

COMMON RANDOM FIXED POINTS OF RANDOM MULTIVALUED OPERATORS ON POLISH SPACES

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ABSTRACT. In this paper, we prove the existence of a common random fixed point of two random multivalued generalized contractions by using functional expressions.

1. Introduction

Random fixed point theorems are stochastic generalization of classical fixed point theorems [6, 15]. Itoh [8, 9] extended several well known fixed point theorems, i.e., for contraction, nonexpansive and condensing, mappings to the random case. Thereafter, various stochastic aspects of Schauder's fixed point theorem have been studied by Sehgal and Singh [14], Papageorgiou [13], Lin [11] and many authors. In a separable metric space, random fixed point theorems for contractive mappings were proved by Spacek [15], Hans [5, 6, 7], Mukherjee [12]. Afterwards, Beg and Shahzad [3, 4], Badshah and Sayyad [1] studied the structure of common random fixed points and random coincidence points of a pair of compatible random operators and proved the random fixed points theorems for contraction random operators in Polish spaces.

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This paper is in continuation of these investigations and proves the existence of a common random fixed point of the random multivalued generalized contractions with functional expressions.

2. Preliminaries

Let (X, d) be a Polish space, that is, a separable complete metric space, and let (Ω, \mathcal{A}) be a measurable space. Let 2^X be a family of all subsets of X and $CB(X)$ denote the family of all non-empty bounded closed subsets of X . A mapping $T : \Omega \rightarrow 2^X$ is called *measurable* if for all open subsets C of X , $T^{-1}(C) = \{w \in \Omega : T(w) \cap C \neq \phi\} \in \mathcal{A}$. A mapping $\xi : \Omega \rightarrow X$ is said to be *measurable selector* of a measurable mapping $T : \Omega \rightarrow 2^X$ if ξ is measurable and $\xi(w) \in T(w)$ for all $w \in \Omega$. A mapping $f : \Omega \times X \rightarrow X$ is called a *random operator* if for all $x \in X$, $f(\cdot, x)$ is measurable. A mapping $T : \Omega \times X \rightarrow CB(X)$ is called a *random multivalued operator* if for every $x \in X$, $T(\cdot, x)$ is measurable. A measurable mapping $T : \Omega \times X$ is called a *random fixed point* of a random multivalued operator $T : \Omega \times X \rightarrow CB(X)$ ($f : \Omega \times X \rightarrow X$) if for every $w \in \Omega$, $\xi(w) \in T(w, \xi(w))$ ($f(w, \xi(w)) = \xi(w)$). Let $T : \Omega \times X \rightarrow CB(X)$ be a random operator and $\{\xi_n\}$ a sequence of measurable mappings $\xi_n : \Omega \rightarrow X$. The sequence $\{\xi_n\}$ is said to be *asymptotically T -regular* if $d(\xi_n(w), T(w, \xi_n(w))) \rightarrow 0$.

Jungck [10] gave the notion of commuting mappings by showing that two self-maps S, T on a complete metric space satisfying a contractive condition have a common fixed point. Beg and Azam [2] further extended it to the case of pairs of multivalued mappings satisfying a more general contractive type condition. Later on, Beg and Shahzad [4] generalized and proved common random fixed point theorems for random multivalued operators on metric spaces.

3. Main results

In this section we give stochastic version of a result of Beg and

Shahzad [4].

THEOREM 1. *Let X be a Polish space. Let $T, S : \Omega \times X \rightarrow CB(X)$ be two continuous random multivalued operators. If there exist measurable mappings $\alpha, \beta : \Omega \rightarrow (0, 1)$ such that*

$$H(S(w, x), T(w, y)) \leq \alpha(w) \frac{d(x, S(w, x))^2 + d(y, T(w, y))^2}{d(x, S(w, x)) + d(y, T(w, y))} + \beta(w)d(x, y)$$

for each $x, y \in X, w \in \Omega$ and $\alpha, \beta \in \mathbb{R}^+$ with $2\alpha(w) + \beta(w) < 1$, there exists a common random fixed point of S and T (hence H represents the Hausdroff metric on $CB(X)$ induced by the metric d).

Proof. Let $\xi_0 : \Omega \rightarrow X$ be an arbitrary measurable mapping and choose a measurable mapping $\xi_1 : \Omega \rightarrow X$ such that $\xi_1(w) \in S(w, \xi_0(w))$ for each $w \in \Omega$. Then for each $w \in \Omega$,

$$\begin{aligned} & H(S(w, \xi_0(w)), T(w, \xi_1(w))) \\ & \leq \alpha(w) \frac{d(\xi_0(w), S(w, \xi_0(w)))^2 + d(\xi_1(w), T(w, \xi_1(w)))^2}{d(\xi_0(w), S(w, \xi_0(w))) + d(\xi_1(w), T(w, \xi_1(w)))} \\ & \quad + \beta(w)d(\xi_0(w), \xi_1(w)). \end{aligned}$$

Further, there exists a measurable mapping $\xi_2 : \Omega \rightarrow X$ such that for all $w \in \Omega$, $\xi_2(w) \in T(w, \xi_1(w))$ and

$$\begin{aligned} & d(\xi_1(w), \xi_2(w)) \leq \alpha(w) \frac{d(\xi_0(w), \xi_1(w))^2 + d(\xi_1(w), \xi_2(w))^2}{d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))} \\ & \quad + \beta(w)d(\xi_0(w), \xi_1(w)) \\ & \leq \alpha(w) \frac{\{d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))\}^2 - 2d(\xi_0(w), \xi_1(w))d(\xi_1(w), \xi_2(w))}{d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))} \\ & \quad + \beta(w)d(\xi_0(w), \xi_1(w)) \\ & \leq \alpha(w)[d(\xi_0(w), \xi_1(w)) + d(\xi_1(w), \xi_2(w))] + \beta(w)d(\xi_0(w), \xi_1(w)). \end{aligned}$$

So

$$(1 - \alpha(w))d(\xi_1(w), \xi_2(w)) \leq (\alpha(w) + \beta(w))d(\xi_0(w), \xi_1(w)).$$

Thus

$$d(\xi_1(w), \xi_2(w)) \leq k(w)d(\xi_0(w), \xi_1(w)),$$

where $k = k(w) = \frac{\alpha(w)+\beta(w)}{1-\alpha(w)} < 1$.

By Beg and Shahzad [3, Lemma 2.3], we obtain a measurable mapping $\xi_3 : \Omega \rightarrow X$ such that for all $w \in \Omega$, $\xi_3(w) \in S(w, \xi_2(w))$ and

$$\begin{aligned} d(\xi_2(w), \xi_3(w)) \leq & \alpha(w) \frac{d(\xi_1(w), \xi_2(w))^2 + d(\xi_2(w), \xi_3(w))^2}{d(\xi_1(w), \xi_2(w)) + d(\xi_2(w), \xi_3(w))} \\ & + \beta(w)d(\xi_1(w), \xi_2(w)). \end{aligned}$$

Hence

$$\begin{aligned} d(\xi_2(w), \xi_3(w)) \leq & \alpha(w)[d(\xi_1(w), \xi_2(w)) + d(\xi_2(w), \xi_3(w))] \\ & + \beta(w)d(\xi_1(w), \xi_2(w)). \end{aligned}$$

So

$$(1 - \alpha(w))d(\xi_2(w), \xi_3(w)) \leq (\alpha(w) + \beta(w))d(\xi_1(w), \xi_2(w)).$$

Thus

$$d(\xi_2(w), \xi_3(w)) \leq k^2 d(\xi_0(w), \xi_1(w)).$$

Similarly, proceeding in the same way, by induction, we get a sequence of measurable mappings $\xi_n : \Omega \rightarrow X$ such that for $n > 0$ and for any $w \in \Omega$,

$$\xi_{2n+1}(w) \in S(w, \xi_{2n}(w)), \quad \xi_{2n+2}(w) \in T(w, \xi_{2n+1}(w))$$

and

$$d(\xi_n(w), \xi_{n+1}(w)) \leq kd(\xi_{n-1}(w), \xi_n(w)) \leq \cdots \leq k^n d(\xi_0(w), \xi_1(w)).$$

Further, for $m > n$,

$$\begin{aligned} d(\xi_n(w), \xi_m(w)) &\leq d(\xi_n(w), \xi_{n+1}(w)) + \cdots + d(\xi_{m-1}(w), \xi_m(w)) \\ &\leq (k^n + k^{n+1} + \cdots + k^{m-1})d(\xi_0(w), \xi_1(w)) \\ &\leq \frac{k^n}{1-k}d(\xi_0(w), \xi_1(w)), \end{aligned}$$

which tends to zero as $n \rightarrow \infty$. It follows that $\{\xi_n(w)\}$ is a Cauchy sequence and there exists a measurable mapping $\xi : \Omega \rightarrow X$ such that $\xi_n(w) \rightarrow \xi(w)$ for each $w \in \Omega$. It implies that $\xi_{2n+1}(w) \rightarrow \xi(w)$ and $\xi_{2n+2}(w) \rightarrow \xi(w)$. Thus we have for any $w \in \Omega$,

$$\begin{aligned} d(\xi(w), S(w, \xi(w))) &\leq d(\xi(w), \xi_{2n+2}(w)) + d(\xi_{2n+2}(w), S(w, \xi(w))) \\ &\leq d(\xi(w), \xi_{2n+2}(w)) + H(T(w, \xi_{2n+1}(w)), S(w, \xi(w))). \end{aligned}$$

Therefore,

$$\begin{aligned} d(\xi(w), S(w, \xi(w))) &\leq d(\xi(w), \xi_{2n+2}(w)) \\ &+ \alpha(w) \frac{d(\xi(w), S(w, \xi(w)))^2 + d(\xi_{2n+1}(w), T(w, \xi_{2n+1}(w)))^2}{d(\xi(w), S(w, \xi(w))) + d(\xi_{2n+1}(w), T(w, \xi_{2n+1}(w)))} \\ &+ \beta(w)d(\xi(w), \xi_{2n+1}(w)). \end{aligned}$$

Taking $n \rightarrow \infty$, we have

$$d(\xi(w), S(w, \xi(w))) \leq \alpha(w)d(\xi(w), S(w, \xi(w))).$$

Hence $\xi(w) \in S(w, \xi(w))$ for all $w \in \Omega$.

Similarly, for any $w \in \Omega$,

$$\begin{aligned} d(\xi(w), T(w, \xi(w))) &\leq \alpha(w)d(\xi(w), \xi_{2n+1}(w)) \\ &+ H(S(w, \xi_{2n}(w)), T(w, \xi(w))) \\ &\leq \alpha(w)d(\xi(w), T(w, \xi(w))). \end{aligned}$$

Hence $\xi(w) \in T(w, \xi(w))$ for all $w \in \Omega$. □

COROLLARY 2. Let X be a Polish space and $T : \Omega \times X \rightarrow CB(X)$ a continuous random multivalued operator. If there exist measurable mappings $\alpha, \beta : \Omega \rightarrow (0, 1)$ such that for each $x, y \in X$ and $w \in \Omega$

$$H(T(w, x), T(w, y)) \leq \alpha(w) \frac{d(x, T(w, x))^2 + d(y, T(w, y))^2}{d(x, T(w, x)) + d(y, T(w, y))} + \beta(w)d(x, y).$$

Then there exists a sequence $\{\xi_n\}$ of measurable mappings $\xi_n : \Omega \rightarrow X$ which is asymptotically T -regular and converges to a random point of T .

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