JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **18**, No.2, October 2005

REDUCIBILITY, MULTIBASIC EXPANSION AND INTEGRAL REPRESENTATION FOR BASIC APPELL FUNCTIONS

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Abstract. We give bibasic expansion for basic Appell functions $\Phi^{(1)}$ and $\Phi^{(2)}$, and their integral representations. We also give a continued fraction representation for $\Phi^{(2)}$.

1. Introduction

The basic analogue of Appell's hypergeometric functions of two variables were first defined and studied by F.H. Jackson [6, 7]. R.P. Agarwal [1, 2] also studied these functions and gave some general identities involving these functions. G.E. Andrews [3] also studied these functions and showed that the first of the Appell series $\Phi^{(1)}$ can be reduced to a series ${}_3\varphi_2$ series.

We defined and considered bibasic Appell series in our paper [10]. This is a new approach. In another paper [11] we have summation formulae and continued fraction representation of the bibasic Appell functions.

In this paper we give a bibasic expansion for Appell functions $\Phi^{(1)}$ and $\Phi^{(2)}$. We also give integral representation for these functions.

By using certain transformations we have reduced the Appell functions to a ${}_2\varphi_1$ series. We then give a relation between $\Phi^{(2)}$ and $\Phi^{(3)}$ series and some summation results. We have also given a continued fraction representation for $\Phi^{(2)}$.

Received by the editors on May 01, 2005.
2000 *Mathematics Subject Classifications* : Primary 33Dxx, 33D50.
Key words and phrases: q-bibasic hypergeometric series, multibasic series.

We shall use the following usual basic hypergeometric notations: For $|q^k| < 1$,

$$\begin{aligned}
(a;q^k)_n &= (1-a)(1-aq^k)\ldots(1-aq^{k(n-1)}), \quad n \geq 1 \\
(a;q^k)_0 &= 1, \\
(a;q^k)_\infty &= \Pi_{j=0}^{\infty}(1-aq^{kj}), \\
(a_1,a_2,\cdots,a_m;q^k)_n &= (a_1;q^k)_n(a_2;q^k)_n\cdots(a_m;q^k)_n, \\
(a;q)_n &= (a)_n,
\end{aligned}$$

$$\phi\left[\begin{matrix} a_1,\cdots,a_r : c_{1,1},\cdots,c_{1,r_1} : \cdots : c_{m,1},\cdots,c_{m,r_m} \\ b_1,\cdots,b_s : e_{1,1},\cdots,e_{1,s_1} : \cdots : e_{m,1},\cdots,e_{m,s_m} \end{matrix} ; q, q_1, \cdots, q_m; z\right]$$

$$= \sum_{n=0}^{\infty} \left(\frac{(a_1,\cdots,a_r;q)_n}{(q,b_1,\cdots,b_s;q)_n} z^n \left[(-1)^n q^{\frac{n^2-n}{2}}\right]^{1+s-r} \right.$$

$$\left. \times\, \Pi_{j=1}^{m} \frac{(c_{j,1},\cdots,c_{j,r_j};q_j)_n}{(e_{j,1},\cdots,e_{j,s_j};q_j)_n} \left[(-1)^n q^{\frac{n^2-n}{2}}\right]^{s_j-r_j} \right),$$

$${}_A\varphi_{A-1}\left[a_1,a_2,\cdots,a_A;b_1,b_2\cdots,b_{A-1};q_1,z\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a_1;q_1)_n\cdots(a_A;q_1)_n z^n}{(b_1;q_1)_n\cdots(b_{A-1};q_1)_n(q_1;q_1)_n}, \qquad |z| < 1.$$

2. Basic Appell series

The four basic Appell series defined by Jackson [6] are

$$\Phi^{(1)}\left[\, a;b,b';c;x,y \,\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(q)_m(q)_n(c)_{m+n}},$$

$$\Phi^{(2)}\left[\, a;b,b';c,c';x,y \,\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_m(b')_n x^m y^n}{(q)_m(q)_n(c)_m(c')_n},$$

$$\Phi^{(3)}\left[\,a;a';b,b';c;x,y\,\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_m(a')_n(b)_m(b')_n x^m y^n}{(q)_m(q)_n(c)_{m+n}},$$

$$\Phi^{(4)}\left[\,a,b;c,c';x,y\,\right]=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n} x^m y^n}{(q)_m(q)_n(c)_m(c')_n}.$$

3. Reducibility as simple basic hypergeometric series

The double series defining Appell functions can be written as a simple series containing basic hypergeometric series. These will be used later in giving summation formulae.

$$(1)\qquad \Phi^{(1)}\left[\,a;b,b';c;x,y\,\right]=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(q)_m(c)_m}{}_2\varphi_1\left[\,aq^m,b';cq^m;y\,\right]x^m.$$

$$(2)\qquad \Phi^{(2)}\left[\,a;b,b';c,c';x,y\,\right]=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(q)_m(c)_m}{}_2\varphi_1\left[\,aq^m,b';c';y\,\right]x^m.$$

$$(3)\qquad \Phi^{(3)}\left[\,a,a';b,b';c;x,y\,\right]=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(q)_m(c)_m}{}_2\varphi_1\left[\,a',b';cq^m;y\,\right]x^m.$$

$$(4)\qquad \Phi^{(4)}\left[\,a;b;c,c';x,y\,\right]=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(q)_m(c)_m}{}_2\varphi_1\left[\,aq^m,bq^m;c';y\,\right]x^m.$$

From this representation the Appell function can be written in q-analogue of Gauss' series :

$$\Phi^{(1)}\left[\,a,b,b';c;x,0\,\right]=\Phi^{(1)}\left[\,a;b,1;c;x,y\,\right]={}_2\varphi_1\left[\,a,b;c;x\,\right].$$

$$\Phi^{(1)}\left[\,a,b,b';c;0,y\,\right]=\Phi^{(1)}\left[\,a;1,b';c;x,y\,\right]={}_2\varphi_1\left[\,a,b';c;y\,\right].$$

$$\Phi^{(2)}\left[\,a;b,b';c,c';x,0\,\right] = \Phi^{(2)}\left[\,a;b,1;c,c';x,y\,\right] = {}_2\varphi_1\left[\,a,b;c;x\,\right].$$

$$\Phi^{(2)}\left[\,a;b,b';c,c';0,y\,\right] = \Phi^{(2)}\left[\,a;1,b';c,c';x,y\,\right] = {}_2\varphi_1\left[\,a,b';c;y\,\right].$$

$$\begin{aligned}\Phi^{(3)}\left[\,a,a';b,b';c;x,0\,\right] &= \Phi^{(3)}\left[\,a,1;b,b';x,y\,\right]\\ &= \Phi^{(3)}\left[\,a,a';b,1;c;x,y\,\right] = {}_2\varphi_1\left[\,a,b;c;x\,\right]\\ \Phi^{(3)}\left[\,a,a';b,b';c;0,y\,\right] &= \Phi^{(3)}\left[\,1,a';b,b';c;x,y\,\right]\\ &= \Phi^{(3)}\left[\,a,a';1,b';c;x,y\,\right] = {}_2\varphi_1\left[\,a,b;c;x\,\right].\end{aligned}$$

$$\Phi^{(4)}\left[\,a;b,b';c,c';x,0\,\right] = {}_2\varphi_1\left[\,a,b;c;x\,\right].$$

$$\Phi^{(4)}\left[\,a;b;c,c';0,y\,\right] = {}_2\varphi_1\left[\,a,b;c';x\,\right].$$

4. Transformations

We have transformed the q-Appell functions defined by (1), (2), (3) and (4) by using Heine's transformation [5, p.9, (1.4.1)], Heine's q-analogue of Eulers transformation [5, p. 10, (1.4.3)], Jackson's q-analogue of Pfaff-Kummer's transformation [5, p.11, (1.5.4)]

$$\begin{aligned}&\Phi^{(1)}\left[\,a;b,b';c;x,y\,\right]\\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (b')_\infty (ayq^m)_\infty}{(q)_m (c)_m (cq^m)_\infty (y)_\infty} {}_2\varphi_1\left[\,cq^m/b',y;b';aq^m y\,\right]\\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (aq^m)_\infty (b'y)_\infty}{(q)_m (c)_m (cq^m)_\infty (y)_\infty} {}_2\varphi_1\left[\,c/a,y;aq^m;b'y\,\right]\\ &= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (aq^m y)_\infty}{(q)_m (c)_m (y)_\infty} {}_2\varphi_2\left[\,aq^m,cq^m/b';cq^m,aq^m y;b'y\,\right].\end{aligned}$$

$$\Phi^{(2)}\left[\,a;b,b';c,c';x,y\,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (b')_\infty (ayq^m)_\infty}{(q)_m(c)_m(c')_\infty (y)_\infty} {}_2\varphi_1 \left[\, c'/b', y; ayq^m; b' \,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (ab'yq^m/c')_\infty}{(q)_m(c)_m(y)_\infty} {}_2\varphi_1 \left[\, c'q^{-m}/a, c'/b'; c'; ab'yq^m/c' \,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (ayq^m)_\infty}{(q)_m(c)_m(y)_\infty} {}_2\varphi_2 \left[\, aq^m, c'/b'; c', ayq^m; b'y \,\right].$$

(5) $\Phi^{(3)} \left[\, a, a'; b, b'; c; x, y \,\right]$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (a'y)_\infty (b')_\infty}{(q)_m(c)_m(cq^m)_\infty (y)_\infty} {}_2\varphi_1 \left[\, c'q^m/b', y; ay; b' \,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (a'b'yq^{-m}/c)_\infty}{(q)_m(c)_m(y)_\infty} {}_2\varphi_1 \left[\, cq^m/a', c'q^m/b'; cq^m; a'b'yq^{-m}/c \,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (a'y)_\infty}{(q)_m(c)_m(y)_\infty} {}_2\varphi_2 \left[\, a', cq^m/b'; cq^m, a'y; b'y \,\right].$$

$\Phi^{(4)} \left[\, a; b; c, c'; x, y \,\right]$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (bq^m)_\infty (ayq^m)_\infty}{(q)_m(c)_m(c')_\infty (y)_\infty} {}_2\varphi_1 \left[\, c'q^{-m}/b, y; ayq^m; bq^m \,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (abyq^{2m}/c')_\infty}{(q)_m(c)_m(y)_\infty} {}_2\varphi_1 \left[\, c'q^{-m}/a, c'q^{-m}/b; c'; abyq^{2m}/c' \,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m (abyq^m)_\infty}{(q)_m(c)_m(y)_\infty} {}_2\varphi_2 \left[\, aq^m, c'q^{-m}/b; c'; ayq^m; b'yq^m \,\right].$$

5. Multibasic expansions

We shall give multibasic expansion for the q-Appell functions. We use the summation formula [5, p 71, (3.6.7)] and [9, Lemma 10, p. 57], to have the bibasic expansion formula

$$\sum_{k=0}^{n} \frac{(1-ap^kq^k)(1-bp^kq^{-k})(a,b;p)_k(c,a/bc;q)_kq^k}{(1-a)(1-b)(q,aq/b;q)_k(ap/c,bcp;p)_k} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$= \sum_{m=0}^{\infty} \frac{(ap,bp;p)_m(cq,aq/bc;q)_m}{(ap/c,bcp;p)_m(q,aq/b;q)_m} \alpha_m. \tag{6}$$

COROLLARY 5.1. *Taking $a=0$, $q \to q^2$ and $p=q$ in (6), we have*

$$\sum_{k=0}^{\infty} \frac{(1-bq^{-k})(b;q)_k(c;q^2)_kq^{2k}}{(1-b)(q^2,q^2)_k(bcq;q)_k} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$= \sum_{m=0}^{\infty} \frac{(bq;q)_n(cq^2;q^2)_n}{(q^2:q^2)_n(bcq;q)_n} \alpha_n. \tag{7}$$

COROLLARY 5.2. *Taking $a=0$, $q \to q^3$ and $p=q$ in (6), we have*

$$\sum_{k=0}^{\infty} \frac{(1-bq^{-2k})(b;q)_k(c;q^3)_kq^{3k}}{(1-b)(q^3,q^3)_k(bcq;q)_k} \sum_{m=0}^{\infty} \alpha_{m+k}$$

$$= \sum_{m=0}^{\infty} \frac{(bq;q)_n(cq^3;q^3)_n}{(q^3;q^3)_n(bcq;q)_n} \alpha_n. \tag{8}$$

THEOREM 5.3. *The multibasic expansion of $\Phi^{(1)}$:*

$$\Phi^{(1)}\left[\,a;b,b';c;x,y\,\right] = \sum_{k=0}^{\infty} \frac{(1-b)(1-xq^{-2k-1})(c/a,x,y;q)_k(aq^3)^k}{(1-xq^{k-1})(1-bq^{3k})(q,bx,b'y;q)_n}$$
$$\times\ \phi\left[\begin{matrix} q,cq^{-k}/a,yq^k:q^{3k+3} \\ q^{k+1},b'yq^k:bq^{3k+3}\end{matrix};q,q^3:a\right]$$

Proof. Putting $k=1$ in Theorem 1 of Srivastava [10, p. 31], we have

$$\Phi^{(1)}\left[\,a;b,b';c;x,y;q\,\right]$$

$$= \frac{(a)_\infty(bx)_\infty(b'y)_\infty}{(c)_\infty(x)_\infty(y)_\infty} {}_3\varphi_2\left[\,c/a,x,y;bx,by;q;a\,\right]. \tag{9}$$

Take $b = x/y$, $c = b$ and $\alpha_n = \frac{(c/a;q)_n(y;q)_n(q^3;q^3)_n a^n}{(q;q)_n(b'y;q)_n(bq^3;q^3)_n}$ in (8) of Corollary 5.2 we have the right-side

$$= \Phi^{(1)}\left[\, a; b, b'; c; x, y \,\right]$$

and the left-side

$$= \sum_{k=0}^{\infty} \frac{(1-xq^{-2k-1})(x/q;q)_k(b;q^3)_k q^{3k}}{(1-x/q)(q^3,q^3)_k(bx;q)_k}$$

$$\times \sum_{m=0}^{\infty} \frac{(c/a;q)_{m+k}(y;q)_{m+k}(q^3;q^3)_{m+k}a^{m+k}}{(q;q)_{m+k}(b'y;q)_{m+k}(bq^3;q^3)_{m+k}}$$

$$= \sum_{k=0}^{\infty} \frac{(1-xq^{-2k-1})(x;q)_{k-1}(b;q^3)_k q^{3k}}{(q^3,q^3)_k(bx;q)_k}$$

$$\times \sum_{m=0}^{\infty} \frac{(c/a;q)_k(cq^{-k}/a:q)_m(y;q)_k(yq^k;q)_m(q^3;q^3)_k(q^{3k+3};q^3)_m a^{m+k}}{(q;q)_k(q^{k+1}:q)_m(b'y;q)_k(b'yq^k;q)_m(bq^3;q^3)_k(bq^{3k+3};q^3)_m}$$

$$= \sum_{k=0}^{\infty} \frac{(1-b)(1-xq^{-2k-1})(c/a,x,y;q)_k(aq^3)^k}{(1-xq^{k-1})(1-bq^{3k})(q,bx,b'y;q)_n}$$

$$\times \phi\left[\begin{matrix} q, cq^{-k}/a, yq^k : q^{3k+3} \\ q^{k+1}, b'yq^k : bq^{3k+3} \end{matrix} ; q, q^3 : a\right],$$

as desired. □

THEOREM 5.4. *The multibasic expansion of* $\Phi^{(2)}$:

$$\Phi^{(2)}\left[\, a; b, b'; c, b'; x, y \,\right] = \frac{(ay)_\infty}{(yq)_\infty} \sum_{k=0}^{\infty} \frac{(1-y)(1-aq^{-k-1})(a,b;q)_k}{(1-yq^{2k})(1-aq^{k-1})(q,ay,c;q)_k}$$

$$\times \ \phi\left[\begin{matrix} q, bq^k : q^{2k+2} \\ q^{k+1}, cq^k : yq^{2k+2} \end{matrix} ; q, q^3 : x\right].$$

Proof. Take $b = a/q$, $c = y$ and $\alpha_n = \frac{(b;q)_n(q^2;q^2)_n x^n}{(q;q)_n(c;q)_n(yq^2;q^2)_n}$ in (7) of Corollary 5.1 and using (2) after a little simplification we have the theorem. □

6. Integral representation

Using the definition of q-integral, we write $\Phi^{(1)}$ and $\Phi^{(2)}$ as q-integrals. Thomae and Jackson [5, p. 19, (1.11.1)] defined q-integral by

$$\int_0^1 f(t)d_qt = (1-q)\sum_{n=0}^{\infty} f(q^n)q^n. \tag{10}$$

Taking $f(t) = t^{x-1}(tq;q)_\infty$, we have

$$\begin{aligned}\int_0^1 t^{x-1}(tq;q)_\infty d_qt &= (1-q)\sum_{n=0}^{\infty}(q^{n+1};q)_\infty q^{nx} \\ &= (1-q)(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{nx}}{(q;q)_n} \\ &= \frac{(1-q)(q;q)_\infty}{(q^x;q)_\infty}.\end{aligned}$$

That is,

$$\frac{1}{(q^x;q)_\infty} = \frac{(1-q)^{-1}}{(q;q)_\infty}\int_0^1 t^{x-1}(tq;q)_\infty d_qt. \tag{11}$$

THEOREM 6.1.

$$\begin{aligned}&\Phi^{(1)}\left[\, a;0,b';c;q^x,y \,\right] \\ &= \frac{(1-q)^{-1}(a)_\infty}{(c)_\infty(q)_\infty}\int_0^1 \frac{t^{x-1}(ct)_\infty(tq)_\infty}{(at)_\infty}\Phi^{(1)}\left[\, at;0,b';ct;0,y \,\right] d_qt\end{aligned}$$

Proof. Putting $b=0$ in (9) we have

$$\begin{aligned}\Phi^{(1)}\left[\, a;0,b';c;x,y \,\right] &= \frac{(a)_\infty(b'y)_\infty}{(c)_\infty(x)_\infty(y)_\infty}\sum_{m=0}^{\infty}\frac{(c/a)_m(x)_m(y)_m a^m}{(q)_m(b'y)_m} \\ &= \frac{(a)_\infty(b'y)_\infty}{(c)_\infty(y)_\infty}\sum_{m=0}^{\infty}\frac{(c/a)_m(y)_m a^m}{(q)_m(b'y)_m(xq^m)_\infty}.\end{aligned}$$

Writing q^x for x, we have

$$\Phi^{(1)}\left[\, a;0,b';c;q^x,y \,\right] = \frac{(a)_\infty (b'y)_\infty}{(c)_\infty (y)_\infty} \sum_{m=0}^{\infty} \frac{(c/a)_m (y)_m a^m}{(q)_m (b'y)_m} \frac{1}{(q^{m+x};q)_\infty}.$$

By (11), we have

$$\begin{aligned}&\Phi^{(1)}\left[\, a;0,b';c;q^x,y \,\right]\\ &= \frac{(1-q)^{-1}(a)_\infty (b'y)_\infty}{(q)_\infty (c)_\infty (y)_\infty} \sum_{m=0}^{\infty} \frac{(c/a)_m (y)_m a^m}{(q)_m (b'y)_m} \int_0^1 t^{m+x-1}(tq;q)_\infty d_q t.\end{aligned}$$

But

$$\Phi^{(1)}\left[\, a;0,b';c;0,y \,\right] = \frac{(a)_\infty (b'y)_\infty}{(c)_\infty (y)_\infty} \sum_{m=0}^{\infty} \frac{(c/a)_m (y)_m a^m}{(q)_m (b'y)_m}.$$

By (9),

$$\Phi^{(1)}\left[\, at;0,b';ct;0,y \,\right] = \frac{(at)_\infty (b'y)_\infty}{(ct)_\infty (y)_\infty} \sum_{m=0}^{\infty} \frac{(c/a)_m (y)_m (at)^m}{(q)_m (b'y)_m}.$$

Hence

$$\begin{aligned}&\Phi^{(1)}\left[\, a;0,b';c;q^x,y \,\right]\\ &= \frac{(1-q)^{-1}(a)_\infty}{(c)_\infty (q)_\infty} \int_0^1 \frac{t^{x-1}(ct)_\infty (tq)_\infty}{(at)_\infty} \Phi^{(1)}\left[\, at;0,b';ct;0,y \,\right] d_q t,\end{aligned}$$

as desired. □

THEOREM 6.2.

$$\begin{aligned}&\Phi^{(2)}\left[\, a;y,b';c,b';x,y \,\right]\\ &= \frac{(1-q)^{-1}(q^y)_\infty}{(q)_\infty} \int_0^1 t^{y-1}(tq;q)_\infty \Phi^{(2)}\left[\, a;0,b';c;b';xt,y \,\right] d_q t.\end{aligned}$$

Proof. The proof is similar to the proof of Theorem 6.1. □

7. Relation between $\Phi^{(2)}$ and $\Phi^{(3)}$

By specializing the parameter we give relation between $\Phi^{(2)}$ and $\Phi^{(3)}$ in the following theorem.

THEOREM 7.1.

$$\Phi^{(2)}\left[\, a;b,b';0,bb';x,y \,\right] = \frac{(ay)_\infty (b')_\infty}{(y)_\infty (a'b')_\infty} \Phi^{(3)}\left[\, a,a';b,y;ab';x,y \,\right].$$

Proof. Putting $c=0, c'=bb'$ in (2), we have

$$\Phi^{(2)}\left[\, a;b,b';0;bb';x,y \,\right]$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m (b)_m x^m}{(q)_m} {}_2\varphi_1\left[\, aq^m, b'; bb'; y \,\right] x^m. \tag{12}$$

Putting $y=b', b'=y$ in (5) and then $c=ay, a=b$, we have

$$\Phi^{(3)}\left[\, a,b;b,y;ay;x,b' \,\right]$$

$$= \frac{(y)_\infty (a'b')_\infty}{(ay)_\infty (b')_\infty} \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(q)_m} {}_2\varphi_1\left[\, aq^m, b'; a'b'; y \,\right] x^m. \tag{13}$$

(12) and (13) give the theorem. □

8. Reducibility of $\Phi^{(2)}$

By taking suitable values of the parameter, we express $\Phi^{(2)}$ as ${}_2\varphi_1$ series and also give a summation formula for $\Phi^{(2)}$. By expressing $\Phi^{(2)}$ as ${}_2\varphi_1$ series , we can have different forms for $\Phi^{(2)}$ by applying transformation formulae for ${}_2\varphi_1$ series.

Putting $b'=c'$ in (2)

$$\Phi^{(2)}\left[\, a;b,b';c,b';x,y \,\right] = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(q)_m (c)_m} {}_1\varphi_0\left[\, aq^m; -; y \,\right] x^m$$

$$= \sum_{m=0}^{\infty} \frac{(a)_m(b)_m(aq^m y)_\infty x^m}{(q)_m(c)_m(y)_\infty}$$

$$= \frac{(ay)_\infty}{(y)_\infty} \sum_{m=0}^{\infty} \frac{(a)_m(b)_m x^m}{(q)_m(c)_m(ay)_m} \tag{14}$$

Putting $c = 0$, we have

$$\Phi^{(2)}\left[\, a; b, b'; 0, b'; x, y \,\right] = \frac{(ay)_\infty}{(y)_\infty} {}_2\varphi_1 \left[\, a, b; ay; x \,\right], \tag{15}$$

which gives the reduction of $\Phi^{(2)}$ into q-analogue of Gauss' series. Similar result is

$$\Phi^{(2)}\left[\, a; b, b'; b, 0; x, y \,\right] = \frac{(ax)_\infty}{(x)_\infty} {}_2\varphi_1 \left[\, a, b'; ax; y \,\right]. \tag{16}$$

Taking $b = y/x$, in (15), we have

$$\begin{aligned} \Phi^{(2)}\left[\, a; y/x, b'; 0, b'; x, y \,\right] &= \frac{(ay)_\infty}{(y)_\infty} {}_2\varphi_1 \left[\, a, y/x; ay; y \,\right] \\ &= \frac{(ax)_\infty}{(x)_\infty}, \end{aligned} \tag{17}$$

which is a summation formulae for

$$\Phi^{(2)}\left[\, a; y/x, b'; 0, b'; x, y \,\right].$$

9. Continued fraction representation

We have expressed $\Phi^{(2)}$ as ${}_2\varphi_1$ series and with this relation we prove a three term relation to have the continued fraction representation.

THEOREM 9.1.

$$\frac{\Phi^{(2)}\left[\, a; y/x, b'; 0, b'; x, y \,\right]}{\Phi^{(2)}\left[\, aq; y/x; b'; 0, b'; x, y \,\right]}$$

$$= -ax + \cfrac{(1-axq)}{-axq + \cfrac{(1-axq^2)}{\cfrac{\Phi^{(2)}\left[\, aq^2; y/x, b'; 0, b'; x, y \,\right]}{\Phi^{(2)}\left[\, aq^3; y/x; b'; 0, b'; x, y \,\right]\cdots}}}$$

Proof. By (17), we have the three term relation

$$\begin{aligned}\Phi^{(2)}\left[\, a; y/x, b'; 0, b'; x, y \,\right] &= -ax\Phi^{(2)}\left[\, aq; y/x, b'; 0, b'; x, y \,\right] \\ &+ (1-axq)\Phi^{(2)}\left[\, aq^2; y/x, b'; 0, b'; x, y \,\right].\end{aligned}$$

This gives

$$\frac{\Phi^{(2)}\left[\, a; y/x, b'; 0, b'; x, y \,\right]}{\Phi^{(2)}\left[\, aq; y/x, b'; 0, b'; x, y \,\right]} = -ax + \cfrac{(1-axq)}{\cfrac{\Phi^{(2)}\left[\, aq; y/x, b'; 0, b'; x, y \,\right]}{\Phi^{(2)}\left[\, aq^2; y/x, b'; 0, b'; x, y \,\right]}}.$$

On iteration, we have the theorem. □

Conclusion

Andrews [3] wrote that a large amount is known about ordinary Appell series, however, the literature of basic Appell series is less extensive. Slater [8, p. 234] wrote that there appears to be no systematic attempt to find summation theorems for basic Appell series. This paper may be helpful in getting more results for basic Appell functions.

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