JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume 18, No. 2, October 2005

GENERALIZED (θ, ϕ)–DERIVATIONS ON POISSON BANACH ALGEBRAS AND JORDAN BANACH ALGEBRAS

Chun-Gil Park*

Abstract. In [1], the concept of generalized (θ, ϕ)-derivations on rings was introduced. In this paper, we introduce the concept of generalized (θ, ϕ)-derivations on Poisson Banach algebras and of generalized (θ, ϕ)-derivations on Jordan Banach algebras, and prove the Cauchy–Rassias stability of generalized (θ, ϕ)-derivations on Poisson Banach algebras and of generalized (θ, ϕ)-derivations on Jordan Banach algebras.

1. Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Th.M. Rassias [10] introduced the following inequality, that we call *Cauchy–Rassias inequality*: Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th.M. Rassias [10] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2 - 2^p}||x||^p$$

Received by the editors on August 15, 2005.
2000 *Mathematics Subject Classifications*: Primary 17C65, 17A36, 39B52, 17B63.
Key words and phrases: Generalized (θ, ϕ)-derivation, Cauchy–Rassias stability, Poisson Banach algebra, Jordan Banach algebra.

for all $x \in X$. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [2] generalized the Rassias' result in the following form: Let G be an abelian group and X a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{k=0}^{\infty} \frac{1}{2^k} \varphi(2^k x, 2^k y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to X$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to X$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2} \widetilde{\varphi}(x, x)$$

for all $x \in G$.

Jun and Lee [5] proved the following: Denote by $\varphi : X \setminus \{0\} \times X \setminus \{0\} \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} \frac{1}{3^j} \varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|2f(\frac{x + y}{2}) - f(x) - f(y)\| \le \varphi(x, y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)\| \le \frac{1}{3} \big(\widetilde{\varphi}(x, -x) + \widetilde{\varphi}(-x, 3x) \big)$$

for all $x \in X \setminus \{0\}$. The stability problem of functional equations has been investigated in several papers (see [8]–[13]).

Throughout this paper, we denote by R the set of real numbers or complex numbers. Let θ, ϕ be endomorphisms of R. An additive mapping $D : R \to R$ is called a (θ, ϕ)-*derivation* on R if $D(xy) = D(x)\theta(y) + \phi(x)D(y)$ holds for all $x, y \in R$. An additive mapping $U : R \to R$ is called a *generalized* (θ, ϕ)-*derivation* on R if there exists a (θ, ϕ)-derivation $D : R \to R$ such that $U(xy) = U(x)\theta(y) + \phi(x)D(y)$ holds for all $x, y \in R$ (see [1], [4]).

In this paper, we are going to introduce the concept of generalized (θ, ϕ)-derivations on Poisson Banach algebras and of generalized (θ, ϕ)-derivations on Jordan Banach algebras. We prove the Cauchy–Rassias stability of generalized (θ, ϕ)-derivations on Poisson Banach algebras and of generalized (θ, ϕ)-derivations on Jordan Banach algebras.

2. Generalized (θ, ϕ)-derivations on Poisson Banach algebras

A *Poisson Banach algebra* B is a Banach algebra with a R-bilinear map $\{\cdot, \cdot\} : B \times B \to B$, called a *Poisson bracket*, such that $(B, \{\cdot, \cdot\})$ is a Lie algebra and

$$\{ab, c\} = a\{b, c\} + \{a, c\}b$$

for all $a, b, c \in B$. Poisson algebras have played an important role in many mathematical areas and have been studied to find sympletic leaves of the corresponding Poisson varieties. It is also important to find or construct a Poisson bracket in the theory of Poisson algebra (see [3], [6], [7]).

Throughout this section, let B be a Poisson Banach algebra over R with norm $\| \cdot \|$.

Definition 2.1. Let θ, $\phi : B \to B$ be additive mappings. An additive mapping $D : B \to B$ is called a *(θ,ϕ)-derivation* on B if $D(\{x,y\}) = \{D(x),\theta(y)\} + \{\phi(x),D(y)\}$ holds for all $x,y \in B$.

An additive mapping $U : B \to B$ is called a *generalized (θ,ϕ)-derivation* on B if there exists a (θ,ϕ)-derivation $D : B \to B$ such that $U(\{x,y\}) = \{U(x),\theta(y)\} + \{\phi(x),D(y)\}$ holds for all $x,y \in B$.

Theorem 2.1. *Let $f,g,h,u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exists a function $\varphi : B\times B \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) < \infty, \tag{2.1}$$

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y), \tag{2.2}$$

$$\|g(x+y) - g(x) - g(y)\| \le \varphi(x,y), \tag{2.3}$$

$$\|h(x+y) - h(x) - h(y)\| \le \varphi(x,y), \tag{2.4}$$

$$\|u(x+y) - u(x) - u(y)\| \le \varphi(x,y), \tag{2.5}$$

$$\|f(\{x,y\}) - \{f(x),g(y)\} - \{h(x),f(y)\}\| \le \varphi(x,y), \tag{2.6}$$

$$\|u(\{x,y\}) - \{u(x),g(y)\} - \{h(x),f(y)\}\| \le \varphi(x,y) \tag{2.7}$$

for all $x,y \in B$. Then there exist unique additive mappings $D,\theta,\phi,U : B \to B$ such that

$$\|f(x) - D(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x), \tag{2.8}$$

$$\|g(x) - \theta(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x), \tag{2.9}$$

$$\|h(x) - \phi(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x), \tag{2.10}$$

$$\|u(x) - U(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x) \tag{2.11}$$

for all $x \in B$. Moreover, $D : B \to B$ is a (θ, ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ, ϕ)-derivation on B.

Proof. By the Găvruta's theorem [2], it follows from (2.1)–(2.5) that there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (2.8)–(2.11). The additive mappings $D, \theta, \phi, U : B \to B$ are given by

$$D(x) = \lim_{l\to\infty} \frac{1}{2^l} f(2^l x), \tag{2.12}$$

$$\theta(x) = \lim_{l\to\infty} \frac{1}{2^l} g(2^l x), \tag{2.13}$$

$$\phi(x) = \lim_{l\to\infty} \frac{1}{2^l} h(2^l x), \tag{2.14}$$

$$U(x) = \lim_{l\to\infty} \frac{1}{2^l} u(2^l x), \tag{2.15}$$

for all $x \in B$.

It follows from (2.6) that

$$\begin{aligned} \frac{1}{2^{2l}} \|f(\{2^l x, 2^l y\}) - \{f(2^l x), g(2^l y)\} - \{h(2^l x), f(2^l y)\}\| \\ \leq \frac{1}{2^{2l}} \varphi(2^l x, 2^l y) \\ \leq \frac{1}{2^l} \varphi(2^l x, 2^l y), \end{aligned}$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.1). By (2.12)–(2.14),

$$D(\{x, y\}) = \{D(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all $x, y \in B$. So the additive mapping $D : B \to B$ is a (θ, ϕ)-derivation on B.

It follows from (2.7) that

$$\begin{aligned}\frac{1}{2^{2l}}\|u(\{2^l x, 2^l y\}) - \{u(2^l x), g(2^l y)\} - \{h(2^l x), f(2^l y)\}\| \\ \le \frac{1}{2^{2l}}\varphi(2^l x, 2^l y) \\ \le \frac{1}{2^l}\varphi(2^l x, 2^l y),\end{aligned}$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.1). Thus

$$U(\{x, y\}) = \{U(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all $x, y \in B$. So the additive mapping $U : B \to B$ is a generalized (θ, ϕ)-derivation on B. □

COROLLARY 2.2. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exist constants $\epsilon \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &\le \epsilon(\|x\|^p + \|y\|^p),\\ \|g(x+y) - g(x) - g(y)\| &\le \epsilon(\|x\|^p + \|y\|^p),\\ \|h(x+y) - h(x) - h(y)\| &\le \epsilon(\|x\|^p + \|y\|^p),\\ \|u(x+y) - u(x) - u(y)\| &\le \epsilon(\|x\|^p + \|y\|^p),\\ \|f(\{x, y\}) - \{f(x), g(y)\} - \{h(x), f(y)\}\| &\le \epsilon(\|x\|^p + \|y\|^p),\\ \|u(\{x, y\}) - \{u(x), g(y)\} - \{h(x), f(y)\}\| &\le \epsilon(\|x\|^p + \|y\|^p)\end{aligned}$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ such that

$$\begin{aligned}\|f(x) - D(x)\| &\le \frac{2\epsilon}{2 - 2^p}\|x\|^p,\\ \|g(x) - \theta(x)\| &\le \frac{2\epsilon}{2 - 2^p}\|x\|^p,\\ \|h(x) - \phi(x)\| &\le \frac{2\epsilon}{2 - 2^p}\|x\|^p,\\ \|u(x) - U(x)\| &\le \frac{2\epsilon}{2 - 2^p}\|x\|^p\end{aligned}$$

for all $x \in B$. Moreover, $D : B \to B$ is a (θ,ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ,ϕ)-derivation on B.

Proof. Define $\varphi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$ to be Th.M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 2.1. □

THEOREM 2.3. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exists a function $\varphi : B \times B \to [0,\infty)$ satisfying (2.6) and (2.7) such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} \frac{1}{3^j}\varphi(3^j x, 3^j y) < \infty, \tag{2.16}$$

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \le \varphi(x,y), \tag{2.17}$$

$$\|2g(\frac{x+y}{2}) - g(x) - g(y)\| \le \varphi(x,y), \tag{2.18}$$

$$\|2h(\frac{x+y}{2}) - h(x) - h(y)\| \le \varphi(x,y), \tag{2.19}$$

$$\|2u(\frac{x+y}{2}) - u(x) - u(y)\| \le \varphi(x,y) \tag{2.20}$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ such that

$$\|f(x) - D(x)\| \le \frac{1}{3}\big(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x)\big), \tag{2.21}$$

$$\|g(x) - \theta(x)\| \le \frac{1}{3}\big(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x)\big), \tag{2.22}$$

$$\|h(x) - \phi(x)\| \le \frac{1}{3}\big(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x)\big), \tag{2.23}$$

$$\|u(x) - U(x)\| \le \frac{1}{3}\big(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x)\big) \tag{2.24}$$

for all $x \in B$. Moreover, $D : B \to B$ is a (θ,ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ,ϕ)-derivation on B.

Proof. By the Jun and Lee's theorem [5, Theorem 1], it follows from (2.16)–(2.20) that there exist unique additive mappings D, θ, ϕ, U :

$B \to B$ satisfying (2.21)–(2.24). The additive mappings $D, \theta, \phi, U : B \to B$ are given by

$$D(x) = \lim_{l\to\infty} \frac{1}{3^l} f(3^l x), \tag{2.25}$$

$$\theta(x) = \lim_{l\to\infty} \frac{1}{3^l} g(3^l x), \tag{2.26}$$

$$\phi(x) = \lim_{l\to\infty} \frac{1}{3^l} h(3^l x), \tag{2.27}$$

$$U(x) = \lim_{l\to\infty} \frac{1}{3^l} u(3^l x), \tag{2.28}$$

for all $x \in B$.

It follows from (2.6) that

$$\begin{aligned} \frac{1}{3^{2l}} \|f(\{3^l x, 3^l y\}) - \{f(3^l x), g(3^l y)\} - \{h(3^l x), f(3^l y)\}\| \\ \leq \frac{1}{3^{2l}} \varphi(3^l x, 3^l y) \\ \leq \frac{1}{3^l} \varphi(3^l x, 3^l y), \end{aligned}$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.16). By (2.25)–(2.28),

$$D(\{x, y\}) = \{D(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all $x, y \in B$. So the additive mapping $D : B \to B$ is a (θ, ϕ)-derivation on B.

It follows from (2.7) that

$$\begin{aligned} \frac{1}{3^{2l}} \|u(\{3^l x, 3^l y\}) - \{u(3^l x), g(3^l y)\} - \{h(3^l x), f(3^l y)\}\| \\ \leq \frac{1}{3^{2l}} \varphi(3^l x, 3^l y) \\ \leq \frac{1}{3^l} \varphi(3^l x, 3^l y), \end{aligned}$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.16). Thus

$$U(\{x, y\}) = \{U(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all $x, y \in B$. So the additive mapping $U : B \to B$ is a generalized (θ, ϕ)-derivation on B. □

COROLLARY 2.4. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}
\|2f(\frac{x+y}{2}) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|2g(\frac{x+y}{2}) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|2h(\frac{x+y}{2}) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|2u(\frac{x+y}{2}) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|f(\{x, y\}) - \{f(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|u(\{x, y\}) - \{u(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p)
\end{aligned}$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ such that

$$\begin{aligned}
\|f(x) - D(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\
\|g(x) - \theta(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\
\|h(x) - \phi(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p, \\
\|u(x) - U(x)\| &\leq \frac{3 + 3^p}{3 - 3^p} \epsilon \|x\|^p
\end{aligned}$$

for all $x \in B$. Moreover, $D : B \to B$ is a (θ, ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ, ϕ)-derivation on B.

Proof. Define $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$, and apply Theorem 2.3. □

THEOREM 2.5. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exists a function $\varphi : B \times B \to [0, \infty)$ satisfying* (2.17)–(2.20), (2.6) *and* (2.7) *such that*

$$\sum_{j=0}^{\infty} 3^{2j} \varphi(\frac{x}{3^j}, \frac{y}{3^j}) < \infty \tag{2.29}$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ such that

$$\|f(x) - D(x)\| \le \widetilde{\varphi}(\frac{x}{3}, -\frac{x}{3}) + \widetilde{\varphi}(-\frac{x}{3}, x), \tag{2.30}$$

$$\|g(x) - \theta(x)\| \le \widetilde{\varphi}(\frac{x}{3}, -\frac{x}{3}) + \widetilde{\varphi}(-\frac{x}{3}, x), \tag{2.31}$$

$$\|h(x) - \phi(x)\| \le \widetilde{\varphi}(\frac{x}{3}, -\frac{x}{3}) + \widetilde{\varphi}(-\frac{x}{3}, x), \tag{2.32}$$

$$\|u(x) - U(x)\| \le \widetilde{\varphi}(\frac{x}{3}, -\frac{x}{3}) + \widetilde{\varphi}(-\frac{x}{3}, x) \tag{2.33}$$

for all $x \in B$, where

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 3^j \varphi(\frac{x}{3^j}, \frac{y}{3^j})$$

for all $x, y \in B$. Moreover, $D : B \to B$ is a (θ, ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ, ϕ)-derivation on B.

Proof. Note that $\sum_{j=0}^{\infty} 3^j \varphi(\frac{x}{3^j}, \frac{y}{3^j}) \le \sum_{j=0}^{\infty} 3^{2j} \varphi(\frac{x}{3^j}, \frac{y}{3^j})$ for all x, $y \in B$. By the Jun and Lee's theorem [5, Theorem 6], it follows from (2.29) and (2.17)–(2.20) that there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (2.30)–(2.33). The additive mappings $D, \theta, \phi, U : B \to B$ are given by

$$D(x) = \lim_{l \to \infty} 3^l f(\frac{x}{3^l}), \tag{2.34}$$

$$\theta(x) = \lim_{l \to \infty} 3^l g(\frac{x}{3^l}), \tag{2.35}$$

$$\phi(x) = \lim_{l \to \infty} 3^l h(\frac{x}{3^l}), \tag{2.36}$$

$$U(x) = \lim_{l \to \infty} 3^l u(\frac{x}{3^l}), \tag{2.37}$$

for all $x \in B$.

It follows from (2.6) that

$$3^{2l}\|f(\{\frac{x}{3^l}, \frac{y}{3^l}\}) - \{f(\frac{x}{3^l}), g(\frac{y}{3^l})\} - \{h(\frac{x}{3^l}), f(\frac{y}{3^l})\}\| \leq 3^{2l}\varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.29). By (2.34)–(2.37),

$$D(\{x, y\}) = \{D(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all $x, y \in B$. So the additive mapping $D : B \to B$ is a (θ, ϕ)-derivation on B.

It follows from (2.7) that

$$3^{2l}\|u(\{\frac{x}{3^l}, \frac{y}{3^l}\}) - \{u(\frac{x}{3^l}), g(\frac{y}{3^l})\} - \{h(\frac{x}{3^l}), f(\frac{y}{3^l})\}\| \leq 3^{2l}\varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.29). Thus

$$U(\{x, y\}) = \{U(x), \theta(y)\} + \{\phi(x), D(y)\}$$

for all $x, y \in B$. So the additive mapping $U : B \to B$ is a generalized (θ, ϕ)-derivation on B. □

COROLLARY 2.6. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exist constants $\epsilon \geq 0$ and $p \in (2, \infty)$ such that*

$$\begin{aligned}
\|2f(\frac{x+y}{2}) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|2g(\frac{x+y}{2}) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|2h(\frac{x+y}{2}) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|2u(\frac{x+y}{2}) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|f(\{x, y\}) - \{f(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\
\|u(\{x, y\}) - \{u(x), g(y)\} - \{h(x), f(y)\}\| &\leq \epsilon(\|x\|^p + \|y\|^p)
\end{aligned}$$

for all $x, y \in B$. *Then there exist unique additive mappings* $D, \theta, \phi, U : B \to B$ *such that*

$$\|f(x) - D(x)\| \leq \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p,$$
$$\|g(x) - \theta(x)\| \leq \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p,$$
$$\|h(x) - \phi(x)\| \leq \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p,$$
$$\|u(x) - U(x)\| \leq \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p$$

for all $x \in B$. *Moreover,* $D : B \to B$ *is a* (θ, ϕ)*-derivation on* B, *and* $U : B \to B$ *is a generalized* (θ, ϕ)*-derivation on* B.

Proof. Define $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$, and apply Theorem 2.5. □

Remark 2.1. A Banach algebra B, endowed with the Lie product $[x, y] = \frac{xy-yx}{2}$ on B, is called a *Lie Banach algebra*. Let θ, ϕ : $B \to B$ be additive mappings. An additive mapping $D : B \to B$ is called a (θ, ϕ)*-derivation* on a Lie Banach algebra B if $D([x, y]) = [D(x), \theta(y)] + [\phi(x), D(y)]$ holds for all $x, y \in B$. An additive mapping $U : B \to B$ is called a *generalized* (θ, ϕ)*-derivation* on a Lie Banach algebra B if there exists a (θ, ϕ)-derivation $D : B \to B$ such that $U([x, y]) = [U(x), \theta(y)] + [\phi(x), D(y)]$ holds for all $x, y \in B$.

When each Poisson bracket $\{\cdot, \cdot\}$ in this section is replaced by the Lie product $[\cdot, \cdot]$, we can show that there exists a unique (θ, ϕ)-derivation $D : B \to B$ on a Lie Banach algebra, and there exists a unique generalized (θ, ϕ)-derivation $U : B \to B$ on a Lie Banach algebra.

3. Generalized (θ, ϕ)-derivations on Jordan Banach algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see

[14]). Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative Banach algebra X with product $x \circ y$ is called a *Jordan Banach algebra*.

Throughout this section, let B be a Jordan Banach algebra over R with norm $\| \cdot \|$.

DEFINITION 3.1. Let θ, $\phi : B \to B$ be additive mappings. An additive mapping $D : B \to B$ is called a (θ, ϕ)-*derivation* on B if $D(x \circ y) = D(x) \circ \theta(y) + \phi(x) \circ D(y)$ holds for all $x, y \in B$.

An additive mapping $U : B \to B$ is called a *generalized* (θ, ϕ)-*derivation* on B if there exists a (θ, ϕ)-derivation $D : B \to B$ such that $U(x \circ y) = U(x) \circ \theta(y) + \phi(x) \circ D(y)$ holds for all $x, y \in B$.

THEOREM 3.1. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exists a function $\varphi : B \times B \to [0, \infty)$ satisfying* (2.1)–(2.5) *such that*

$$\|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| \leq \varphi(x, y), \tag{3.1}$$

$$\|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| \leq \varphi(x, y) \tag{3.2}$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (2.8)–(2.11). *Moreover, $D : B \to B$ is a (θ, ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ, ϕ)-derivation on B.*

Proof. By the Găvruta's theorem [3], it follows from (2.1)–(2.5) that there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (2.8)–(2.11).

It follows from (3.1) that

$$\frac{1}{2^{2l}}\|f((2^l x)\circ(2^l y)) - f(2^l x)\circ g(2^l y) - h(2^l x)\circ f(2^l y)\| \\ \leq \frac{1}{2^{2l}}\varphi(2^l x, 2^l y) \\ \leq \frac{1}{2^l}\varphi(2^l x, 2^l y),$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.1). By (2.12)–(2.15),

$$D(x\circ y) = D(x)\circ\theta(y) + \phi(x)\circ D(y)$$

for all $x, y \in B$. So the additive mapping $D : B \to B$ is a (θ, ϕ)-derivation on B.

It follows from (3.2) that

$$\frac{1}{2^{2l}}\|u((2^l x)\circ(2^l y)) - u(2^l x)\circ g(2^l y) - h(2^l x)\circ f(2^l y)\| \\ \leq \frac{1}{2^{2l}}\varphi(2^l x, 2^l y) \\ \leq \frac{1}{2^l}\varphi(2^l x, 2^l y),$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.1). Thus

$$U(x\circ y) = U(x)\circ\theta(y) + \phi(x)\circ D(y)$$

for all $x, y \in B$. So the additive mapping $U : B \to B$ is a generalized (θ, ϕ)-derivation on B. □

COROLLARY 3.2. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),$$
$$\|g(x+y) - g(x) - g(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),$$
$$\|h(x+y) - h(x) - h(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),$$
$$\|u(x+y) - u(x) - u(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),$$

$$\|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p),$$
$$\|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ such that

$$\|f(x) - D(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p,$$
$$\|g(x) - \theta(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p,$$
$$\|h(x) - \phi(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p,$$
$$\|u(x) - U(x)\| \leq \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in B$. Moreover, $D : B \to B$ is a (θ, ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ, ϕ)-derivation on B.

Proof. Define $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$ to be Th.M. Rassias upper bound in the Cauchy–Rassias inequality, and apply Theorem 3.1. □

THEOREM 3.3. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exists a function $\varphi : B \times B \to [0, \infty)$ satisfying* (3.1), (3.2) *and* (2.16)–(2.20). *Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying* (2.21)–(2.24). *Moreover, $D : B \to B$ is a (θ, ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ, ϕ)-derivation on B.*

Proof. By the Jun and Lee's theorem [5, Theorem 1], it follows from (2.16)–(2.20) that there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (2.21)–(2.24).

It follows from (3.1) that

$$\frac{1}{3^{2l}}\|f((3^l x) \circ (3^l y)) - f(3^l x) \circ g(3^l y) - h(3^l x) \circ f(3^l y)\|$$
$$\leq \frac{1}{3^{2l}}\varphi(3^l x, 3^l y) \leq \frac{1}{3^l}\varphi(3^l x, 3^l y),$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.16). By (2.25)–(2.28),

$$D(x \circ y) = D(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all $x, y \in B$. So the additive mapping $D : B \to B$ is a (θ, ϕ)-derivation on B.

It follows from (3.2) that

$$\begin{aligned} \frac{1}{3^{2l}} \|u((3^l x) \circ (3^l y)) - u(3^l x) \circ g(3^l y) - h(3^l x) \circ f(3^l y)\| & \\ \leq \frac{1}{3^{2l}} \varphi(3^l x, 3^l y) & \\ \leq \frac{1}{3^l} \varphi(3^l x, 3^l y), & \end{aligned}$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.16). Thus

$$U(x \circ y) = U(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all $x, y \in B$. So the additive mapping $U : B \to B$ is a generalized (θ, ϕ)-derivation on B. □

Corollary 3.4. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|2f(\frac{x+y}{2}) - f(x) - f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2g(\frac{x+y}{2}) - g(x) - g(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2h(\frac{x+y}{2}) - h(x) - h(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|2u(\frac{x+y}{2}) - u(x) - u(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p), \\ \|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| &\leq \epsilon(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ such that

$$\|f(x) - D(x)\| \le \frac{3+3^p}{3-3^p}\epsilon\|x\|^p,$$
$$\|g(x) - \theta(x)\| \le \frac{3+3^p}{3-3^p}\epsilon\|x\|^p,$$
$$\|h(x) - \phi(x)\| \le \frac{3+3^p}{3-3^p}\epsilon\|x\|^p,$$
$$\|u(x) - U(x)\| \le \frac{3+3^p}{3-3^p}\epsilon\|x\|^p$$

for all $x \in B$. Moreover, $D : B \to B$ is a (θ,ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ,ϕ)-derivation on B.

Proof. Define $\varphi(x,y) = \epsilon(\|x\|^p + \|y\|^p)$, and apply Theorem 3.3. □

THEOREM 3.5. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exists a function $\varphi : B \times B \to [0,\infty)$ satisfying (2.17)–(2.20), (3.1), (3.2) and (2.29). Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (2.30)–(2.33). Moreover, $D : B \to B$ is a (θ,ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ,ϕ)-derivation on B.*

Proof. By the Jun and Lee's theorem [5, Theorem 6], it follows from (2.29) and (2.17)–(2.20) that there exist unique additive mappings $D, \theta, \phi, U : B \to B$ satisfying (2.30)–(2.33).

It follows from (3.1) that

$$3^{2l}\|f(\frac{x}{3^l} \circ \frac{y}{3^l}) - f(\frac{x}{3^l}) \circ g(\frac{y}{3^l}) - h(\frac{x}{3^l}) \circ f(\frac{y}{3^l})\| \le 3^{2l}\varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.29). By (2.34)–(2.37),

$$D(x \circ y) = D(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all $x, y \in B$. So the additive mapping $D : B \to B$ is a (θ, ϕ)-derivation on B.

It follows from (3.2) that

$$3^{2l}\|u(\frac{x}{3^l} \circ \frac{y}{3^l}) - u(\frac{x}{3^l}) \circ g(\frac{y}{3^l}) - h(\frac{x}{3^l}) \circ f(\frac{y}{3^l})\| \le 3^{2l}\varphi(\frac{x}{3^l}, \frac{y}{3^l}),$$

which tends to zero as $l \to \infty$ for all $x, y \in B$ by (2.29). Thus

$$U(x \circ y) = U(x) \circ \theta(y) + \phi(x) \circ D(y)$$

for all $x, y \in B$. So the additive mapping $U : B \to B$ is a generalized (θ, ϕ)-derivation on B. □

Corollary 3.6. *Let $f, g, h, u : B \to B$ be mappings with $f(0) = g(0) = h(0) = u(0) = 0$ for which there exist constants $\epsilon \ge 0$ and $p \in (2, \infty)$ such that*

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$
$$\|2g(\frac{x+y}{2}) - g(x) - g(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$
$$\|2h(\frac{x+y}{2}) - h(x) - h(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$
$$\|2u(\frac{x+y}{2}) - u(x) - u(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$
$$\|f(x \circ y) - f(x) \circ g(y) - h(x) \circ f(y)\| \le \epsilon(\|x\|^p + \|y\|^p),$$
$$\|u(x \circ y) - u(x) \circ g(y) - h(x) \circ f(y)\| \le \epsilon(\|x\|^p + \|y\|^p)$$

for all $x, y \in B$. Then there exist unique additive mappings $D, \theta, \phi, U : B \to B$ such that

$$\|f(x) - D(x)\| \le \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p,$$
$$\|g(x) - \theta(x)\| \le \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p,$$
$$\|h(x) - \phi(x)\| \le \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p,$$
$$\|u(x) - U(x)\| \le \frac{3^p + 3}{3^p - 3}\epsilon\|x\|^p$$

for all $x \in B$. Moreover, $D : B \to B$ is a (θ, ϕ)-derivation on B, and $U : B \to B$ is a generalized (θ, ϕ)-derivation on B.

Proof. Define $\varphi(x, y) = \epsilon(\|x\|^p + \|y\|^p)$, and apply Theorem 3.5. □

References

1. M. Ashraf, A. Ali and S. Ali, *On Lie ideals and generalized (θ, ϕ)-derivations in prime rings*, Comm. Algebra **32** (2004), 2977–2985.
2. P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.
3. K.R. Goodearl and E.S. Letzter, *Quantum n-space as a quotient of classical n-space*, Trans. Amer. Math. Soc. **352** (2000), 5855–5876.
4. B. Hvala, *Generalized derivations in rings*, Comm. Algebra **26** (1998), 1147–1166.
5. K. Jun and Y. Lee, *A generalization of the Hyers–Ulam–Rassias stability of Jensen's equation*, J. Math. Anal. Appl. **238** (1999), 305–315.
6. S. Oh, C. Park and Y. Shin, *Quantum n-space and Poisson n-space*, Comm. Algebra **30** (2002), 4197–4209.
7. ______, *A Poincaré–Birkhoff–Witt theorem for Poisson enveloping algebras*, Comm. Algebra **30** (2002), 4867–4887.
8. C. Park, *Homomorphisms between Lie JC^*-algebras and Cauchy–Rassias stability of Lie JC^*-algebra derivations*, J. Lie Theory **15** (2005), 393–414.
9. M. Rajagopalan, Th.M. Rassias and K. Sundaresan, *Generalized backward shifts on Banach spaces*, Bulletin des Sciences Mathématiques **124** (2000), 685–693.
10. Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.
11. ______, *New characterizations of inner product spaces*, Bulletin des Sciences Mathématiques **108** (1984), 95–99.
12. ______, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. **251** (2000), 264–284.
13. Th.M. Rassias and P. Šemrl, *On the Hyers–Ulam stability of linear mappings*, J. Math. Anal. Appl. **173** (1993), 325–338.
14. H. Upmeier, *Jordan Algebras in Analysis, Operator Theory, and Quantum Mechanics*, Regional Conference Series in Mathematics No. 67, Amer. Math. Soc., Providence, 1987.

*

Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea

E-mail: cgpark@cnu.ac.kr