JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **18**, No.2, October 2005

THE FUNDAMENTAL GROUP OF THE CONNECTED SUM OF MANIFOLDS AND THEIR FOLDINGS

M. El-Ghoul*, A.E. El-Ahmady**, H. Rafat*** and M. Abu-Saleem****

Abstract. In this paper, we will introduce the folding on the connected sum of some types of manifolds which are determined by their fundamental group. Also, the fundamental group of the unfolding of the connected sum will be deduced some types of conditional foldings restricted on the elements of the fundamental groups are deduced. Theorems governing these relations will be achieved. Some applications in manufacturing are presented.

1. Introduction

When a plane sheet of paper is crumpled gently in the hand , and then is crushed flat a against a desk-top. The effect is to cross-cross, the sheet with a pattern of creases, which persist even when the sheet is unfolded and smoothed out again to it is original planar form. At first slight, the pattern may seem random and chaotic. However, a closer inspection will lead to the following observation. First of all, the creases appear to be composed of straight line segments. Secondly, if p is the end-point of such segment, then the total number of crease-segment that end at p is even. Thirdly, the sum of alternate angle between creases at each such point p is equal to π. This physical process can be modelled mathematically as follows. Let us replace both the sheet of paper and the desk-top by Euclidean plane R^2, equipped with its standard flat Riemannian tensor field. We model the crumpling and

Received by the editors on June 06, 2005.
2000 *Mathematics Subject Classifications* : Primary 51H10, 57N10.
Key words and phrases: Manifolds, folding, connected sum, fundamental group.

crushing process by a map $f : R^2 \longrightarrow R^2$ that send each piecewise-straight path in R^2 to piecewise-straight path of the same length [12].

More studies on the folding of manifolds are studied in [3–5, 7, 8]. Various folding problems arising in the physics of membrane and polymers reviewed by Francesco [2]. The unfolding of a manifold introduced in [7]. The deformation retracts of a manifolds defined and discussed in [1, 3, 6].

Poincare' [1895] was the first to construct an algebraic group which is a topological invariant of the space Y to which it is associated, called the fundamental group [11], or first homotopy group. The fundamental groups of some types of a manifold are discussed in [9, 10].

Definition 1.1. The set of homotopy classes of loops based at the point $x_\circ$ with the product operation$[f][g] = [f \cdot g]$ is called the fundamental group and denoted by $\pi_1(X, x_\circ)$ [10].

Definition 1.2. A subset A of a topological space X is called a retract of X if there exist a continuous map $r : X \to A$ (called a retraction) such that $r(a) = a, \ \forall\, a \in A$ [11].

Definition 1.3. A subset A of a topological space X is a deformation retract of X if there exist a retraction $r : X \to A$ and a homotopy $f : X \times I \to X$ such that

$$\left.\begin{array}{l} f(x,0) = x \\ f(x,1) = r(x) \end{array}\right\} \forall\, x \in X, \text{ and}$$

$$f(a,t) = a \qquad \forall\, a \in A,\ t \in [0,1] \ \ [11].$$

Definition 1.4. Let M and N be two smooth manifolds of dimension m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if for every piecewise geodesic path $\gamma : I \to M$ the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ . If f does not preserve length it is called topological folding [9].

Definition 1.5. Let M and N be two smooth manifolds of the same dimension. A map $g : M \to N$ is said to be unfolding of M

into N if every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ [5].

Definition 1.6. Given spaces X and Y with chosen points $x_\circ \in X$ and $y_\circ \in Y$, then the wedge sum $X \vee Y$ is the quotient of the disjoint union $Y \cup X$ obtained identifying $x_\circ$ and $y_\circ$ to a single point [11].

Definition 1.7. Let X and Y be two compact surfaces, then the connected sum $X \# Y$ is the compact surfaces constructed by deleting a small open disc from each of them and pasting the remaining surface pieces together along the edge of discs [11].

2. The main results

Aiming to our study we will introduce the following:

Theorem 2.1. *If $\#_m S^n$ is a connected sum of m-spheres. Then the fundamental group of any folding of $\#_m S^n$ into itself is either isomorphic to Z or identity group.*

Proof. Let $\#_m S^n$ be a connected sum of m-spheres, then any folding of $\#_m S^n$ into itself is either a folding without singularity or a folding with singularity.

Thus if $F\,(\#_m S^n)$ is a folding without singularity, then $F(\#_m S^n)$ is a manifold homeomorphic to S^n, and so $\pi_1(\#_m S^n) \approx \pi_1(\#_m S^n$. Hence, $\#_m S^n$ is either isomorphic to Z or identity group. Also, if $F(\#_m S^n)$ is a folding with singularity, then every loop in $F(\#_m S^n)$ is homotopic to the identity loop and so $\pi_1(F(\#_m S^n)) \approx 0$. □

Lemma 2.2. *Let $\#_m S^n$ be a connected sum of m-spheres and D_n the disjoint union of n discs. Then $\pi_1(\#_m S^n - D_n)$ is either a free group of rank $n-1$ or identity group.*

Proof. If $n = 1$ then every loop in $\#_m S^n - D_n$ is homotopic to the identity loop, and so $\pi_1(\#_m S^n - D_n) \approx 0$. Also, if $n > 1$, from follows it rose leaved $(n-1)$ is a deformation retract of $\#_m S^n - D_n$, that

$\pi_1(\#_m S^n - D_n) \approx \pi_1(\underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{n-1\, term})$, and so $\pi_1(\#_m S^n - D_n) \approx \underbrace{Z * Z * \cdots * Z}_{n-1\, term}$. Hence, $\pi_1(\#_m S^2 - D_n)$ is a free group of rank $n-1$. Therefore $\pi_1(\#_m S^2 - D_n)$ is either a free group of rank $n-1$ or identity group. □

COROLLARY 2.3. *Let $\#_m S^2$ be a connected sum of m-spheres, and D_n the disjoint union of n discs. Then the fundamental group of any folding of $\pi_1(\#_m S^2 - D_n)$ into itself is either a free group of rank $\leq n-1$ or identity group.*

THEOREM 2.4. *Let $\#_m S^n$ be a connected sum of m-spheres, and let D_n be the disjoint union of n discs. Then there are unfoldings* unf : $\#_m S^2 - D_n \to \#_m S^2 - D_n$ *such that $\pi_1(\lim_{n\to\infty}(unf_n(\#_m S^2 - D_n))) \approx 0$.*

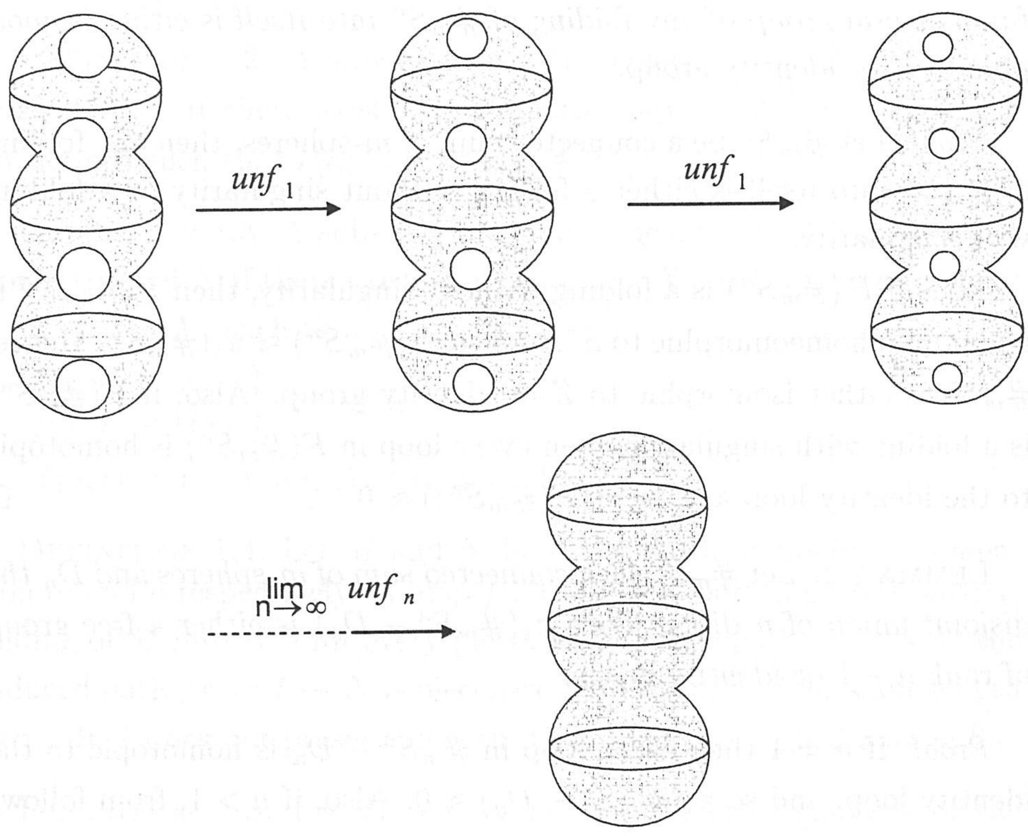

Fig.(1)

Proof. Let $\#_m S^2$ be a connected sum of m-spheres, and D_n the disjoint union of n discs. Then, we can define a sequence of unfoldings

$unf_1 : \#_m S^2 - D_n \to M_1, M_1 \subseteq \#_m S^2$

$unf_2 : M_1 \to M_2, M_1 \subseteq M_2 \subseteq \#_m S^2$

$\vdots$

and so $\lim_{n\to\infty} unf_n(\#_m S^2 - D_n) = \#_m S^2$ as in Fig.(1) for $m = 3, n = 4$.

Hence $\pi_1(\lim_{n\to\infty}(unf_n(\#_m S^2 - D_n))) = \pi_1(\#_m S^2)$.

Therefore, $\pi_1(\lim_{n\to\infty}(unf_n(\#_m S^2 - D_n))) = 0$. $\square$

THEOREM 2.5. *Let $\#_m S^2$ be a connected sum of m-spheres, and let D_n be the disjoint union of n discs. Then there are unfoldings $unf : \#_m S^2 \to \#_m S^2$ such that $\pi_1(\lim_{k\to\infty} unf_k(\#_m S^2))$ is either a free group of rank $n-1$ or identity group.*

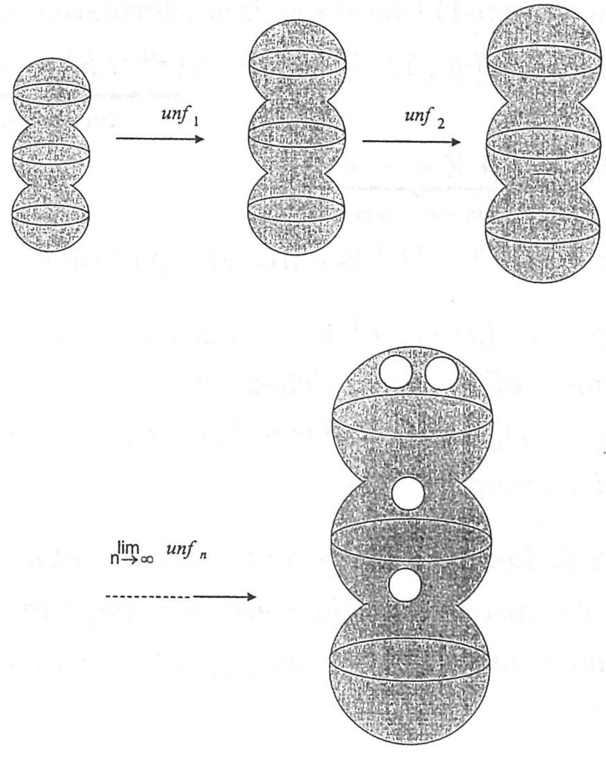

Fig.(2)

Proof. Let $\#_m S^2$ be a connected sum of m-spheres, and D_n the disjoint union of n discs. Then we can define a sequence of unfoldings

$unf_2 : \#_m S^2_1 \to \#_m S^2_2$

$unf_2 : \#_m S^2_1 \to \#_m S^2_2$

$\vdots$

$unf_k : \#_m S^2_{k-1} \to \#_m S^2_k$ and $\lim_{k\to\infty} unf_k(\#_m S^2) = \#_m S^2 - D_n$, as in Fig.(2) for $m = 3, n = 4$.

Hence $\pi_1(\lim_{k\to\infty} unf_k(\#_m S^2)) = \pi_1(\#_m S^2 - D_n)$.

Therefore, $\pi_1(\lim_{k\to\infty} unf_k(\#_m S^2))$ is either a free group of rank $n-1$ or identity group. □

Lemma 2.6. *Let $\#_m T^1$ be a connected sum of m-tori, and let D_n be the disjoint union of n discs. Then $\pi_1(\#_m T^1 - D_n)$ is a free group of rank$= 2m+n-1$.*

Proof. Since $(2m+n\text{-}1)$ leaved rose is a deformation retract of $\#T^1 - D_n$, it follows that $\pi_1(\#_m T^n - D_n) \approx \pi_1(\underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{2m+n-1\, term})$, and so

$\pi_1(\#_m T^n - D_n) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n-1\, term}$.

Therefore, $\pi_1(\#_m T^n - D_n)$ is a free group of rank$= 2m+n-1$. □

Corollary 2.7. *Let $\#_m T^1$ be a connected sum of m-tori, and D_n the disjoint union of n discs. Then the fundamental group of any folding of $\#_m T^1 - D_n$ into itself is either a free group of rank $\leq 2m+n-1$ or identity group.*

Theorem 2.8. *Let $\#_m T^1$ be a connected sum of m-tori, and D_n the disjoint union of n discs. Then there are unfoldings $unf : \#_m T^1 - D_n \to \#_m T^1 - D_n$ such that $\pi_1(\lim_{n\to\infty}(unf_n(\#_m T^1 - D_n)))$ is a free abelian group of rank $2m$.*

Proof. Let $\#_m T^1$ be a connected sum of m-tori and D_n the disjoint union of n discs. Then we can define a sequence of unfoldings

$unf_1 : \#_m T^1 - D_n \to M_1, M_1 \subseteq \#_m T^1$
$unf_2 : M_1 \to M_2, M_1 \subseteq M_2 \subseteq \#_m T^1$
$\vdots$
$unf_n : M_{n-1} \to M_n, M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n \subseteq \#_m T^1$
$\lim_{n\to\infty} unf_n(\#_m T^1 - D_n) = \#_m T^1$ as in Fig.(3) for $m = 4, n = 4$.

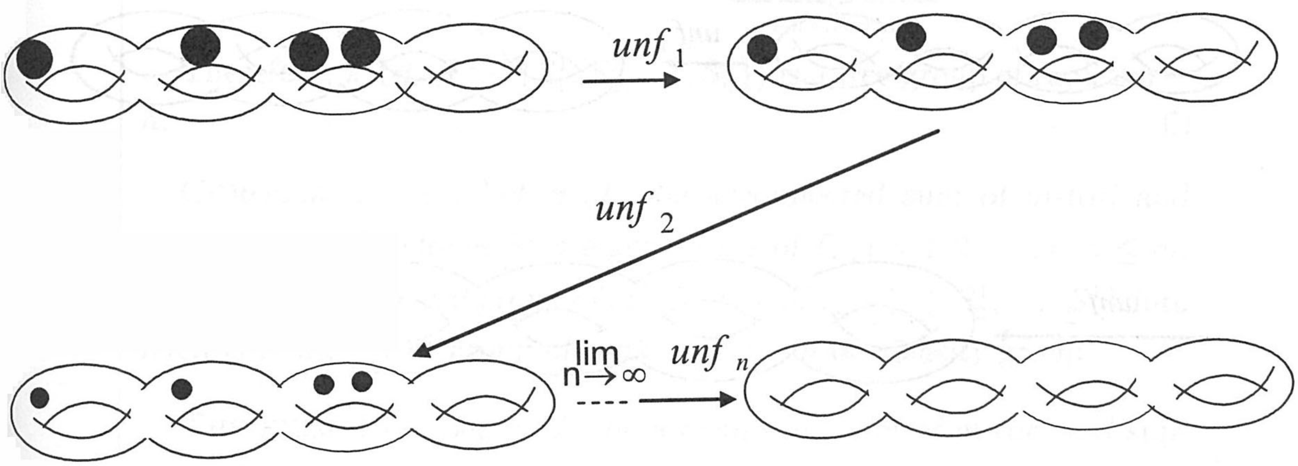

Fig.(3)

Hence $\pi_1(\lim_{n\to\infty} unf_n(\#_m T^1 - D_n)) = \pi_1(\#_m T^1)$.

Therefore, $\pi_1(\lim_{n\to\infty} unf_n(\#_m T^1 - D_n)))$ is a free abelian group of rank $2m$. □

THEOREM 2.9. *Let $\#_m T^1$ be a connected sum of m-tori, and D_n the disjoint union of n discs. Then there are unfoldings $unf : \#_m T^1 \to \#_m T^1$ such that $\pi_1(\lim_{k\to\infty}(unf_k(\#_m T^1))$ is a free group of rank $2m + n - 1$.*

Proof. Let $\#_m T^1$ be a connected sum of m-tori, and D_n the disjoint union of n discs. Then, we can define a sequence of unfoldings

$unf_1 : \#_m T^1 \to \#_m T_1^1$

$unf_2 : \#_m T_1^1 \to \#_m T_2^1$

$\vdots$

$unf_k : \#_m T_{k-1}^1 \to \#_m T_k^1$,

$\lim_{k\to\infty} unf_k(\#_m T^1) = \#_m T^1 - D_n$ as in Fig.(4) for $m = 4, n = 4$.

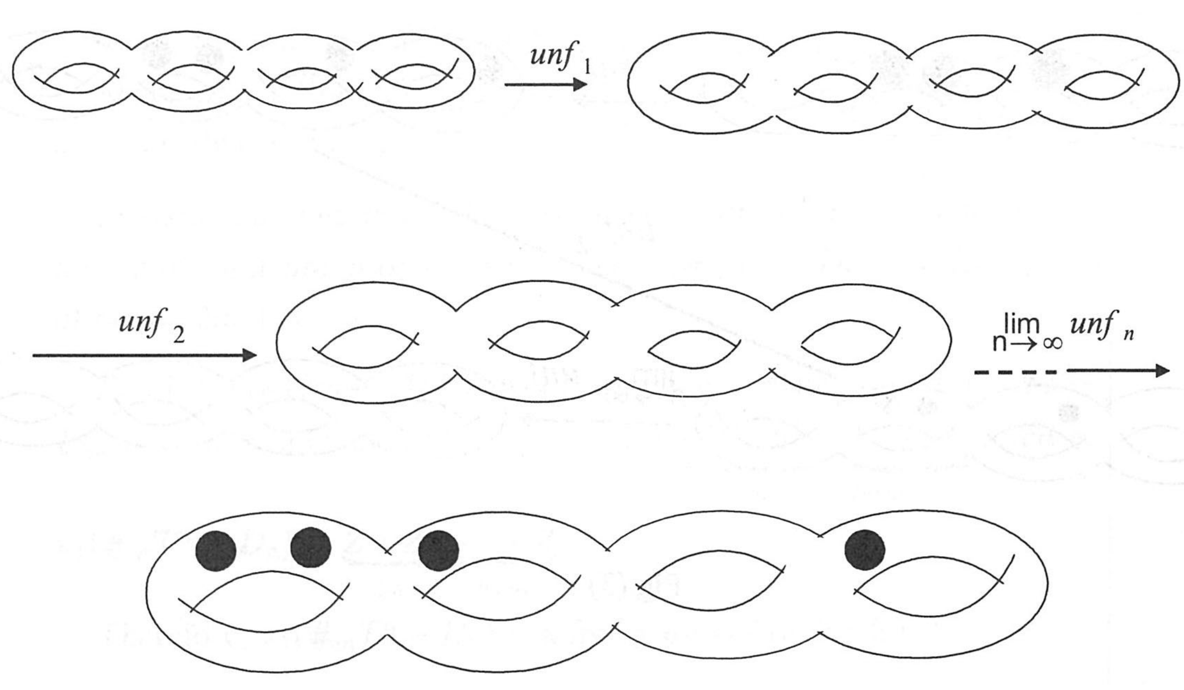

Fig. (4)

Hence $\pi_1(\lim_{k\to\infty} unf_k(\#_m T^1)) = \pi_1(\#_m T^1 - D_n)$.

Therefore, $\pi_1(\lim_{k\to\infty} unf_k(\#_m T^1))$ is a free group of rank $2m + n - 1$. $\square$

THEOREM 2.10. *Let $\#_m T^1$ be a connected sum of m-tori and suppose that the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, n, n \leq m$. Then $\pi_1(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})$ is a free group of rank $2m - n$.*

Proof. Since $(2m-n)$ leaved rose is a deformation retract of

$$\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\},$$

it follows that

$$\pi_1(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\}) \approx \pi_1(\underbrace{(S^1 \vee S^1 \vee \cdots \vee S^1)}_{2m+n\,term})$$

and so $\pi_1(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\}) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n\,term}$.

Therefore, $\pi_1(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})$ is a free group of rank $2m-n$. ∎

COROLLARY 2.11. *Let $\#_m T^1$ be a connected sum of m-tori and suppose that the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, n, n \leq m$. Then the fundamental group of any folding of $T_n - \{S_1^1, S_2^1, \ldots, S_n^1\}$ into itself is either a free group of rank $\leq 2m-n$ or identity group.*

THEOREM 2.12. *Let $\#_m T^1$ be a connected sum of m-tori and suppose that the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, m$. Then there are unfoldings*

$$unf : \#_m T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\} \rightarrow \#_m T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\}$$

such that $\pi_1(\lim_{n\to\infty} unf_n(F(\#_m T^1)))$ is a free abelian group of rank $2m$.

Proof. Let $\#_m T^1$ be a connected sum of m-tori and suppose that the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, m$. Then we can define a sequence of unfoldings

$unf_1 : \#_m T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\} \rightarrow M_1, \#_m T^1 - \{S_1^1, S_2^1, \ldots, S_m^1\} \subseteq M_1 \subseteq \#_m T^1$

$unf_2 : M_1 \rightarrow M_2$, $M_1 \subseteq M_2 \subseteq \#_m T^1$

$\vdots$

$unf_n : M_{n-1} \rightarrow M_n, M_1 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n \subseteq \#_m T^1$.

$\lim_{n\to\infty} unf_n(F(\#_m T^1)) = \#_m T^1$. Hence

$$\pi_1(\lim_{n\to\infty} unf_n(F(\#_m T^1))) \approx \#_m T^1.$$

Therefore, $\pi_1(\lim_{n\to\infty} unf_n(F(\#_m T^1)))$ is a free abelian group of rank $2m$. $\square$

THEOREM 2.13. *If $\#_m T^1$ is a connected sum of m-tori, $\#_k S^2$ is a connected sum of k spheres, and D_n is the disjoint union of n discs $D_n \in \#_m T^1$ or $\#_k S^2$. Then $\pi_1(((\#_m T^1)\#(\#_k S^2 - D_n)$ is a free group of rank$= 2m + n - 1$.*

Proof. Since $(2m + n - 1)$ leaved rose is a deformation retract of $(\#_m T^1)\#(\#_k S^2) - D_n$, it follows that

$$\pi_1(((\#_m T^1)\#(\#_k S^2 - D_n) \approx \pi_1(\underbrace{(S^1 \vee S^1 \vee \cdots \vee S^1)}_{2m+n-1\, term}),$$

and so $\pi_1(((\#_m T^1)\#(\#_k S^2 - D_n) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n-1\, term}$.

Therefore, $\pi_1(((\#_m T^1)\#(\#_k S^2 - D_n)$ is a free group of rank $= 2m + n - 1$. $\square$

COROLLARY 2.14. *If $\#_m T^1$ is a connected sum of m-tori, and $\#_k S^2$ is a connected sum of k spheres, and D_n is the disjoint union of n discs $D_n \in \#_m T^1$ or $\#_k S^2$. Then the fundamental group of any folding of $\pi_1(((\#_m T^1)\#(\#_k S^2 - D_n)$ into itself is a free group of rank $\leq 2m+n-1$ or identity group.*

THEOREM 2.15. *Let $\#_m T^1$ be a connected sum of m-tori, and let $\#_k S^2$ be a connected sum of k spheres. Suppose that the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, n, n \leq m$. Then $\pi_1(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\} \# (\#_k S^2))$ is a free group of rank $2m - n$.*

Proof. Since $(2m - n)$ leaved rose deformation retract of $(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\} \# (\#_k S^2))$, it follows that

$$\pi_1(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\#_k S^2) \approx \pi_1(\underbrace{(S^1 \vee S^1 \vee \cdots \vee S^1)}_{2m+n\, term})$$

so we have $\pi_1((\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\#_k S^2)) \approx \underbrace{Z * Z * \cdots * Z}_{2m+n\ term}.$

Therefore, $\pi_1((\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\#_k S^2))$ is a free group of rank $2m - n$. $\square$

COROLLARY 2.16. *Let $\#_m T^1$ be a connected sum of m-tori, and $\#_k S^2$ a connected sum of k spheres, and suppose that the circles S_i^1 are generators of $T_i^1, i = 1, 2, \ldots, n, n \leq m$. Then the fundamental group of any folding of $(\#_m T^1 - \{S_1^1, S_2^1, \ldots, S_n^1\})\#(\#_k S^2)$ into itself is a free group of rank$\leq 2m - n$ or identity group.*

3. Applications

1. According to the elasticity of wall of S^2 the energy inside S^2, if it is increasing, the sphere will be unfolded until some quantity of the internal energy, after that some explosion will happen causing some holes on the wall as in Fig .(5).

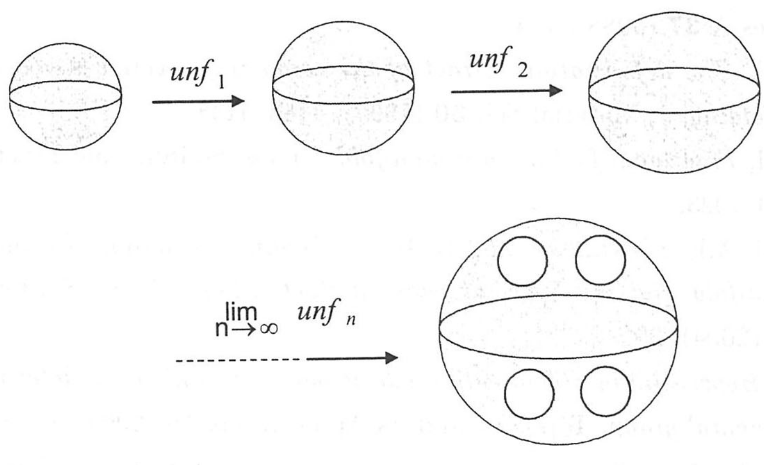

Fig.(5)

2. During the form of cream in the milk, it has a small holes on its surface, these holes gradually decreasing by the time to arrive to a similar shape of a section of sphere.

3. In industry, mechanics use some kinds of hit and heat to close the holes in the cars bodies surface. Also, in wheels which has holes the man closes these holes by iron.

References

1. P.P. Covarrubias, *Retraction in hyperspaces,* Topology and its Applications **135** (2004), 277–291.
2. P. Difrancesco, *Folding and coloring problems in mathematics and physics,* Bull. Amer. Math. Soc. **37** (2000), 251–307.
3. A.E. El-Ahmady, *The deformation retract and topological folding of Buchdahi space,* Periodica Mathematica Hungarica **28**, (1994), 19–30.
4. A.E. El-Ahmady, *Fuzzy Lobachevskian space and its folding,* J. Fuzzy Math. **12** (2004), 255–260.
5. M. El-Ghoul, *Unfolding of Riemannian manifolds,* Commun. Fac. Sci. Univ Ankara Series A **37** (1988), 1–4.
6. M. El-Ghoul, *The deformation retract of the complex projective space and its topological folding,* J. Material Sci. **30** (1995), 4145–4148.
7. M. El-Ghoul, *Fractional folding of a manifold,* Chaos, Solitons and Fractals **12** (2001), 1019–1023.
8. M. El-Ghoul, A.E. El-Ahmady and H. Rafat, *Folding-retraction of chaotic dynamical manifold and the VAK of vacuum fluctuation,* Chaos, Solitons and Fractals **20** (2004), 209–217.
9. R. Frigerio, *Hyperbolic manifold with geodesic boundary which are determine by their fundamental group,* Topology and its Applications **45** (2004), 69–81.
10. A. Hatcher, *Algebraic Topology,* http: /www.math.coronell.edu/hatcher.
11. W.S. Massey, *Algebraic Topology: An Introduction,* Harcourt Brace and World, New York, 1967.
12. S.A. Robertson, *Isometric folding of Riemannian manifolds,* Proc. Roy. Soc. Edinburgh **77** (1977), 275–289.

*
Department of Mathematics
Faculty of Science
Tanta University, Tanta
Egypt

**
Department of Mathematics
Faculty of Science
Tanta University, Tanta
Egypt

***
Department of Mathematics
Faculty of Science
Tanta University, Tanta, Egypt
E-mail: hishamrafat2005@yahoo.com

****
Department of Mathematics
Faculty of Science
Tanta University, Tanta
Egypt