JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **18**, No.1, April 2005

COHOMOLOGY AND GENERALIZED GOTTLIEB GROUPS

Kee Young Lee*

Abstract. In this paper, we observe the relation between the concept of generalized Gottlieb groups and the Hurewicz homomorphism.

1. Introduction and preliminary

Let X be a pointed CW-complex. Consider a continuous map $\phi : X \times S^n \to X$ such that $\phi(x, *) = x$, where $*$ is a base point of S^n. Then $g : S^n \to X$ defined by $g(s) = \phi(*, s)$ represents an element $[g] \in \pi_n(X)$. In this case, ϕ is called an *affiliated map* of g and g is a *cyclic map*. The set of all element $[g] \in \pi_n(X)$ obtained in the above manner from ϕ is denoted by $G_n(X)$ and called a *Gottlieb group* or an *evaluation subgroup* of the homotopy group [1]. That is, the n-th Gottlieb group $G_n(X)$ consists of those $\alpha \in \pi_n(X)$ for which there is a map $\phi : X \times S^n \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} X \times S^n & \xrightarrow{\phi} & X \\ \uparrow J & & \uparrow \nabla \\ X \vee S^n & \xrightarrow{1_X \vee f} & X \vee X \end{array}$$

where $f : S^n \to X$ is a representative of α and ∇ is a folding map.

The Gottlieb groups of a space have been generalized to certain subgroups of the homotopy groups by Woo and Kim[14].

Received by the editors on November 17, 2004.
2000 *Mathematics Subject Classifications* : Primary 55E05; Secondary 55E40, 55B20.
Key words and phrases: Gottlieb group, evaluation subgroup, cyclic map.

Let $(X,*)$ and $(A,*)$ be any two pointed topological spaces and $f:(A,*)\to(X,*)$ be a fixed map. Consider the continuous map $\phi : A\times S^n \to X$ such that $\phi(a,*)=f(a)$. Then $g:S^n\to X$ defined by $g(s)=\phi(*,s)$ represents an element $[g]\in\pi_n(X)$. The set of all element $[g]\in\pi_n(X)$ obtained in the above manner from ϕ is denoted by $G_n^f(X,A)$ and called generalized Gottlieb groups. Especially, if $f=i:A\to X$ is an inclusion, then $G_n^f(X,A)$ is denoted by $G_n(X,A)$.

In [7], the author and Woo have defined and studied relative evaluation subgroups $G_n^{Rel}(X,A)$ of relative homotopy groups $\pi_n(X,A)$. Moreover, we showed that for a CW-pair (X,A), $G_n(X)$, $G_n(X,A)$ and $G_n^{Rel}(X,A)$ make a sequence

$$\begin{aligned}\cdots\to G_n(A)\xrightarrow{i_*}G_n(X,A)\xrightarrow{j_*}G_n^{\mathrm{Rel}\,l}(X,A)\xrightarrow{\partial}\cdots\\ \to G_1^{\mathrm{Rel}\,l}(X,A)\to G_0(A)\to G_0(X,A),\end{aligned}$$

where i_*, j_* and ∂ are restrictions of the usual homomorphisms of the homotopy sequence

$$\cdots\to\pi_n(A)\xrightarrow{i_*}\pi_n(X)\xrightarrow{j_*}\pi_n(X,A)\xrightarrow{\partial}\cdots\to\pi_0(A)\to\pi_0(X).$$

This sequence is called the *G-sequence* of (X,A). It was shown that if the inclusion $i:A\to X$ has a left homotopy inverse[7] or is homotopic to a constant map [7], then the G-sequence of the CW-pair (X,A) is exact.

Let $h:\pi_n(X)\to H_n(X;Z)$ be the Hurewicz homomorphism. We shall defined $h_p:\pi_n(X)\to H_n(X;Z)\to H_n(X;Z_p)$ as composition of h tensored with Z_p. h_p will be called *the mod p Hurewicz homomorphism.* We shall let h_∞ stands for the Hurewicz homomorphism $h_\infty:\pi_n(X)\to H_n(X;Q)$, where Q is the rational field.

In [6], the author and Woo have studied algebraic structure induced by $\phi: A\times S^n\to X$ affiliated to some $\alpha\in G_n^f(X,A)$ on the homology and proved following theorems.

THEOREM 1.1. *Let X and A be topological spaces and A has a finitely generated integer homology and $f : A \to X$ be a map which has a left homotopy inverse r. If n is an odd integer, then $G_n^f(X, A)$ is contained in the kernel of $r_* h_p$, for any prime number p or ∞ provided $\chi(A) \neq 0$.*

THEOREM 1.2. *Let X be a topological space and A be topological space with finitely generated integer homology and $f : A \to X$ be a map with left homotopy inverse r. Suppose p is a prime number which does not divide $\chi(A)$. Then $G_n^f(X, A) \subset \ker r_* h_p$ for even n.*

THEOREM 1.3. *Let A be a retract of CW-complex X. Then $G_n^i(X, A) \subset \ker r_* h_p$ and $G_n^{Rel}(X, A) \subset \operatorname{Ker} k_p$ if and only if $G_n^i(X, A) \subset \ker h_p$ where $i : A \to X$ be an inclusion and r is a retraction and h_p and k_p are Hurewicz homomorphisms tensored with Z_p for all prime number p.*

In this paper, we study algebraic structure induced by $\phi : A \times S^n \to X$ affiliated to some $\alpha \in G_n^f(X, A)$ on the cohomology and prove the following theorem.

THEOREM 1.4. *Let X and A be CW-complexes and $f : A \to X$ be a map which has a left homotopy inverse r. If A has only a finite number of nonzero homology group, then $G_{2n}^f(X, A) \subset \ker r_* h_\infty$.*

As a corollary, we have the following result due to Gottlieb [1].

COROLLARY 1.5. *Let X be CW-complex which has only a finite number of nonzero homology group, then $G_{2n}(X) \subset \ker h_\infty$.*

Moreover, we have the following corollary. Here we denote the relative version of h_∞ by k_∞.

COROLLARY 1.6. *Let A be a retract of X and have a finite number of nonzero homology groups. Then $G_n^{Rel}(X, A) \subset \operatorname{Ker} k_\infty$ if and only if $G_{2n}(X, A) \subset \operatorname{Ker} h_\infty$.*

Throughout this paper, all spaces are connected and based CW-complexes, all maps and all homotopies are based.

2. Some consequences of an affiliated map on cohomology groups

In this section, we study the cohomology effect on $G_n^f(X,A)$ and $G_n^{Rel}(X,A)$. Let $\phi : A \times S^n \to X$ be a map such that $\phi|_A = f$.

In [6], we studied the homology effect on $G_n^f(X,A)$ and $G_n^{Rel}(X,A)$. By the Künneth formula and the fact that $H_*(S^n;Z)$ has no torsion, we have

$$\mu : H_*(A;G) \otimes H_*(S^n;Z) \cong H_*(A \times S^n;G)$$

Thus if $x \in H_*(A \times S^n;G)$, $x = \mu(y \otimes 1 + z \otimes \lambda)$, where $\lambda \in H_n(S^n;Z)$ is a fundamental class and $y, z \in H_*(A;G)$. We shall denote $\mu(z \otimes z')$ by $z \times z'$. Furthermore, we showed that for the affiliated map $\phi : A \times S^n \to X$ with respect to f with trace g, $\phi_*(1 \times \lambda) = g_*(\lambda)$ and if f has a left homotopy inverse r, then $r\phi$ induces a homomorphism

$$K_\lambda : H_q(A;G) \to H_{q+n}(A;G)$$

given by $K_\lambda(x) = r_*\phi_*(x \times \lambda)$.

Similarly, we obtain their dualities on cohomology groups.

By the Künneth formula and the fact that $H^*(S^n;Z)$ has no torsion, we have

$$\Theta : H^*(A;G) \otimes H^*(S^n;Z) \cong H^*(A \times S^n;G)$$

Thus if $x \in H^0(A \times S^n;G)$, $x = \Theta(y \otimes 1 + z \otimes \overline{\lambda}$ where $\overline{\lambda} \in H^n(S^n;Z)$ is a fundamental class of S^n dual to λ. We shall denote $\Theta(z \otimes z')$ by $z \times z'$.

Proposition 2.1. *Let $i_1 : A \to A \times S^n$ be the map given by $i_1(x) = (x,*)$ and let $i_2 : S^n \to A \times S^n$ given by $i_2(s) = (*,s)$ and p_1, p_2 be the natural projections from $A \times S^n$ to A and S^n,respectively. Then $p_1^*(x) = x \times 1$, $p_2^*(\overline{\lambda}) = 1 \times \overline{\lambda}$ and $i_1^*(z \times 1) = z$, $i_1^*(z \times z') = 0$ unless $z' \in H^0(S^n;Z)$ and $i_2^*(1 \times z') = z'$, $i_2^*(z \times z') = 0$ unless $z \in H^0(A;G)$.*

Proof. See p.249 [13]. □

PROPOSITION 2.2. *Let $\phi : A \times S^n \to X$ be a map such that $\phi i_1 = f$ and r is a left homotopy inverse of f. Then $r\phi$ induces a homomorphism $K_{\overline{\lambda}} : H^p(A;G) \to H^{p-n}(A;G)$, where G is a field.*

Proof. Since $\phi^* r^*(x) \in H^q(A \times S^n;G)$, $\phi^* r^*(x) = z \times 1 + y \times \overline{\lambda}$. Moreover, we have

$$x = i_1^* \phi^* r^*(x) = i_1^*(z \times 1) + i_1^*(y \times \overline{\lambda})$$

because $i_1^* \phi^* r^* = (r\phi i_1)^* = (rf)^* = 1$. By Proposition 2.1, $x = z$ and thus $\phi^* r^*(x) = x \times 1 + y \times \overline{\lambda}$ for some $y \in H^*(A;G)$ is a free module over G. y is determined uniquely. Define $K_{\overline{\lambda}}(x) = y$. Then $K_{\overline{\lambda}} : H^p(A;G) \to H^{p-n}(A;G)$ is a homomorphism. In fact, if $K_{\overline{\lambda}}(x) = y$ and $K_{\overline{\lambda}}(x') = y'$, then

$$\phi^* r^*(x + x') = \phi^* r^*(x) + \phi^* r^*(x') = x \times 1 + y \times \overline{\lambda} + x' \times 1 + y' \times \overline{\lambda}.$$

So we have $K_{\overline{\lambda}}(x + x') = y + y' = K_{\overline{\lambda}}(x) + K_{\overline{\lambda}}(x')$. □

Here we define $K_{\overline{\lambda}}^n(x) = K_{\overline{\lambda}}(K_{\overline{\lambda}}^{n-1}(x))$.

PROPOSITION 2.3. *$K_{\overline{\lambda}} : H^p(A;G) \to H^{p-n}(A;G)$ is dual to $k_\lambda : H_q(A;G) \to H_{q+n}(A;G)$.*

Proof. Let us denote the Kronecker product of x and y by $\langle x, y \rangle$. Then we have

$$\begin{aligned}\langle u, K_\lambda(v) \rangle &= \langle u, r_* \phi_*(v \times \lambda) \rangle \\ &= \langle \phi^* r^*(u), v \times \lambda \rangle \\ &= \left\langle u \times 1 + K_{\bar{\lambda}}(u) \times \bar{\lambda}, v \times u \right\rangle \\ &= \langle u \times 1, v \times \lambda \rangle + \left\langle K_{\bar{\lambda}}(u) \times \bar{\lambda}, v \times \lambda \right\rangle \\ &= \langle u, v \rangle \langle 1, \lambda \rangle + \langle K_{\bar{\lambda}}(u), v \rangle \left\langle \bar{\lambda}, \lambda \right\rangle \\ &= \langle K_{\bar{\lambda}}(u), v \rangle,\end{aligned}$$

for $\langle u, v \rangle = 0$ and $\langle \bar{\lambda}, \lambda \rangle = 1$. □

The cup product in $A \times S^n$ is given by

$$(x \times y) \cup (x' \times y') = (-1)^{qr}((x \cup x') \times (y \cup y'))$$

where $x' \in H^q(A;G)$ and $y \in H^r(S^n;Z)$. Thus

$$\begin{aligned}\phi^*r^*(u \cup v) &= \phi^*r^*(u) \cup \phi^*r^*(v)\\ &= (u \times 1 + K_{\bar\lambda}(u) \times \bar\lambda) \cup (v \times 1 + K_{\bar\lambda}(v) \times \bar\lambda)\\ &= (u \cup v) \times 1 + [(u \cup K_{\bar\lambda}(v)) + (-1)^{n \dim v}(K_{\bar\lambda}(u) \cup v)] \times \bar\lambda.\end{aligned}$$

The map $g : S^n \to X$ given by $g = \phi i_2$ plays an important role. Let $x \in H^n(A;Z)$. Then

$$\begin{aligned}g^*r^*(x) &= i_2^*\phi^*r^*(x)\\ &= i_2^*(x \times 1 + K_{\bar\lambda}(x) \times \bar\lambda)\\ &= i_2^*(K_{\bar\lambda}(x) \times \bar\lambda)\\ &= m \cdot \bar\lambda \in H^n(S^n;Z),\end{aligned}$$

for some integer m. Since $K_{\bar\lambda}(x) \in H^0(A;Z)$, $K_{\bar\lambda}(x) = s \cdot 1$ for some integer s, where 1 is the generator of $H^0(A;Z)$. But

$$m \cdot \bar\lambda = i_2^*(K_{\bar\lambda}(x) \times \bar\lambda) = i_2^*(s \cdot 1 \times \bar\lambda) = i_2^*(1 \times s \cdot \bar\lambda) = s \cdot \bar\lambda.$$

so we have $K_{\bar\lambda}(x) = m \cdot 1$.

3. Proof of the main theorem

In this section we prove Theorem 1.4 and corollaries.

Proof of Theorem 1.4. Suppose $\alpha \in G_{2n}^f(X,A)$ is not contained in the kernel of r_*h_∞. Let $\phi : A \times S^n \to X$ be an affiliated map of α with respect to f. Then by Lemma 4 [8], we have $k_\lambda(1) = r_*h_\infty(\alpha) \in H_{2n}(A;Q)$ and $1 = u(1 \times 1)$, where Q is the field of rational numbers and $u : H_0(A) \otimes Q \to H_0(A;Q)$ is a natural isomorphism. Let $\bar\beta \in H^{2n}(A;Q)$ be dual to $K_\lambda(1)$. Then since $\langle K_{\bar\lambda}(\bar\beta), 1\rangle = \langle \bar\beta, K_\lambda(1)\rangle = 1$. So we have $K_{\bar\lambda}(\bar\beta) = 1 \in H_0(A;Q)$.

Now we shall prove that $\bar\beta^s \neq 0$ for all integers s, where $\bar\beta^s = \bar\beta \cup \overset{s}{\cdots} \cup \bar\beta$ (n-times cup product of $\bar\beta$). Note $\bar\beta^0 = 1$. First we show that

$$\phi^*r^*(\bar\beta^s) = \bar\beta^s \times 1 + s\bar\beta^{s-1} \times \bar\lambda.$$

When $s = 1$, it is true. In fact,

$$\begin{aligned} \phi^* r^*(\bar{\beta}) &= \bar{\beta} \times 1 + K_{\overline{\lambda}}(\bar{\beta}) \times \overline{\lambda} \\ &= \bar{\beta} \times 1 + 1 \times \overline{\lambda} \\ &= \bar{\beta} \times 1 + 1 \cdot \bar{\beta}^0 \times \overline{\lambda} \end{aligned}$$

Now suppose it is true for $s - 1$. Then

$$\begin{aligned} \phi^* r^*(\bar{\beta}^s) &= \phi^* r^*(\bar{\beta}^{s-1} \cup \bar{\beta}) \\ &= \phi^* r^*(\bar{\beta}^{s-1}) \cup \phi^* r^*(\bar{\beta}) \\ &= (\bar{\beta}^{s-1} \times 1 + (s-1)\bar{\beta}^{s-2} \times \overline{\lambda}) \cup (\bar{\beta} \times 1 + 1 \times \overline{\lambda}) \\ &= (\bar{\beta}^{s-1} \cup \bar{\beta}) \times 1 + (-1)^{2n^2}(s-1)(\bar{\beta}^{s-2} \cup \bar{\beta}) \times (\overline{\lambda} \cup 1) \\ &\quad + (\bar{\beta}^{s-1} \cup 1) \times (1 \cup \overline{\lambda}) \\ &= \bar{\beta}^s \times 1 + ((-1)^{2n^2}(s-1)\bar{\beta}^{s-1} + \bar{\beta}^{s-1}) \times \overline{\lambda} \\ &= \bar{\beta}^s \times 1 + s\bar{\beta}^{s-1} \times \overline{\lambda}. \end{aligned}$$

If $\bar{\beta}^s = 0$, then

$$0 = \phi^* r^*(\bar{\beta}^s) = \bar{\beta}^s \times 1 + s\bar{\beta}^{s-1} \times \overline{\lambda}.$$

So we have $\bar{\beta}^{s-1} = 0$. By continuous calculation, we have $\bar{\beta} = 0$. It is a contradiction to the fact that $\langle \bar{\beta}, K_\lambda(1) \rangle = 1$. Consequently, $\bar{\beta}^s \neq 0$ for all s. But $\bar{\beta}^s \neq 0$ leads to a contradiction. Since $0 \neq \bar{\beta}^s \in H^{2sn}(A; Q)$, $H^{2sn}(A; Q) \neq 0$ for all $s \in Z$. This implies $Hom(H_{2sn}(A; Q), Q) \neq 0$. So $H_{2sn}(A; Q) \neq 0$. This is a contradiction to the hypothesis. Consequently, α is contained in the kernel of $r_* h_\infty$. □

In the case $f = 1_X$, we have $G_n^f(X, A) = G_n(X)$. So we the following corollary.

COROLLARY 3.1. *Let X be CW-complex which has only a finite number of nonzero homology group, then $G_{2n}(X) \subset \ ker\ h_\infty$.*

Let A be a retract of X with retraction r, h_∞ Hurewicz homomorphism with tensor Q and k_∞ is the relative version of it, that is, $k_\infty : G_n^{Rel}(X, A) \to H_n(X, A; Q)$. Then we have the following commutative ladder;

$$\begin{array}{ccccc} \to G_n(A) & \xrightarrow{i_\#} & G_n(X,A) & \xrightarrow{j_\#} & G_n^{\mathrm{Re}\,l}(X,A) \to \\ \downarrow h_\infty & & \downarrow h_\infty & & \downarrow k_\infty \\ \to H_n(X,A) & \xrightarrow{i_*} & H_n(X;Q) & \xrightarrow{j_*} & H_n(X,A;Q) \to \end{array}$$

Since $j_\#$ is surjective, $G_n(X,A) \subset$ Ker h_∞ implies $G_n^{Rel}(X,A) \subset$ Ker k_∞ and $G_n(X,A) \subset$ Ker $r_* h_\infty$. Conversely, suppose $G_n^{Rel}(X,A) \subset$ Ker k_∞ and $G_n(X,A) \subset$ Ker $r_* h_\infty$. Then

$$j_* h_\infty(G_n(X,A) = k_\infty j_\#(G_n(X,A)) = k_\infty(G_n^{Rel}(X,A)) = 0.$$

Thus $h_\infty(G_n(X,A)) \subset$ Ker$j_* =$ Imi_*. So, for every $\alpha \in G_n(X,A)$, there is a $\beta \in H_n(X;Q)$ such that $i_*(\beta) = h_\infty(\alpha)$. But $\beta = r_* i_*(\beta) = r_* h_\infty(\alpha) = 0$. Hence $h_\infty(\alpha) = i_*(\beta) = 0$. Consequently $G_n(X,A) \subset$ Ker h_∞. Thus we have the following corollary.

Corollary 3.2. *Let A be a retract of X and have a finite number of nonzero homology groups. Then $G_{2n}(X,A) \subset$ Ker $r_* h_\infty$ and $G_{2n}^{Rel}(X,A) \subset$ Ker k_∞ if and only if $G_{2n}(X,A) \subset$ Ker h_∞.*

References

1. D. H. Gottlieb, *Evaluation subgroups of homotopy groups*, Amer. J. Math. **91** (1969), 729–755.
2. D. H. Gottlieb, *Witness, transgressions, and the evaluation map*, Indiana Univ. Math. J. **24** (1975), 825–836.
3. Sze-Tsen Hu, *Homotopy Theory*, Academic Press, New York, 1989.
4. G. E. Lang, *Evaluation subgroups of factor spaces*, Pacific. J. Math. **42** (1972), 701–709.
5. G. Lupton and S. Smith, *Rationalized evaluation subgroups of a map and the rationalized G-sequence*, preprint.
6. K. Y. Lee and M. H. Woo, *Homology and generalized evaluation subgroups of homotopy groups*, J. Korean Math. Soc. **25** (1988), 333–342.
7. K. Y. Lee and M. H. Woo, *The G-sequence and the ω-homology of a CW-pair*, Topology Appl. **52** (1993), 221–236.

8. K. Y. Lee and M. H. Woo, *Cyclic morphisms in the category of pairs and generalized G-sequences*, J. Math. Kyoto Univ. **38** (1998), 271–285.
9. K. Y. Lee and M. H. Woo, *Cocyclic morphisms and dual G-sequences*, Topology Appl. **116** (2001), 123–136.
10. J. Oprea, *Gottlieb groups, group actions, fixed points and rational homotopy*, Lecture Notes Series, Seoul National Univ. Research Inc. Math. Global Analysis Research Center, Seoul **29**, 1995.
11. J. Siegel, *G-spaces, W-spaces and H-spaces*, Pacific J. Math. **31** (1970), 209–214.
12. S. B. Smith, *Rational evaluation subgroups*, Math. Zeit. **221** (1996), 387–400.
13. E. Spanier, *Algebraic Topology*, McGraw-Hill Book Company, New York, 1981.
14. M. H. Woo and J. R. Kim, *Certain subgroups homotopy groups*, J. Korean Math. Soc. **21** (1984), 109–120.

*
Department of Information and Mathematics
Korea University
Chochiwon 339-701, Korea
E-mail: keyolee@korea.ac.kr