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A NOTE ON CONDUCTANCE METHOD IN R^n

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Abstract. We introduce the conductance and examine its properties. We study the local behavior of quasiconformal mappings on the boundary of a domain $D \subset \overline{R}^n$ and present some geometric applications of conductance.

1. Introduction

One of the problems in the theory of quasiconformal mappings is to determine whether or not two given homeomorphic domains can be mapped quasiconformally onto each other. The problem is closely related to the behavior of the mappings near the boundaries. This problems have mostly been studied in the special case where one of the domains is a ball. For example,

"If f is a quasiconformal mapping of B^3 onto a domain D which is m-connected at a given point $b \in \partial D$, then at most m points of ∂B^3 correspond to b under f."

Using the method of conductance, F. W. Gehring ([4]) established the above theorem. The method of conductance is a basic tool in the theory of quasiconformal mappings. In this note, we introduce the concept of conductance of a curve family and examine some basic properties of conductance. And we study the local behavior of quasiconformal mappings on the boundary of a domain $D \subset \overline{R}^n$, ([4], [9], [10]).

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Throughout this note, as a measure in R^n we use the n-dimensional Lebesque measure m_n , where the n may be omitted. we let $B^n(x_0, r)$ denote the n-dimensional ball $|x-x_0| < r$ and $S^{n-1}(x_0, r)$ its $(n-1)$-dimensional boundary sphere $|x - x_0| = r$. Let d be the Euclidean distance and ∂D the boundary of D.

2. Conductance of a curve family

Given a family, Γ, of curves γ in $\overline{R}^n$, we let $adm(\Gamma)$ denote the family of non-negative Borel measurable functions $\rho(x) : R^n \to [0, \infty]$ such that

$$\int_\gamma \rho\, ds \quad \geq 1, \tag{2.1}$$

for every rectifiable curve $\gamma \in \Gamma$. We define the *conductance* of Γ by

$$C(\Gamma) = \inf_{\rho \in adm(\Gamma)} \int_{R^n} \rho^n\, dm. \tag{2.2}$$

Obviously $0 \leq C(\Gamma) \leq \infty$, (ref. [3]).

Let D and D^* be two domains in $\overline{R}^n$. A homeomorphism $f : D \to D^*$ is said to be k-*quasiconformal*, $1 \leq k < \infty$, if it satisfies the double inequality

$$\frac{1}{k}C(\Gamma) \leq C(f(\Gamma)) \leq kC(f(\Gamma))$$

for each curve family Γ in D. A homeomorphism f is said to be *quasiconformal* if it is k-quasiconformal for some k, ([9]).

We let $\Gamma = \Gamma_D(F_1, F_2)$ denote the family of all curves which join F_1 and F_2 in a domain $D \subset \overline{R}^n$. We consider a geometric quantity $C(\Gamma_D(F_1, F_2))$, which in terms of the 'sizes' of F_1 and F_2, (ref. [10]).

The numerical value of the conductance is known only for a few curve families. Therefore good estimates are of importance.

PROPOSITION 2.1 [3]. *Let T be the rectangular parallelepiped with two parallel faces E_1 and E_2 of area A and distance h apart, then*

$$C(\Gamma) = C(\Gamma_T(E_1, E_2)) = \frac{A}{h^{n-1}}. \tag{2.3}$$

Proof. Choose $\rho \in adm(\Gamma)$ and let γ_y be the vertical segment in T which join E_1 and point y in the base E_2. Then $\gamma_y \in \Gamma$ and

$$1 \leq (\int_\gamma \rho ds)^n \leq h^{n-1} \int_{\gamma_y} \rho^n ds,$$

for all such y. Hence

$$\int_T \rho^n dm \geq \int_{E_2} (\int_{\gamma_y} \rho^n ds) dm_{n-1} \geq \frac{A}{h^{n-1}}.$$

Since ρ is arbitrary,

$$C(\Gamma) \geq \frac{A}{h^{n-1}}. \tag{2.4}$$

On the other hand, set $\rho = 1/h$, then $\rho \in adm(\Gamma)$ and

$$C(\Gamma) \leq \int_T \rho^n dm = \frac{A}{h^{n-1}}. \tag{2.5}$$

Therefore by (2.4) and (2.5), we obtain (2.3). □

PROPOSITION 2.2 [1]. *Let R be the spherical ring $R: r_1 < x < r_2$ and let R_1 and R_2 denote the bounded component and unbounded component of the complement of R, respectively. Then*

$$C(\Gamma) = C(\Gamma_R(\partial R_1, \partial R_2)) = \frac{\omega_{n-1}}{(log(r_2/r_1))^{n-1}}, \tag{2.6}$$

where $\omega_{n-1} = m_{n-1}(S^{n-1})$ denote the surface area of the unit sphere in R^n.

Proof. Let $\rho \in adm(\Gamma)$ and let $\gamma_e = \{x \in R^n : x = re,\ r_1 < r < r_2\}$ be the radial segment in R which joins ∂R_1 and ∂R_2 and is parallel to the unit vector $e \in S^{n-1}(0,1)$. Using Hölder's inequality (see [5], Thorem 189, p.140) we obtain

$$1 \le (\int_{\gamma_e} \rho ds)^n \le \int_{r_1}^{r_2} \rho^n r^{n-1} dr (\int_{r_1}^{r_2} \frac{1}{r} dr)^{n-1}$$

$$= (log\frac{r_2}{r_1})^{n-1} \int_{r_1}^{r_2} \rho^n r^{n-1} dr.$$

Integrating over all e we obtain by Fubini's theorem in polar coordinates

$$\omega_{n-1} \le (log\frac{r_2}{r_1})^{n-1} \int_R \rho^n dm.$$

Taking the infimum over all $\rho \in adm(\Gamma)$, we obtain

$$C(\Gamma) \ge \frac{\omega_{n-1}}{(log(r_2/r_1))^{n-1}}. \tag{2.7}$$

On the other hand, set

$$\rho = \frac{1}{|x| log(r_2/r_1)}$$

for $x \in R$, then $\rho \in adm(\Gamma)$ and thus

$$C(\Gamma) \le \int_R \rho^n dm = \frac{\omega_{n-1}}{(log(r_2/r_1))^{n-1}}. \tag{2.8}$$

□

THEOREM 2.3 [2]. *If every curve of Γ_1 has a subcurve belonging to Γ_2 (Γ_1 is said to be minorized by Γ_2 and denote $\Gamma_2 < \Gamma_1$), then*

$$C(\Gamma_2) \ge C(\Gamma_1).$$

That is, $C(\Gamma)$ is big when the curves are plentiful or short, small when the curves are few or long.

Proof. Obviously $adm(\Gamma_2) \subset adm(\Gamma_1)$. □

PROPOSITION 2.4 [3]. *$C(\Gamma)$is an outer measure on the collections of curve families Γ. That is, for curve families $\Gamma_1, \Gamma_2, \cdots, \Gamma_k, \cdots$ in $\overline{R}^n$,*

(a) $C(\emptyset) = 0$

(b) *If $\Gamma_1 \subset \Gamma_2$, then $C(\Gamma_1) \leq C(\Gamma_2)$.*

(c) $C(\cup_k \Gamma_k) \leq \sum_k C(\Gamma_k)$.

Proof. (c) Let $\rho_k \in adm(\Gamma_k)$. We set $\rho = \sup_k \rho_k$, hence $\rho \in adm(\cup_k \Gamma_k)$ and it follows that

$$C(\cup_k \Gamma_k) \leq \int \rho^n \, dm \leq \sum_k \int (\rho_k)^n \, dm. \quad \square$$

REMARK. ([3], [9]). (i) If $\Gamma_1 \subset \Gamma_2$, then $\Gamma_1 > \Gamma_2$. Thus Proposition 2.4 (b) is a special case of Theorem 2.3.

(ii) We can define the notion of measurable curve family.

(1) Γ is measurable if $C(\Gamma) = 0$,

(2) Γ is not measurable if $0 < C(\Gamma) < \infty$,

(3) Γ may or may not be measurable if $C(\Gamma) = \infty$.

THEOREM 2.5 [3]. *If all the curves in Γ pass through the fixed the point x_0 ($\neq \infty$), then*

$$C(\Gamma) = 0.$$

Proof. Let Γ_k be the subfamily of Γ which intersect x_0 and S^{n-1} $(x_0, \frac{1}{k})$ and Γ' the family of all curves in $B^n(x_0, \frac{1}{k})$ which join x_0 and $S^{n-1}(x_0, \frac{1}{k})$. Then $\Gamma' < \Gamma_k$. Hence

$$C(\Gamma_k) \leq \frac{\omega_{n-1}}{(log((1/k)/0))^{n-1}} = 0.$$

by Proposition 2.2 and Theorem 2.3 above. Since $\Gamma = \cup_k \Gamma_k$,

$$C(\Gamma) \leq \sum_k C(\Gamma_k) = 0$$

by Proposition 2.4(c) above. □

As immediate consequence of Theorem 2.5 we have the following Corollary.

COROLLARY 2.6 *A continuum is a compact connected set which contains more than one point. Let A_1 and A_2 be two continua in a domain D, then*

$$C(\Gamma_D(A_1, A_2)) > 0.$$

In particular, if $A_1 \cap A_2 \neq \emptyset$ or $D = \overline{R}^n$, then $C(\Gamma_D(A_1, A_2)) = \infty$.

Just as measure theory provides the notion of negligible point set(measure zero), modulus provides the notion of a negligible curve family.

PROPOSITION 2.7 [8]. *For curve families $\Gamma_1, \Gamma_2, \cdots, \Gamma_j, \cdots$ in $\overline{R}^n$,*

(a) $C(\Gamma_1) = 0$ *implies* $C(\Gamma_1 \cup \Gamma_j) = C(\Gamma_j)$.

(b) $C(\Gamma_j) = 0$ *implies* $C(\cup_j \Gamma_j) = 0$.

3. Some geometric applications of conductance

Our first example states that isolated boundary points are removable singularities. We give some topological notions.

DEFINITION 3.1[9]. Let $f : D \to \overline{R}^n$ be a mapping and b a point of ∂D. The *cluster set* $cl(f, b)$ of f at b is the set of all points $b^* \in \overline{R}^n$ for which there exist a sequence $\{x_k\}$ in D such that $x_k \to b$ and $f(x_k) \to b^*$.

Thus f has a limit b^* at b if and only if $cl(f, b) = \{b^*\}$. Since $\overline{R}^n$ is compact, the cluster set is never empty. The cluster sets of a homeomorphism $f : D \to D^*$ are always subsets of ∂D^*.

THEOREM 3.2 [9]. *Let $f : D \to D^*$ be a quasiconformal mapping and b an isolated point of ∂D, then f has a limit b^* at b.*

Proof. We choose a ball neighborhood U of b such $\overline{U} \cap \partial D = \{b\}$. Then $A = U - \{b\}$ is a ring and

$$C(\Gamma_A) = C[\Gamma_A(\partial U, \{b\})] = 0.$$

$f(A)$ is also a ring with boundary components $f(\partial U)$ and $cl(f, b)$. Since f is quasiconformal mapping,

$$C(\Gamma_{f(A)}) = C[\Gamma_{f(A)}(f(\partial U), \{f(b)\})] = 0.$$

By Proposition 2.2, $cl(f, b)$ consists of a single point b^*. □

We next introduce a number of concepts which describe the behavior of a domain at a boundary point. All point sets considered lie in $\overline{R}^n$.

DEFINITION 3.3 [9]. Let b be a boundary point of a domain D.

(i) D is *locally connected* at b if b has arbitrarily small neighborhoods U such that $U \cap D$ connected.

(ii) D is *m-connected* at b, $(m = 1, 2, \cdots)$, if there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of m components each of which is locally connected at b.

(iii) D has *property* P_1 at b if the following condition is satisfied : If F_1 and F_2 are connected subsets of D such that $b \in \overline{F_1} \cap \overline{F_2}$, then $C(\Gamma_D(F_1, F_2)) = \infty$.

(iv) D is *finitely connected* at b if b has arbitrarily small neighborhoods U such that $U \cap D$ has a finite number of components.

THEOREM 3.4. *Let D be m-connected at $b(\neq \infty)$ and $f : D \to D^*$ a quasiconformal mapping. Then*

(a) *$cl(f, b)$ contains at most m components.*

(b) *If D^* has property P_1 at every point of $cl(f, b)$, then $cl(f, b)$ contains at least m points.*

Proof. (a) From the m-connectedness property and the definition of $cl(f,b)$, we readily obtain the above result.
(b) Since the case of $m=1$ is trivial, we discuss the case of $m \geq 2$. Assume that the statement is not true, that is, suppose that $cl(f,b)$ contained at most $(m-1)$ points. Then there exist i and j, $(1 \leq i < j \leq m)$ and a point c in $cl(f,b)$ such that $c \in \overline{f(F_i)} \cap \overline{f(F_j)}$. But since $f(F_i)$ and $f(F_j)$ are connected sets in D^*. Since D^* has property P_1 at c,

$$C(f(\Gamma_D)) = C[\Gamma_{D^*}(f(F_i), f(F_j))] = \infty. \tag{3.1}$$

On the other hand, let U be a neighborhood of b with each of the components, $E_1, \cdots, E_m$, of $U \cap D$ being locally-connected at b. Set $d = d(b, \partial U)$. Then there is a connected set $F_i \subset E_i \cap B^n(b, d/2)$ with $b \in \overline{F_i}, (i = 1, \cdots, m)$. Denote

$$\Gamma_D = \Gamma_D(F_i, F_j),$$

$$\Gamma_R = \Gamma_R[S^{n-1}(b, d/2), S^{n-1}(b, d)],$$

where R is the ring :$(d/2) < |x-b| < d$. Since the curve family Γ_D is minorized by the family Γ_R, by Proposition 2.2 and Theorem 2.3, we obtain

$$C(\Gamma_D) \leq C(\Gamma_R) = \frac{\omega_{n-1}}{(log\, 2)^{n-1}} < \infty, \tag{3.2}$$

which contradicts the quasiconformality of f. □

An argument similar to that employed in the proof of Theorem 3.4(b) yields the following result.

COROLLARY 3.5. *Let D be finitely connected at $b(\neq \infty) \in \partial D$ without being m-connected for any integer m and $f : D \to D^*$ a quasiconformal mapping. If D^* has property P_1 at every point of $cl(f,b)$, then $cl(f,b)$ is infinite.*

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