{n+p-1}, X_m) \end{pmatrix}$$

$$\leq \phi \;\; Max\begin{pmatrix} \delta(X_{n-1}, X_{n+p-1}, X_{m-1}), \delta(X_n, X_{n+p}, X_{m-1}), \\ \delta(X_{n-1}, X_n, X_{m-1}), \delta(X_{n+p-1}, X_{n+p}, X_{m-1}), \\ \delta(X_{n-1}, X_{n+p}, X_{m-1}), \delta(X_n, X_{n+p-1}, X_{m-1}), \\ \delta(X_{n-1}, X_{n+p-1}, X_m), \delta(X_{n-1}, X_n, X_m), \\ \delta(X_{n+p-1}, X_{n+p}, X_m), \delta(X_{n-1}, X_{n+p}, X_m), \\ \delta(X_n, X_{n+p-1}, X_m) \end{pmatrix}$$

$$\leq \phi \;\; Max(\gamma_{n-1}, \gamma_n, \gamma_{n+p-1}) = \phi\ (\gamma_{n-1}).$$

Thus

$$\delta(X_n, X_{n+p}, X_{n+p+t}) \leq \phi(\gamma_{n-1}) \tag{1}$$

Taking supremum over p and t, we get

$$\gamma_n \leq \phi(\gamma_{n-1}).$$

Letting $n \to \infty$, we get

$$\gamma \leq \phi(\gamma) < \gamma, \text{ if } \gamma > 0,$$

which is a contradiction. Hence $\gamma = 0$, i.e., $\gamma_n \to 0$, as $n \to \infty$.

Using (1),

$$\rho(y_n, y_{n+p}, y_{n+p+t}) \leq \delta(X_n, X_{n+p}, X_m) \leq \phi(\gamma_{n-1}).$$

Letting $n \to \infty$, we get

$$Lim_{n\to\infty}\ \rho(y_n, y_{n+p}, y_{n+p+t}) \leq Lim_{n\to\infty}\phi(\gamma_{n-1}) = 0.$$

Hence $\{y_n\}$ is a D-Cauchy sequence in $O(Fg^{-1}, Fx_0)$.

(ii) Let $gx_n \to u$. Using (1),

$$\begin{aligned} Lim_{n\to\infty}\ \delta(Fx_n, Fx_{n+p}, u) &= Lim_{n\to\infty}\ \delta(Fx_n, Fx_{n+p}, gx_m) \\ &\leq Lim_{n\to\infty}\delta(Fx_n, Fx_{n+p}, X_m), \\ &\leq Lim_{n\to\infty}\delta(X_{n+1}, X_{n+p+1}, X_m)\ , \\ &\leq Lim_{n\to\infty}\phi(\gamma_n). \qquad \square \end{aligned}$$

Therefore, $Lim_{n\to\infty}\delta(Fx_n, Fx_{n+p}, u) = 0$, and we get $Fx_n \to \{u\}$ in the D-metric space $(B(X), \delta)$.

3. Main results

The following is a unique common fixed point theorem for a weak-compatible pair of multivalued map and a self-map, both non-continuous, on an unbounded and incomplete D-metric space.

THEOREM 3.1. *Let g be a self-map in a D-metric space (X, ρ) and let $F : X \to CB(X)$ be such that $F(X) \subseteq g(X)$.*

(3.1) *For some $x_0 \in X$ and some $\phi \in \Phi$,*

$$\delta(Fx, Fy, Fz) \leq \phi\ Max \begin{pmatrix} \rho(gx, gy, gz), \delta(Fx, Fy, gz), \delta(gx, Fx, gz), \\ \delta(gy, Fy, gz), \delta(gx, Fy, gz), \delta(gy, Fx, gz), \\ \delta(gx, gy, Fz), \delta(gx, Fx, Fz), \delta(gy, Fy, Fz), \\ \delta(gx, Fy, Fz), \delta(gy, Fx, Fz) \end{pmatrix}$$

for all $x, y \in O(g^{-1}F, x_0)$ and all $z \in X$.

(3.2) *The pair (F, g) is weak-compatible.*

As above, for some $x_0 \in X$, define sequences $\{x_n\}, \{y_n\}$ in X and $\{X_n\}$ in $CB(X)$, by $y_n = gx_n \in Fx_{n-1} = X_n, \forall n \in N$. If, for some $r \in N, y_r = y_{r+1}$, then

(I) $y_r = y_{r+1} = y_{r+2} = \cdots = y_{r+k} = \cdots, \forall k \in N.$

(II) *If $\alpha = y_{r+k}$ for all $k \in N$, then α is the unique common fixed point of F and g.*

Proof. Let $y_r = y_{r+1}$. Then $gx_r = gx_{r+1}$. Let

(2) $$\alpha = gx_{r+1} = gx_r \in Fx_{r+1}.$$

Step I. Using (2) and (3.1), we have

$$\delta(Fx_r, Fx_r, \alpha) = \delta(Fx_r, Fx_r, Fx_r)$$
$$\leq \phi \;\; Max \begin{pmatrix} \rho(gx_r, gx_r, gx_r), \delta(Fx_r, Fx_r, gx_r), \delta(gx_r, Fx_r, gx_r), \\ \delta(gx_r, Fx_r, gx_r), \delta(gx_r, Fx_r, gx_r), \delta(gx_r, Fx_r, gx_r), \\ \delta(gx_r, gx_r, Fx_r), \delta(gx_r, Fx_r, Fx_r), \delta(gx_r, Fx_r, Fx_r), \\ \delta(gx_r, Fx_r, Fx_r), \delta(gx_r, Fx_r, Fx_r) \end{pmatrix}$$
$$\leq \phi \;\; Max \left(0, \delta(Fx_r, Fx_r, \alpha), \delta(Fx_r, \alpha, \alpha) \right)$$
$$\leq \phi \; \left(\delta(Fx_r, Fx_r, \alpha) \right)$$
$$< \delta(Fx_r, Fx_r, \alpha) \;, \quad \text{if} \quad \delta(Fx_r, Fx_r, \alpha) > 0,$$

which is a contradiction. Therefore, $\delta(Fx_r, Fx_r, \alpha) = 0$, which gives $Fx_r = \{\alpha\}$.

Now, using (2), we have

$$\{gx_r\} = Fx_r = \alpha \tag{3}$$

Since (F, g) is weak-compatible, we get

$$F\alpha = g\alpha. \tag{4}$$

Step II. Putting $x = \alpha, y = \alpha$ and $z = x_r$ in (3.1), we get

$$\delta(F\alpha, F\alpha, Fx_r) \leq \phi \;\; Max \begin{pmatrix} \rho(g\alpha, g\alpha, gx_r), \delta(F\alpha, F\alpha, gx_r), \\ \delta(g\alpha, F\alpha, gx_r), \delta(g\alpha, F\alpha, gx_r), \\ \delta(g\alpha, F\alpha, gx_r), \delta(g\alpha, F\alpha, gx_r), \\ \delta(g\alpha, g\alpha, Fx_r), \delta(g\alpha, F\alpha, Fx_r), \\ \delta(g\alpha, F\alpha, Fx_r), \delta(g\alpha, F\alpha, Fx_r), \\ \delta(g\alpha, F\alpha, Fx_r) \end{pmatrix}.$$

Using (3) and (4), we have

$\rho(g\alpha, g\alpha, \alpha) \leq \phi(\rho(g\alpha, g\alpha, \alpha)) < \rho(g\alpha, g\alpha, \alpha)$, if $\rho(g\alpha, g\alpha, \alpha) > 0$, which is not true. Hence $\rho(g\alpha, g\alpha, \alpha) = 0$, which gives $g\alpha = \alpha$. Thus $F\alpha = g\alpha = \alpha$. Therefore, α is a common fixed point of F and g.

Step III. Putting $x = \alpha, y = \alpha$ and $z = x_{r+1}$ in (3.1), we get

$$\delta(F\alpha, F\alpha, Fx_{r+1}) \leq \phi Max \begin{pmatrix} \rho(g\alpha, g\alpha, gx_{r+1}), \delta(F\alpha, F\alpha, gx_{r+1}), \\ \delta(g\alpha, F\alpha, gx_{r+1}), \delta(g\alpha, F\alpha, gx_{r+1}), \\ \delta(g\alpha, F\alpha, gx_{r+1}), \delta(g\alpha, F\alpha, gx_{r+1}), \\ \delta(g\alpha, g\alpha, Fx_{r+1}), \delta(g\alpha, F\alpha, Fx_{r+1}), \\ \delta(g\alpha, F\alpha, Fx_{r+1}), \delta(g\alpha, F\alpha, Fx_{r+1}), \\ \delta(g\alpha, F\alpha, Fx_{r+1}) \end{pmatrix}$$

implies

$\delta(\alpha, \alpha, Fx_{r+1}) \leq \phi \;\; Max \left(\rho(\alpha, \alpha, gx_{r+1}), \delta(\alpha, \alpha, Fx_{r+1}) \right).$

Using (1), we have

$\rho(\alpha, \alpha, Fx_{r+1}) \leq \phi \;\; \{(\delta(\alpha, \alpha, Fx_{r+1})\} < \delta(\alpha, \alpha, Fx_{r+1}),$

if $\delta(\alpha, \alpha, Fx_{r+1}) > 0$. Thus $\delta(\alpha, \alpha, Fx_{r+1}) = 0$, which gives $Fx_{r+1} = \{\alpha\}$. Since $y_{r+2} \in Fx_{r+1}$, we have $y_{r+2} = \alpha$. Therefore, $y_r = y_{r+1} = y_{r+2} = \alpha$.

Similarly, we shall have $y_r = y_{r+1} = y_{r+2} = \cdots = \alpha$. Thus $y_{r+k} = \alpha$ for all $k \in N$.

Step IV. (Uniqueness) Let w be another common fixed point of F and g. Then

$$w = Fw = gw. \tag{5}$$

Putting $x = \alpha, y = \alpha$ and $z = w$ in (3.1) and using (5), we get

$$\delta(F\alpha, F\alpha, Fw) \leq \phi \;\; Max \begin{pmatrix} \rho(g\alpha, g\alpha, gw), \delta(F\alpha, F\alpha, gw), \\ \delta(g\alpha, F\alpha, gw), \delta(g\alpha, F\alpha, gw), \\ \delta(g\alpha, F\alpha, gw), \delta(g\alpha, F\alpha, gw), \\ \delta(g\alpha, g\alpha, Fw), \delta(g\alpha, F\alpha, Fw), \\ \delta(g\alpha, F\alpha, Fw), \delta(g\alpha, F\alpha, Fw), \\ \delta(g\alpha, F\alpha, Fw) \end{pmatrix}$$

implies

$\delta(\alpha, \alpha, w) \leq \phi(\delta(\alpha, \alpha, w)) < \delta(\alpha, \alpha, w), \qquad$ if $\delta(\alpha, \alpha, w) > 0$,

which is a contradiction. Therefore, $\delta(\alpha, \alpha, w) = 0$, i.e., $\alpha = w$. Hence α is the unique common fixed point of F and g. □

In [4], Dhage, Jennifer and Kang proved the following:

Theorem 3.2. ([4]) *Let X be a D -metric space and let $F : X \to CB(X)$ and $g : X \to X$ be two mappings satisfying, for some positive number r,*

$$\delta^r(Fx, Fy, Fz) \leq \phi \quad Max \begin{pmatrix} \rho^r(gx, gy, gz), \delta^r(Fx, Fy, gz), \\ \delta^r(gx, Fx, gz), \delta^r(gy, Fy, gz), \\ \delta^r(gx, Fy, gz), \delta^r(gy, Fx, gz) \end{pmatrix}$$

for all $x, y, z \in X$, where $\phi : R^+ \to R^+$ is non-decreasing, $\phi(t) < t, t > 0$, and $\sum \phi^n(t) < \infty$ for each $t \in R^+$. Further, suppose that

(a) $F(X) \subseteq g(X)$,

(b) $g(X)$ *is bounded and complete,*

(c) $\{F, g\}$ *is coincidentally commuting.*

Then F and g have a unique fixed point $u \in X$ such that $Fu = \{u\} = gu$.

The following theorem generalizes the result of [4] significantly for a weak-compatible pair of a multivalued map and a self-map, on an unbounded and incomplete D-metric space.

Theorem 3.3. *Let g be a self map in a D-metric space (X, ρ) and $F : X \to CB(X)$ with δ continuous in two variables satisfying* (3.1) *and*

(3.3) $F(X) \subseteq g(X)$,

(3.4) *one of $F(X)$ or $g(X)$ is complete,*

(3.5) *the pair (F, g) is weak compatible,*

(3.6) *there exists $x_0 \in X$ such that $F(\{x_n\}) = \bigcup_{i \in X} X_i$ is bounded, where $y_{n+1} = gx_{n+1} \in Fx_n = X_{n+1}$ for all $n \in N$.*

Then F and g have the unique common fixed point in X.

Proof. For $x_0 \in X$, construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_n = gx_n \in Fx_{n-1}, \forall n \in N$. Therefore, by Lemma 2.2, $\{y_n\} = \{gx_n\}$ is a D-Cauchy sequence in $g(X)$.

CASE 1. ($g(X)$ is complete) Since $g(X)$ is complete,

$$y_n = gx_n \to u \in g(X). \tag{6}$$

Therefore, there exists $v \in X$ such that

$$u = gv. \tag{7}$$

Step1. Putting $x = x_n, y = x_n, z = v$ in condition (3.1), we get
$\delta(Fx_n, Fx_n, Fv)$

$$\leq \phi \;\; Max \begin{pmatrix} \rho(gx_n, gx_n, gv), \delta(Fx_n, Fx_n, gv), \delta(gx_n, Fx_n, gv), \\ \delta(gx_n, Fx_n, gv), \delta(gx_n, Fx_n, gv), \delta(gx_n, Fx_n, gv), \\ \delta(gx_n, gx_n, Fv), \delta(gx_n, Fx_n, Fv), \delta(gx_n, Fx_n, Fv), \\ \delta(gx_n, Fx_n, Fv), \delta(gx_n, Fx_n, Fv) \end{pmatrix}$$

Letting $n \to \infty$ and using (6), (7) and Lemma 2.2, we get
$\delta(u, u, Fv) \leq \phi\delta(u, u, Fv) < \delta(u, u, Fv), \qquad \text{if } \delta(u, u, Fv) > 0,$
which is a contradiction. Thus $\delta(u, u, Fv) = 0$, which gives $u = Fv$. Hence $u = gv = Fv$. Since (F, g) is weak-compatible, we obtain

$$Fu = gu. \tag{8}$$

Step 2. Putting $x = x_n, y = x_n and z = u$ in condition (3.1), we get
$\delta(Fx_n, Fx_n, Fu)$

$$\leq \phi \;\; Max \begin{pmatrix} \rho(gx_n, gx_n, gu), \delta(Fx_n, Fx_n, gu), \delta(gx_n, Fx_n, gu), \\ \delta(gx_n, Fx_n, gu), \delta(gx_n, Fx_n, gu), \delta(gx_n, Fx_n, gu), \\ \delta(gx_n, gx_n, Fu), \delta(gx_n, Fx_n, Fu), \delta(gx_n, Fx_n, Fu), \\ \delta(gx_n, Fx_n, Fu), \delta(gx_n, Fx_n, Fu) \end{pmatrix}.$$

Letting $n \to \infty$ and using (6), (8) and Lemma 2.2, we get
$\delta(u, u, Fu) \leq \phi\delta(u, u, gu) < \delta(u, u, Fu), \qquad \text{if } \delta(u, u, Fu) > 0,$
which is a contradiction. Thus $\delta(u, u, Fu) = 0$, which gives $u = Fu$. Hence $u = gu = Fu$. Therefore, u is a common fixed point of F and g.

CASE 2. (When $F(X)$ is complete) Since $y_n \in Fx_{n-1}, y_n \in F(X)$ for all $n \in N$. $\{y_n\}$ is a D-Cauchy sequence in $F(X)$, which is complete. Therefore, $\{y_n\} \to u \in F(X) \subseteq g(X)$. Hence $u \in g(X)$, i.e., $u = gv$ for some $v \in X$. The rest follows as in Case 1.

Step 3. (Uniqueness) Let w be another common fixed point of F and g. Then

$$w = Fw = gw. \tag{9}$$

Since $y_n \to u, gx_n \to u$. Hence by using (ii) of Lemma 2.2,

$$F x_n \to \{u\} \tag{10}$$

Taking $x = x_n, y = x_n$ and $z = w$ in condition (3.1), we get

$$\delta(Fx_n, Fx_n, Fw) \le \phi \ \ Max \begin{pmatrix} \rho(gx_n, gx_n, gw), \delta(Fx_n, Fx_n, gw), \delta(gx_n, Fx_n, gw), \\ \delta(gx_n, Fx_n, gw), \delta(gx_n, Fx_n, gw), \delta(gx_n, Fx_n, gw), \\ \delta(gx_n, gx_n, Fw), \delta(gx_n, Fx_n, Fw), \delta(gx_n, Fx_n, Fw), \\ \delta(gx_n, Fx_n, Fw), \delta(gx_n, Fx_n, Fw) \end{pmatrix}.$$

Letting $n \to \infty$ and using (6), (9) and Lemma 2.2, we get

$\delta(u, u, w) \le \phi\delta(u, u, w) < \delta(u, u, w), \qquad$ if $\delta(u, u, w) > 0$,

which is a contradiction. Thus $\delta(u, u, w) = 0$, which gives $u = w$. Therefore, u is the unique common fixed point of F and g. □

Note that (1) if (3.1) holds for all $x, y, z \in X$, then continuity of g at u implies continuity of F at u in view of the uniqueness of the fixed point and of (ii) of Lemma 2.2,

(2) the power of r in ρ and δ in the result of [4] gets cancelled throughout. Hence it is insignificant.

Remark 1. Theorem 3.3 generalizes the result of [4] in the following sense: (a) The contractive condition of theorem 3.3 contains eleven factors in the right. Therefore, the contraction taken in our Theorem 3.3 is more general than that of [4].

(b) The function ϕ taken in Theorem 3.3 is less restrictive than that of [5] as $\sum \phi^n(t)$ need not to be summable in our Theorem 3.3.

(c) In Theorem 3.3 , $F(\{x_n\}) = \bigcup_i X_i = \bigcup_i F(x_{i-1}) = \bigcup_n F(x_n) \subseteq F(X) \subseteq g(X)$ is assumed to be bounded. Hence the domain $g(X)$ of boundedness of [4] is larger than that one in our theorem 3.3.

In [3], Dhage has established the following result for two single valued maps:

THEOREM 3.4. ([3]) *Let f and g be any two self-maps of a D-metric space X satisfying*

$$\rho(fx, fy, fz) \leq \lambda\rho(gx, gy, gz),$$

for all $x, y, z \in X$ *and for* $0 \leq \lambda < 1$. *Further, suppose that*

(a) $f(X) \subseteq g(X)$,

(b) *any one of* $f(X)$ *or* $g(X)$ *is complete,*

(c) f *and* g *are coincidentally commuting.*

Then f *and* g *have a unique common fixed point.*

Taking F to be a single-valued map, we have the following corollary of Theorem 3.3:

Corollary 3.5. *Let* F *and* g *be self-maps on a D-metric space* (X, ρ), *where* ρ *is continuous in two variables satisfying* (3.3), (3.4), (3.5), (3.6) *and*

$$\rho(Fx, Fy, Fz \leq \phi\left\{\rho(gx, gy, gz)\right\}$$

for all $x, y \in O(g^{-1}F, x_0), z \in X$. *Then* F *and* g *have a unique common fixed point in* X.

Proof. The result follows from Theorem 3.3, by restricting maximum to only first factor of (3.1). □

Remark 2. Even Corollary 3.5 generalizes the result of [3] by taking $\phi(t) = \lambda t, \forall t \in R^+$, for some $0 \leq \lambda < 1$. Generalization is in the sense of domains of variables x and y and non-summability of ϕ.

In [11], Rhoades proved the following:

Theorem 3.6. ([11]) *Let* X *be a complete and bounded D-metric space, and let* f *be a self map of* X *satisfying*

$$\rho(fx, fy, fz) \leq q\ Max \begin{pmatrix} \rho(x, y, z), \rho(fx, x, z), \rho(fy, y, z), \\ \rho(x, fy, z), \rho(y, fx, z), \end{pmatrix}$$

for all $x, y, z \in X$, *where* $0 \leq q < 1$. *Then* f *has a unique fixed point* p *in* X *and* f *is continuous at* p.

The following corollary of Theorem 3.3 is a significant generalization of it:

COROLLARY 3.7. *Let F be a self map on a complete D-metric space (X,ρ), in which ρ is continuous in two variables, such that for some $x_0 \in X$, orbit $O(F,x_0)$ is bounded and*

$$\rho(Fx,Fy,Fz) \leq \phi \;\; Max \begin{pmatrix} \rho(x,y,z), \rho(Fx,Fy,z), \rho(x,Fx,z), \\ \rho(y,Fy,z), \rho(x,Fy,z), \rho(y,Fx,z), \\ \rho(x,y,Fz), \rho(x,Fx,Fz), \\ \rho(y,Fy,Fz), \\ \rho(x,Fy,Fz), \rho(y,Fx,Fz) \end{pmatrix}$$

for all $x,y \in O(F,x_0)$ and all $z \in X$. Then F has a unique fixed point in X.

Proof. The result follows from Theorem 3.3 by taking $g = I$. Since F is a single valued, $\delta = \rho$. □

Remark 3. The above corollary generalizes the result of [11] by taking $\phi(t) = \lambda t, \forall t \in R^+$. Here,

(a) ϕ is less restrictive (not requiring summability) than q of [11].

(b) Contractive condition of Corollary 3.7 is more general than that of the contractive condition of the result of [11].

(c) Domains of x, y and of boundedness in above corollary is less than that of result of [11].

It is to be noted that the mentioned continuity of a D-metric ρ in two variables is necessary, as there are examples of D-metric spaces in which ρ is not continuous even in one variable.

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