4각형 골조의 고유치와 고유치 간의 관계

The Eigenvalues and Their Relationships for the Rectangular Frame

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Abstract

Finite element method is applied to the determinations of the two eigenvalues(the elastic critical load and the natural frequence of lateral vibrations) of single story-3 equal bay rectangular frame. The analysis parameters are taper parameter α for column, and beam span to column height ratio, β and second moment area ratio of beam to column, ϒ. Support condition at the column base and sway condition at the column top are also considered in the stability analysis of frame. The changes in the coefficient of eigenvalue are represented by algebraic function of analysis parameter. The coefficients estimated by the proposed algebraic function show good agreement with those determined by finite element method, which suggest the design aid role of the proposed function. By increasing the column axial forces step by step, the corresponding frequencies are also determined, which makes one examine or confirm the relationship suggested by other studies.

요 지

유한요소법을 이용하여 기둥단면이 직선형태로 변화하는 3경간 단층 골조의 두 고유치 (탄성임계하중과 횡방향 기본진동수)를 산정하였다. 수치해석에서 고려한 변수는 기둥의 taper 비 (=α), 경간대 층고비 (=β), 보와 기둥의 단면2차 모멘트 비(=Y)이다. 또한 주각의 지지상태와 주두의 수평동(side-sway)유무가 고유치에 미치는 영향도 고려하였다. 하나의 연속함수로부터 고유치의 변화 추정이 가능한 대수 함수식을 제안하였다. 대수함수식의 변수는 수치해석에서의 변수 즉 α, β 및 Y 이다. 골조에 작용하는 축방향력의 크기를 점차 증가시켜 가면서 여기에 대응하는 진동수의 감소현상을 검토하였다.

Keywords : Tapered Column, Elastic Critical Load, Fundamental Frequency of Lateral Vibration, Unbraced Frame, Braced Frame

핵심 용어 : Taper진 기둥, 탄성 임계하중, 횡 방향 기본 진동수, 비 가세 골조, 가세 골조

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● 본 논문에 대한 토의를 2005년 9월 30일까지 학회로 보내주시면 2006년 1월호에 토론결과를 게재하겠습니다.

1. Introduction

The elastic critical load (or buckling load) is closely related to the magnification factor when the flexural member is under static loading and the prime factor to be considered is the critical load when one is involved in the structural design of a slender compression member. Meanwhile, dynamic response of a structure is usually expressed in terms of the 1st mode natural frequency (also called the resonance factor). For an independent or isolated member, two seperate eigenvalues(elastic critical load and fundamental frequency) and the relationship between axial thrust and reduced frequency can be determined by adopting conventional analysis tools, such as neutral equilibrium method and Rayleigh-Ritz method. For the framed structures, however, the references are sparse and the determination of them is possible only when one relies on numerical methods.

In the study, two eigenvalues for the rectangular frame shown in Fig. 1 are determined by the finite element method. The changes in the coefficient of eigenvalues are represented by algebraic functions of analysis parameters. To investigate the axial thrust vs frequency relationship, column axial force is increased step by step and corresponding fundamental frequency is determined. On the whole, the relationship between axial thrust vs the square of frequency is approximately linear, and is exactly linear in the case of an isolated prismatic member with simply supported ends.

2. Scope of the Study

For an isolated prismatic member with simply supported ends, the axial thrust, P, and the reduced lateral vibration frequency, w, is generally related by the following equation:

$$\frac{P}{P_{cr}}+\left(\frac{w}{w_0}\right)^2=1.0 \quad (1)$$

where P_{cr} and w_0 denote the elastic critical load and the fundamental frequency of lateral vibration, respectively. Experimental tests on the plane trusses and rigid rectangular frames with prismatic columns reveal that the above relationship is also applicable with minimal error.

This study aims to examine the validity Eq.(1) for the single story-3 equal bay frame with linearly tapered columns shown in Fig. 1. Together with the frame, the parameters and other factors adopted in the finite-element eigenvalue analysis are also shown in the figure.

- column

$$A(x)=A_0\left(1+\alpha\frac{x}{L}\right)^m \quad (2a)$$

$$I(x)=I_0\left(1+\alpha\frac{x}{L}\right)^n \quad (2a)$$

(m, n) (=sectional property parameter) = $(1,3)$

α (= *taper parameter*) = 0.0, 0.2, ⋯, 2.0

- beam

β (=beam span to column height ratio) =1, 2, 3, 4

γ (=second moment of area ratio) =0.5, 1.0, 1.5, 2.0

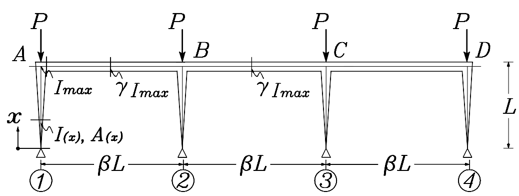
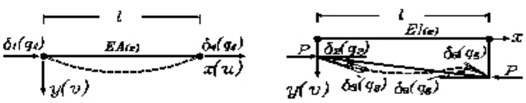


Fig. 1 Frame and analysis parameters and factors

- support condition
  hinged-hinged-hinged bases
  fixed-fixed-fixed bases
- frame classification
  unbraced (side sway permitted) frame
  braced (side sway prevented) frame

3. Element stiffness matrices

Fig. 2 shows a linear element having 3 degrees of freedom at each node. In this case,



(a) axial deformation (b) lateral deformation

Fig. 2 Beam-column element

element displacement functions, $u(x)$ and $v(x)$, can be represented by the combinations of shape function set, $[f]$, and nodal displacement vectors, $\{\delta\}$,

$$u=u(x)=[\,f_1 \quad f_4\,]\begin{Bmatrix}\delta_1\\ \delta_4\end{Bmatrix}=[f_a]\begin{Bmatrix}\delta_1\\ \delta_4\end{Bmatrix} \tag{3a}$$

$$v=v(x)=[\,f_2\ \ f_3\ \ f_5\ \ f_6\,]\begin{Bmatrix}\delta_2\\ \delta_3\\ \delta_5\\ \delta_6\end{Bmatrix}=[f_b]\begin{Bmatrix}\delta_2\\ \delta_3\\ \delta_5\\ \delta_6\end{Bmatrix} \tag{3b}$$

where shape functions $f_i\ (i=1,2,\cdots,6)$ are given by

$$[f_a]=[\,f_1 \quad f_4\,],\ f_1=1-x/l,\ f_4=x/l \tag{4a}$$

$$[f_b]=[\,f_2\ \ f_3\ \ f_5\ \ f_6\,],\ f_2=1-3\left(\frac{x}{l}\right)^2+2\left(\frac{x}{l}\right)^3,$$
$$f_3=x\left(1-\frac{x}{l}\right)^2, f_5=3\left(\frac{x}{l}\right)^2, f_6=\frac{x^2}{l}\left(\frac{x}{l}-1\right) \tag{4b}$$

The strain energy stored in an element can be represented by the following expressions:

$$\begin{aligned}U&=\frac{1}{2}\int_0^l EA(x)\left(\frac{du}{dx}\right)^2dx+\frac{1}{2}\int_0^l EI(x)\left(\frac{d^2v}{dx^2}\right)^2dx\\ &=\frac{1}{2}\{\delta\}^T\Bigg[\int_0^l\left[\frac{df_a}{dx}\right]^T EA(x)\left[\frac{df_a}{dx}\right]dx\\ &\quad+\int_0^l\left[\frac{d^2f_b}{dx^2}\right]^T EI(x)\left[\frac{d^2f_b}{dx^2}\right]dx\Bigg]\{\delta\}\end{aligned} \tag{5}$$

The external work done by the constant axial force, P, the nodal force vector, $\{q\}$, and inertia force are given by

$$\begin{aligned}W&=\frac{P}{2}\int_0^l\left(\frac{dv}{dx}\right)^2dx+\frac{1}{2}\{q\}^T\{\delta\}\\ &\quad+\frac{\rho w^2}{2}\int_0^l(u^2+v^2)\cdot\frac{\rho A(x)}{g}dx\\ &=\frac{1}{2}\{\delta\}^T\Bigg[P\int_0^l\left[\frac{df_b}{dx}\right]^T\left[\frac{df_b}{dx}\right]+[k]\\ &\quad+\frac{\rho w^2}{g}\Big(\int_0^l[f_a]^TA(x)[f_a]dx\\ &\quad+\int_0^l[f_b]^TA(x)[f_b]dx\Big)\Bigg]\{\delta\}\end{aligned} \tag{6}$$

By equating strain energy, U, to the external work, W, and arranging the terms, one can obtain element stiffness matrices

$$[k]=[k]_t-P[k]_g-w^2[m]_c \tag{7}$$

where

$$[k]_t=[k]_a+[k]_b\doteqdot$$

$$\frac{EI(e)l}{l^3}\begin{vmatrix}A(e)l^2/I(e) & & & & & symm.\\ 0 & 12 & & & & \\ 0 & 6l & 4l^2 & & & \\ A(e)l^2/I(e) & 0 & 0 & A(e)l^2/I(e) & & \\ 0 & -12 & -6l & 0 & 12 & \\ 0 & -6l & 2l^2 & 0 & -6l & 4l^2\end{vmatrix}$$

(axial and flexural stiffness matrix) (8a)

$$P[k]_g = \frac{P}{30l}\begin{vmatrix} 0 & & & & & \\ 0 & 36 & & & & \\ 0 & 3l & 4l^2 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -36 & -3l & 0 & 36 & \\ 0 & 3l & -l^2 & 0 & -3l & 4l^2 \end{vmatrix} \quad (8b)$$

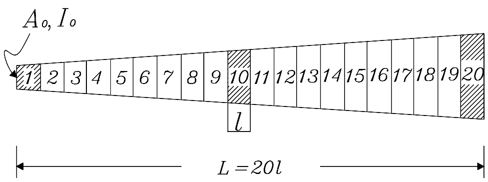
(initial or geometric stiffness matrix)

$$[m]_c \fallingdotseq \frac{\rho A(e) l}{420g}\begin{vmatrix} 140 & & & & & \\ 0 & 156 & & & & \\ 0 & 22l & 4l^2 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 13l & 0 & 156 & \\ 0 & -13l & -3l^2 & 0 & -22l & 4l^2 \end{vmatrix} \quad (8c)$$

(consistent mass matrix)

To avoid complicated calculations appearing in Eq.(8a) and (8c), the element area, $A(x)$, and its second moment of area, $I(x)$, are replaced by constant $A(e)$ and $I(e)$, respectively. Here, $A(e)$ and $I(e)$ denote $A(x)$ and $I(x)$ at the mid-length of each element. The replacement will result in inevitable errors. As can be seen in Table 1, however, the errors are not so significant.

Table 1(a) Example of flexural stiffness errors



$$I(x) = I_0\left(1+\frac{x}{L}\right)^3$$

$$A(x) = A_0\left(1+\frac{x}{L}\right)$$

$$\alpha = 1, \quad (m, n) = (1, 3)$$

1st Element

$[k]_b$	Exact EI_0/l^3	Approx EI_0/l^3	Error (%)
$k_{2,2}$	12.936	12.923	-0.10
$k_{3,2}$	-6.311 l^2	-6.461 l^2	2.38
$k_{5,2}$	-12.936 l^2	-12.923 l^2	-0.10
$k_{5,3}$	6.311 l	6.461 l	2.38
$k_{5,5}$	12.936 l^2	12.923 l^2	2.39
$k_{6,2}$	6.621 l	6.461 l	-2.49
$k_{6,3}$	2.157	2.164	0.32
$k_{6,5}$	-6.311 l^2	-6.461 l^2	-2.48
$k_{6,6}$	4.514 l	4.308 l	-4.56

10thElement

$[k]_b$	Exact EI_0/l^3	ApproxEI_0/l^3	Error (%)
$k_{2,2}$	37550.098	37468.831	-0.22
$k_{3,2}$	18428.046 l^2	18387.839 l^2	-0.22
$k_{3,3}$	-37550.098	-37468.831	-0.22
$k_{5,2}$	19122.052 l^2	19080.993 l^2	-0.21
$k_{5,3}$	9043.732 l	9023.840 l	-0.22
$k_{6,2}$	9384.315 l	9363.999 l	-0.22
$k_{5,5}$	-18428.046l^2	-18387.839l^2	-0.22
$k_{6,3}$	37550.098	37468.831	-0.22
$k_{6,5}$	-19122.052l^2	-19080.993l^2	-0.21
$k_{6,6}$	9737.738 l	9716.994 l	-0.21

Table 1(b) Example of mass matrix errors 1st Element

$[m]_b$	Exact $\rho A_0 l^3/g$	Approx $\rho A_0 l^3/g$	Error (%)
$m_{2,2}$	0.380 l	0.390 l	+2.63
$m_{3,2}$	0.054 l^2	0.055 l^2	+1.85
$m_{3,3}$	0.010	0.010	0.0
$m_{5,2}$	0.135 l	0.135 l	0.0
$m_{5,3}$	0.033 l^2	0.033 l^2	0.0
$m_{5,5}$	0.380 l	0.390 l	+2.63
$m_{6,2}$	-0.0321 l^2	-0.033 l^2	-3.13
$m_{6,3}$	-0.008	-0.008	0.0
$m_{6,5}$	-0.056 l^2	-0.055 l^2	+1.78
$m_{6,6}$	0.010	0.010	0.0

20th Element

$[m]_b$	Exact $\rho A_0 l^3/g$	Approx $\rho A_0 l^3/g$	Error (%)

$m_{2,2}$	1.086 l	1.096 l	+0.90
$m_{3,2}$	0.154 l^2	0.155 l^2	+0.65
$m_{3,3}$	0.028	0.028	0.0
$m_{5,2}$	0.379 l	0.379 l	0.0
$m_{5,3}$	0.091 l^2	0.091 l^2	0.0
$m_{5,5}$	0.154 l	0.155 l	+0.65
$m_{6,2}$	-0.091 l^2	-0.091 l^2	0.0
$m_{6,3}$	-0.021	-0.021	0.0
$m_{6,5}$	-0.156 l^2	-0.155 l^2	+0.60
$m_{6,6}$	0.028	0.028	0.0

Newmark reported that the final eigenvalue (in his case, the natural frequency) errors due to this type of replacement are insignificant.

4. Determination of eigenvalues

The element matrices are transformed and assembled for the whole structure by using the transformation matrix, $[T]$.

$$[T]=\begin{vmatrix} \cos\Theta & \sin\Theta & 0 & 0 & 0 & 0 \\ -\sin\Theta & \cos\Theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\Theta & \sin\Theta & 0 \\ 0 & 0 & 0 & -\sin\Theta & \cos\Theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} \qquad (9)$$

Enforcement of boundary condition at the column bases and sway condition at the column tops to the assembled matrices lead to the following form of the matrix equation:

$$([K]_t - P[K]_g - w^2[M]_c)\{\triangle\}=\{0\} \qquad (10)$$

To obtain the elastic critical load, one lets $w^2=0$ in Eq. (10). In the same way, one lets $P=0$ to determine the natural frequency of lateral vibration. For the determination of the first mode eigenvalues by iterative procedure, the above equation is transformed into following forms:

(critical load),

$$\left([K]_t^{-1}[K]_g - \frac{1}{P}[I] \right)\{\triangle\}=\{0\} \qquad (11)$$

(natural frequency),

$$\left([K]_t^{-1}[M]_c - \frac{1}{w^2}[I] \right)\{\triangle\}=\{0\} \qquad (12)$$

where $[I]$ denotes the unit matrix. In the present study, each tapered column is divided into 20 equal segments and prismatic beam into 2 equal segments with axial force, $P=0.0$. Due to the space limit, only some changes in the eigenvalue coefficient are listed in Tables 2 and 3.

Table 2 Elastic critical load

(a) Unbraced frame with hinged bases ($Pcr=C\cdot EI_0/L^2$)

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	1.490	1.514	1.913	1.917	2.079	2.174	2.167	2.286
0.2	2.276	2.250	2.906	2.832	3.127	3.185	3.250	3.311
1.0	7.357	7.392	9.219	9.002	9.716	9.861	9.970	9.970
1.2	9.127	9.228	11.377	11.172	11.983	12.188	12.235	12.275
1.4	11.183	11.284	13.865	13.594	14.547	14.779	14.814	14.836
1.6	13.535	13.560	16.673	16.268	17.398	17.632	17.792	17.653
1.8	15.941	16.055	19.510	19.192	20.396	20.749	20.747	20.726
2.0	18.795	18.772	22.814	22.368	23.706	24.129	24.124	24.055
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	0.963	0.964	1.490	1.402	1.759	1.695	1.913	1.843
0.2	1.451	1.451	2.276	2.133	2.686	2.587	2.906	2.813
1.0	4.813	5.099	7.357	7.214	8.638	8.578	9.219	9.193
1.2	6.018	6.435	9.127	9.023	10.706	10.682	11.377	11.412
1.4	7.442	7.941	11.183	11.047	13.049	13.028	12.865	13.881
1.6	9.200	9.617	13.535	13.288	15.777	15.616	16.673	16.600
1.8	10.907	11.462	15.941	15.744	18.470	18.446	19.510	19.569

2.0	13.003	13.477	18.794	18.416	21.637	21.519	22.814	22.787
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	0.688	0.780	1.176	1.231	1.490	1.536	1.689	1.695
0.2	1.040	1.041	1.767	1.756	2.276	2.244	2.580	2.503
0.4	1.485	1.443	2.539	2.474	3.239	3.175	3.664	3.547
0.6	2.097	1.986	3.513	3.383	4.395	4.330	5.042	4.827
0.8	2.662	2.670	4.501	4.485	5.710	5.709	6.518	6.343
1.0	3.501	3.496	5.822	5.779	7.357	7.312	8.344	8.095
1.8	8.155	8.212	13.042	12.875	15.941	15.959	17.784	17.463
2.0	9.816	9.744	15.440	15.129	18.795	18.680	20.710	20.395
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	0.541	0.688	0.963	1.145	1.268	1.456	1.490	1.621
0.2	0.815	0.808	1.451	1.541	1.939	2.045	2.276	2.321
0.4	1.152	1.053	2.052	2.114	2.739	2.846	3.239	3.247
0.6	1.652	1.423	2.864	2.866	3.790	3.859	4.394	4.400
0.8	2.086	1.918	3.672	3.796	4.849	5.084	5.710	5.781
1.0	2.776	2.537	4.813	4.904	6.280	6.521	7.357	7.388
1.8	6.483	6.264	10.907	11.118	13.860	14.393	15.941	16.088
2.0	7.860	7.508	13.003	13.117	16.441	16.891	18.794	18.830

(b) Unbraced frame with fixed bases ($Pcr = C \cdot EI_0/L^2$)

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	5.429	5.631	7.255	7.055	8.130	8.188	8.575	9.029
0.2	7.220	7.365	9.814	9.312	10.931	10.838	11.503	11.940
0.4	9.270	9.457	12.566	11.998	14.121	13.950	14.829	15.312
0.6	11.685	11.909	15.945	15.113	17.785	17.525	18.610	19.145
0.8	14.336	14.721	19.396	18.656	21.808	21.562	22.791	23.439
1.0	17.456	17.892	23.491	22.628	26.343	26.062	27.468	28.193
1.2	20.892	21.422	28.000	27.029	31.335	31.025	32.589	33.409
2.0	38.579	39.136	50.508	48.922	55.816	55.502	57.728	58.879
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	4.121	4.166	5.429	5.347	6.402	6.237	7.255	6.836
0.2	5.344	5.444	7.220	7.124	8.620	8.381	9.817	9.216
0.4	6.734	6.970	9.270	9.241	11.166	10.922	12.566	12.014
0.6	8.359	8.742	11.685	11.697	14.083	13.861	15.935	15.233
0.8	10.145	10.762	14.336	14.494	17.319	17.197	19.396	18.870

1.0	12.257	13.029	17.456	17.631	21.009	20.930	23.491	22.927
1.2	14.571	15.544	20.892	21.109	25.124	25.062	27.999	27.403
2.0	26.748	28.073	38.579	38.419	45.913	45.561	50.509	49.499
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	3.609	3.677	4.586	4.778	5.429	5.587	6.136	6.105
0.8	8.373	8.197	11.723	11.861	14.336	14.495	16.443	16.101
1.0	10.010	9.814	14.205	14.371	17.456	17.625	19.972	19.577
1.2	11.856	11.627	16.972	17.178	20.892	21.116	23.899	23.443
1.4	13.789	13.635	20.042	20.280	24.675	24.969	28.161	27.701
1.6	16.153	15.838	23.564	23.679	28.957	29.182	33.018	32.349
1.8	18.457	18.236	27.208	27.373	33.480	33.757	38.074	37.388
2.0	21.234	20.830	31.373	31.363	38.579	38.693	43.823	42.818
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	3.340	3.433	4.121	4.493	4.814	5.262	5.429	5.740
0.8	7.424	6.680	10.145	10.310	12.429	12.911	14.336	14.482
1.0	8.810	7.908	12.257	12.442	15.105	15.674	17.456	17.603
1.2	10.331	9.302	14.571	14.845	18.053	18.777	20.892	21.096
1.4	11.968	10.862	17.108	17.520	21.292	22.220	24.675	24.964
1.6	13.885	12.589	20.112	20.465	25.034	26.004	28.957	29.206
1.8	15.771	14.482	23.195	23.681	28.914	30.128	33.480	33.821
2.0	18.084	16.542	26.748	27.169	33.402	34.592	38.579	38.810

(c) Braced frame with hinged bases ($Pcr = C \cdot EI_0/L^2$)

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	12.134	12.971	13.673	13.287	14.759	14.374	15.552	16.233
0.2	16.293	16.931	18.428	18.102	19880	19.632	20.917	21.526
0.4	20.998	21.468	23.798	23.476	25.653	25.475	27.223	27.464
0.6	26.238	26.565	29.780	29.409	32.060	31.902	33.579	34.047
0.8	32.002	32.227	36.349	35.899	39.084	38.915	40.730	41.275
1.0	38.297	38.453	43.507	42.947	46.708	46.512	48.607	49.149
1.8	68.578	69.012	77.750	76.721	83.189	82.750	86.452	87.097
2.0	77.407	78.065	87.682	86.560	93.446	93.271	97.079	98.197
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	21.478	12.802	12.133	12.252	12.971	12.473	13.673	13.466
0.2	14.743	16.080	16.296	16.440	17.469	17.164	18.399	18.252
0.4	18.985	19.901	20.996	21.145	22.551	22.379	23.799	23.604

0.6	23.956	24.265	26.240	26.367	28.219	28.119	29.781	29.522
0.8	28.831	29.173	32.003	32.106	34.446	34.384	36.349	36.005
1.0	34.470	34.625	38.299	38.363	41.243	41.172	34.505	43.053
1.8	61.151	61.867	68.579	68.559	73.863	73.571	77.748	76.902
2.0	68.803	70.036	77.409	77.401	83.351	82.982	87.687	86.778
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	10.901	12.746	11.440	11.907	12.149	11.839	12.714	12.544
0.2	14.603	15.615	15.386	15.707	16.292	16.163	17.104	16.983
0.4	19.113	19.017	19.834	20.006	21.009	20.986	22.091	21.957
1.2	38.631	37.950	42.119	42.202	45.099	45.287	47.528	47.205
1.4	44.742	44.015	48.911	49.001	52.423	52.614	55.256	54.855
1.6	45.180	50.613	56.169	56.299	60.266	60.442	63.516	63.041
1.8	56.966	57.743	63.940	64.097	68.581	68.770	72.292	71.762
2.0	65.669	65.406	72.124	72.395	77.406	77.598	81.585	81.018
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	10.661	12.718	10.960	11.734	11.671	11.523	12.123	12.083
0.2	14.168	15.349	14.706	15.307	15.617	15.628	16.297	16.314
0.4	18.130	18.507	18.951	19.369	20.058	20.222	20.992	21.065
1.2	38.262	36.404	40.361	40.524	42.921	40.477	45.098	45.263
1.4	44.370	42.194	47.434	47.039	49.868	50.511	52.424	52.611
1.6	47.893	48.511	53.739	54.044	57.294	58.033	60.254	60.478
1.8	56.900	55.355	61.141	61.539	65.162	66.043	68.577	68.865
2.0	66.114	62.725	68.858	69.525	73.594	74.541	77.405	77.771

(d) Braced frame with fixed bases ($Pcr = C \cdot EI_0/L^2$)

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	23.677	23.665	26.573	26.408	28.582	28.506	30.109	29.959
0.2	31.944	32.079	35.761	35.505	38.493	38.399	40.617	40.762
0.4	41.097	41.456	46.132	45.705	49.658	49.468	52.213	52.746
0.6	51.263	51.798	57.668	57.006	62.048	61.711	65.172	65.912
1.4	101.838	102.801	115.080	113.237	123.364	122.436	129.109	130.397
1.6	116.951	117.961	132.206	130.051	141.622	140.555	147.332	149.474
1.8	133.041	134.085	150.644	147.967	160.965	159.849	168.017	169.732
2.0	150.110	151.173	169.672	166.986	179.621	180.319	189.333	191.172
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	22.579	22.324	23.825	24.203	25.322	25.438	26.571	26.027
0.2	30.305	29.957	31.575	32.3001	34.029	34.113	35.763	35.393
0.4	38.553	38.432	41.072	41.399	43.853	43.859	46.131	45.835
0.6	47.304	47.814	51.934	51.498	54.800	54.678	57.666	57.355
1.4	91.985	93.891	101.814	101.899	109.278	108.670	115.076	114.203
1.6	105.383	107.570	116.889	117.002	125.561	124.848	132.209	131.108
1.8	119.408	122.112	133.053	133.104	142.900	142.097	150.424	149.090
2.0	134.558	137.517	150.118	150.208	161.411	160.418	169.703	168.149
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	3.435	3.176	2.610	2.820	2.259	2.473	2.057	2.136
0.2	3.890	3.728	2.976	3.174	2.588	2.745	2.365	2.440
0.4	4.328	4.252	3.332	3.528	2.909	3.024	2.667	2.741
0.6	4.752	4.748	3.678	3.881	3.223	3.312	2.964	3.041
0.8	5.165	5.216	4.018	4.233	3.533	3.607	3.256	3.339
1.0	5.569	5.656	4.352	4.585	3.838	3.911	3.546	3.636
1.2	5.965	6.067	4.682	4.936	4.140	4.223	3.832	3.930
2.0	7.494	7.433	5.964	6.333	5.322	5.552	4.959	5.090
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	21.447	21.653	22.548	23.101	22.948	23.903	23.885	24.061
0.8	54.388	53.460	54.958	57.145	59.851	60.243	62.385	62.751
1.0	64.601	63.363	66.358	67.948	71.382	71.795	74.528	74.903
1.2	75.616	74.046	79.987	79.668	83.839	84.335	87.741	88.046
1.4	87.899	85.511	91.371	92.305	97.323	97.862	101.807	102.181
1.6	100.538	97.756	84.689	105.859	111.686	112.376	116.942	117.308
1.8	113.440	110.781	119.469	120.329	126.943	127.877	133.044	133.426
2.0	127.931	124.587	141.909	135.716	143.264	144.366	150.128	15.535

Table 3 Natural Frequency ($w_0 = C \cdot \sqrt{\frac{gEI_0}{L^4 \rho A_0}}$)

(a) Unbraced frame with hinged bases

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	1.476	1.478	1.350	1.344	1.222	1.250	1.223	1.195
0.2	1.704	1.704	1.539	1.539	1.389	1.407	1.274	1.308
0.4	1.920	1.923	1.719	1.727	1.545	1.560	1.415	1.421
0.6	2.148	2.136	1.901	1.910	1.716	1.710	1.556	1.535
0.8	2.329	2.343	2.058	2.086	1.840	1.855	1.679	1.649
1.0	2.540	2.544	2.225	2.257	1.984	1.997	1.808	1.764
1.8	3.277	3.283	2.889	2.878	2.512	2.527	2.281	2.229

α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
2.0	3.463	3.452	2.980	3.018	2.637	2.650	2.394	2.346
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	1.040	0.992	0.947	0.823	0.857	0.793	0.787	0.752
0.2	1.213	1.164	1.088	1.034	0.981	0.938	0.897	0.874
0.4	1.376	1.330	1.221	1.189	1.100	1.076	0.999	0.991
0.6	1.548	1.492	1.362	1.337	1.214	1.208	1.103	1.104
0.8	1.687	1.650	1.473	1.479	1.313	1.333	1.192	1.212
1.0	1.851	1.803	1.603	1.613	1.419	1.401	1.287	1.316
1.8	2.422	2.368	2.059	2.087	1.810	1.858	1.633	1.684
2.0	2.563	2.498	2.175	2.188	1.908	1.944	1.718	1.764
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	0.798	0.832	0.733	0.719	0.670	0.645	0.619	0.611
0.2	0.933	0.945	0.849	0.831	0.769	0.750	0.708	0.703
1.0	1.447	1.399	1.266	1.252	1.130	1.133	1.027	1.041
1.2	1.570	1.512	1.362	1.351	1.211	1.219	1.099	1.118
1.4	1.686	1.626	1.462	1.446	1.294	1.301	1.171	1.193
1.6	1.814	1.740	1.561	1.538	1.376	1.380	1.246	1.265
1.8	1.913	1.854	1.646	1.627	1.450	1.454	1.311	1.334
2.0	2.033	1.968	1.737	1.714	1.532	1.524	1.382	1.400
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	0.641	0.752	0.598	0.643	0.552	0.573	0.513	0.542
0.2	0.737	0.821	0.696	0.716	0.636	0.645	0.589	0.606
1.0	1.055	1.136	1.051	1.015	0.945	0.922	0.865	0.855
1.2	1.135	1.225	1.099	1.091	1.015	0.988	0.927	0.916
1.4	1.219	1.318	1.213	1.168	1.084	1.054	0.990	0.975
1.6	1.310	1.415	1.299	1.245	1.156	1.118	1.052	1.035
1.8	1.376	1.516	1.372	1.322	1.219	1.181	1.110	1.093
2.0	1.468	1.622	1.452	1.400	1.201	1.243	1.172	1.151

(b) Unbraced frame with fixed bases

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	3.314	3.295	2.837	2.859	2.517	2.590	2.287	2.489
0.2	3.487	3.492	2.997	3.029	2.658	2.708	2.414	2.528
0.4	3.669	3.685	3.154	3.197	2.795	2.829	2.538	2.580
0.6	3.860	3.876	3.313	3.362	2.936	2.952	2.662	2.643
0.8	4.038	4.063	3.468	3.525	3.069	3.077	2.783	2.720
1.6	4.784	4.784	4.088	4.148	3.609	3.603	3.267	3.149
1.8	4.961	4.957	4.233	4.298	3.737	3.741	3.380	3.287
2.0	5.144	5.127	4.387	4.445	3.865	3.882	3.496	3.438
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	2.366	2.147	1.989	1.756	1.757	1.533	1.593	1.477
0.2	2.496	2.328	2.106	1.955	1.861	1.722	1.686	1.631
0.4	2.629	2.502	2.223	2.140	1.963	1.898	1.777	1.776
0.6	2.768	2.668	2.343	2.313	2.066	2.061	1.869	1.911
0.8	2.903	2.826	2.456	2.472	2.166	2.209	1.958	2.036
1.6	3.469	3.385	2.927	2.980	2.570	2.665	2.317	2.440
1.8	3.603	3.506	3.038	3.074	2.664	2.744	2.400	2.516
2.0	3.743	3.620	3.153	3.155	2.765	2.809	2.489	2.583
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	1.743	1.769	1.521	1.398	1.372	1.195	1.252	1.159
0.2	1.861	1.847	1.637	1.512	1.458	1.319	1.329	1.268
0.4	1.972	1.929	1.733	1.624	1.541	1.438	1.403	1.371
0.6	2.083	2.016	1.830	1.732	1.628	1.550	1.479	1.470
0.8	2.183	2.109	1.926	1.837	1.708	1.655	1.553	1.564
1.0	2.293	2.207	2.023	1.939	1.794	1.755	1.628	1.654
1.2	2.406	2.310	2.119	2.038	1.877	1.848	1.702	1.738
2.0	2.836	2.774	2.495	2.406	2.202	2.157	1.991	2.026
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	1.183	1.582	0.991	1.223	0.913	1.032	0.869	1.008
0.2	1.272	1.569	1.087	1.259	1.012	1.089	0.970	1.061
0.4	1.354	1.573	1.177	1.301	1.105	1.149	1.067	1.116
0.6	1.437	1.594	1.267	1.351	1.199	1.211	1.162	1.174
0.8	1.511	1.632	1.352	1.409	1.288	1.276	1.252	1.234
1.0	1.593	1.687	1.441	1.473	1.379	1.343	1.344	1.296
1.2	1.672	1.759	1.526	1.545	1.468	1.412	1.423	1.361
2.0	1.984	2.218	1.870	1.907	1.817	1.715	1.675	1.643

(c) Braced frame with hinged bases

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	10.086	10.499	10.227	10.412	10.326	10.255	10.400	10.029
0.8	14.639	15.066	15.351	15.210	15.839	15.537	16.201	16.046
1.0	15.712	16.190	16.569	16.397	17.151	16.826	17.581	17.477

1.2	16.765	17.308	17.766	17.579	18.442	18.103	18.940	18.879
1.4	17.807	18.418	18.951	18.756	19.719	19.367	20.283	20.252
1.6	18.834	19.522	20.122	19.927	20.982	20.619	21.612	21.596
1.8	19.848	20.619	21.280	21.094	22.231	21.858	22.925	22.910
2.0	20.853	21.709	22.429	22.256	23.470	23.085	24.228	24.196
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	5.418	4.718	4.440	4.544	4.001	4.301	3.746	3.989
0.8	8.179	7.343	6.955	6.935	6.401	6.709	6.083	6.665
1.0	8.831	7.942	7.559	7.500	6.984	7.279	6.653	7.281
1.2	9.470	8.518	8.156	8.051	7.562	7.837	7.220	7.875
1.4	10.101	9.072	8.748	8.590	8.136	8.382	7.784	8.448
1.6	10.725	9.602	9.335	9.115	8.706	8.914	8.345	8.999
1.8	11.340	10.110	9.917	9.628	9.274	9.434	8.904	9.529
2.0	11.949	10.595	10.496	10.127	9.838	9.941	9.461	10.037
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	3.007	2.782	2.334	2.570	2.046	2.288	1.881	1.937
0.2	3.412	3.273	2.669	2.930	2.352	2.595	2.171	2.268
0.4	3.804	3.741	2.996	3.282	2.652	2.898	2.456	2.588
0.6	4.186	4.187	3.317	3.625	2.948	3.195	2.738	2.899
0.8	4.559	4.611	3.632	3.958	3.239	3.487	3.017	3.199
1.0	4.925	5.013	3.943	4.283	3.528	3.775	3.293	3.488
1.8	6.337	6.398	5.155	5.491	4.659	4.873	4.380	4.544
2.0	6.680	6.689	5.453	5.771	4.937	5.136	4.648	4.782
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	1.936	1.810	1.470	1.575	1.457	1.270	1.158	0.896
0.2	2.195	2.183	1.677	1.784	1.639	1.392	1.332	1.008
0.4	2.445	2.536	1.878	1.989	1.817	1.516	1.502	1.118
0.6	2.688	2.870	2.076	2.189	1.993	1.641	1.670	1.226
0.8	2.925	3.184	2.269	2.385	2.167	1.768	1.836	1.334
1.0	3.158	3.479	2.460	2.577	2.338	1.897	2.000	1.439
1.8	4.054	4.461	3.201	3.301	3.012	2.430	2.645	1.848
2.0	4.271	4.657	3.382	3.472	1.457	2.568	2.803	1.947

(d) Braced frame with fixed bases

β=1	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	14.129	14.441	13.361	13.610	12.879	12.788	12.554	11.975
0.2	15.560	15.832	14.991	15.064	14.625	14.420	14.374	13.900
0.4	16.916	17.201	16.539	16.501	16.295	16.022	16.124	15.764
0.6	18.216	18.548	18.028	17.922	17.904	17.594	17.816	17.566
1.4	23.069	23.713	23.587	23.439	23.932	23.586	24.180	24.156
1.6	24.224	24.949	24.911	24.776	25.368	25.010	25.697	25.649
1.8	25.362	26.162	26.216	26.097	26.783	26.403	27.192	27.081
2.0	26.484	27.353	27.502	27.401	28.178	27.767	28.667	28.451
β=2	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	6.334	5.965	4.994	5.463	4.412	4.971	4.078	4.488
0.2	7.164	6.669	5.706	6.049	5.072	5.553	4.708	5.183
0.4	7.961	7.351	6.398	6.634	5.717	6.138	5.327	5.863
0.6	8.731	8.013	7.074	7.219	6.351	6.725	5.937	6.528
1.4	11.633	10.450	9.669	9.558	8.809	9.089	8.318	9.041
1.6	12.327	11.006	10.299	10.142	9.409	9.684	8.902	9.633
1.8	13.011	11.542	10.923	10.726	10.006	10.282	9.484	10.209
2.0	13.686	12.057	11.541	11.310	10.599	10.881	10.062	10.770
β=3	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	3.435	3.176	2.610	2.820	2.259	2.473	2.057	2.136
0.2	3.890	3.728	2.976	3.174	2.588	2.745	2.365	2.440
0.4	4.328	4.252	3.332	3.528	2.909	3.024	2.667	2.741
0.6	4.752	4.748	3.678	3.881	3.223	3.312	2.964	3.041
0.8	5.165	5.216	4.018	4.233	3.533	3.607	3.256	3.339
1.0	5.569	5.656	4.352	4.585	3.838	3.911	3.546	3.636
1.2	5.965	6.067	4.682	4.936	4.140	4.223	3.832	3.930
2.0	7.494	7.433	5.964	6.333	5.322	5.552	4.959	5.090
β=4	γ=0.5		γ=1.0		γ=1.5		γ=2.0	
α	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}	C_{fem}	C_{est}
0.0	2.206	1.796	1.649	1.527	1.411	1.267	1.274	1.017
0.2	2.496	2.302	1.876	1.787	1.612	1.397	1.461	1.131
0.4	2.775	2.774	2.096	2.045	1.808	1.537	1.642	1.250
0.6	3.045	3.212	2.310	2.300	1.999	1.687	1.821	1.372
0.8	3.308	3.615	2.520	2.552	2.187	1.846	1.996	1.498
1.0	3.564	3.983	2.726	2.800	2.371	2.015	2.169	1.628
1.2	3.816	4.316	2.928	3.045	2.554	2.194	2.340	1.761
2.0	4.786	5.305	3.715	3.996	3.266	3.005	3.012	2.332

In these tables, one can notice that the increase in taper parameter, α, leads to the increase in eigenvalues, but the increase in two parameters, β and γ, lead to opposite results, which can also be deemed the expected results.

To make the changes in the eigenvalue coefficient a continuous function, the changes are represented by algebraic functions of analysis parameters

$$\begin{aligned} C_{fem} \fallingdotseq C_{est} = & A_0 + A_1\alpha + A_2\alpha^2 + (B_0 + B_1\alpha + B_2\alpha) \\ & \cdot (\beta^{-0.5}) + (C_0 + C_1\alpha + C_2\alpha^2) \cdot (\gamma) \\ & + (D_0 + D_A\alpha + D_2\alpha^2) \cdot (\beta^{-1}) \\ & + (E_0 + E_1\alpha + E_2\alpha^2) \cdot (\beta^{-0.5} \cdot \gamma) \\ & + (F_0 + F_1\alpha + F_2\alpha^2) \cdot (\gamma^2) \end{aligned} \tag{13}$$

The numerical values of coefficients, $A_0, A_1, \cdots$, are determined by regression technique and the results are listed in Table 4.

Table 4 Regression constants

· Unbraced frame hinged bases

	Pcr	ω_0		Pcr	ω_0		Pcr	ω_0
A_0	1.470	1.512	C_0	2.752	-1.066	E_0	0.970	-0.309
A_1	-5.211	-6.775	C_1	5.845	1.760	E_1	0.747	-1.589
A_2	-1.037	2.446	C_2	4.247	-0.673	E_2	-1.177	0.501
B_0	2.446	0.001	D_0	0.000	2.387	F_0	-0.583	0.335
B_1	22.670	17.273	D_1	-12.704	-9.522	F_1	-1.134	-0.288
B_2	5.864	-5.780	D_2	-1.647	3.353	F_2	-0.890	0.120

· unbraced frame fixed bases

	Pcr	ω_0		Pcr	ω_0		Pcr	ω_0
A_0	1.470	1.512	C_0	2.752	-1.066	E_0	0.970	-0.309
A_1	-5.211	-6.775	C_1	5.845	1.760	E_1	0.747	-1.589
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B_2	5.864	-5.780	D_2	-1.647	3.353	F_2	-0.890	0.120

The columns C_{est} in Tables 2 and 3 denote the eigenvalue coefficients estimated by the proposed Eq. (13) together with Table 4. The differences between C_{fem} and C_{est} are insignificant in any case.

To examine the correlation between C_{fem} and C_{est} more quantitatively, the correlation coefficients are calculated by using following formula:

$$\rho = \frac{\sum(XY)}{\sqrt{\sum(X)^2 \cdot (Y)^2}}$$

$$X = C_{fem} - \overline{C}_{fem}, \quad Y = C_{est} - \overline{C}_{est} \tag{14}$$

where $\overline{C}_{fem}$ and $\overline{C}_{est}$ denote mean values of C_{fem} and C_{est}, respectively. Table 5 in the next page shows an example of correlation coefficient for natural frequency.

The correlation coefficient value for elastic critical load reveals $\rho = 0.9942$, which suggest that proposed algebraic function can be used in the prediction of eigenvalue changes.

Table 5 Example of correlation coefficient of natural frequency for unbraced frame with fixed bases and $(m, n) = (1, 3)$, $\beta = 1.0$

γ	α	C_{fem}	C_{est}	X	Y	X^2	XY	Y^2
0.5	0.2	3.487	3.492	-0.075	-0.075	0.006	0.006	0.006
	0.4	3.669	3.685	0.107	0.118	0.012	0.013	0.014
	0.6	3.860	3.876	0.298	0.309	0.089	0.092	0.095
	1.8	4.961	4.957	1.399	1.390	1.958	1.945	1.931
	2.0	5.144	5.127	1.582	1.560	2.504	2.468	2.432
1.0	0.2	2.997	3.029	-0.565	-0.538	0.319	0.304	0.289
	0.4	3.154	3.197	-0.408	-0.370	0.166	0.151	0.137
	0.6	3.313	3.362	-0.249	-0.205	0.062	0.051	0.042
	1.8	4.233	4.298	0.672	0.731	0.451	0.491	0.534
	2.0	4.387	4.445	0.826	0.878	0.682	0.725	0.770
1.5	0.2	2.658	2.708	-0.904	-0.859	0.816	0.776	0.738
	0.4	2.795	2.829	-0.766	-0.738	0.587	0.566	0.545
	0.6	2.936	2.952	-0.626	-0.616	0.392	0.385	0.379
	1.8	3.737	3.741	0.176	0.174	0.031	0.031	0.030
	2.0	3.865	3.882	0.304	0.315	0.092	0.096	0.099
2.0	0.2	2.414	2.528	-1.148	-1.039	1.318	1.193	1.080
	0.4	2.538	2.580	-1.024	-0.988	1.049	1.011	0.975
	0.6	2.662	2.643	-0.900	-0.924	0.810	0.831	0.853
	1.8	3.380	3.287	-0.182	-0.280	0.033	0.051	0.078
	2.0	3.496	3.438	-0.065	-0.130	0.004	0.009	0.017
$\sum Cest$ = 3.562			3.567 = $\sum Cest$		$\sum =$	9.304	9.499	9.812

$X = \overline{C}_{fem} - C_{fem}$, $Y = \overline{C}_{est} - C_{est}$, $\rho = \dfrac{\sum(XY)}{\sqrt{\sum(X)^2 \cdot (Y)^2}} = \dfrac{9.499}{9.555} = 0.9942$

5. Axial Thrust vs Frequency Elation

To examine the reduced frequency changes of the structure under the variable column axial force, P, Eq.(10) is transformed into the following form

$$([K]_t - RP_{cr}[K]_g - (\Omega w_0)^2[M]_c)\{\triangle\} = \{0\} \quad (15)$$

in which R denotes the load ratio defined by $R = P/P_{cr}$. By changing R from zero to 1.0 with subinterval $\triangle R = 0.2$, the corresponding frequency ratio Ω ($= w/w_0$; w=the reduced frequency) is determined. Table 6 shows some examples of reduced frequency due to the increases of load ratios and to visualize the changes in frequency reduction, they are represented by Fig. 3.

Table 6 Unbraced frame with fixed bases
($\alpha = 1.0$, $\beta = 1.0$, $\gamma = 1.0$)
($R = P/Pcr$, $\Omega = \omega/\omega_0$), $(m, n) = (1, 3)$

R	ω^2	$R + \Omega^2$	R	ω^2	$R + \Omega^2$
0.0	13.138	1.0	0.6	6.256	1.076
0.2	10.880	1.028	0.8	3.687	1.081
0.4	8.619	1.056	1.0	0.585	1.004

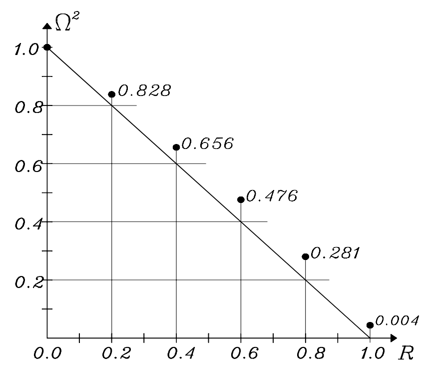

Fig. 3 Axial thrust vs frequency relation

As can be seen the figure, $R+\Omega^2$ shows some errors from the unity (=1.0) with maximum error 8.1% when R=0.8.

6. Conclusions

The first mode eigenvalues (=elastic critical load and natural frequency) of single story-3 equal bay frame with linearly tapered columns are determined by the finite element method. The parameters considered in the numerical analysis are taper parameter, α, beam span to column height ratio, β, and second moment area ratio, γ, for the beam.

The inclusion of axial rigidity term, $EA(x)$, in the element stiffness matrix resulted in the reduced elastic critical load of frame. In other words critical loads determined by the present study are less than those determined by the modified slope-deflection method, where only the flexural rigidity of column is considered. The algebraic function proposed for the eigenvalue changes can predict near exact coefficients which are determined by the finite element method.

The axial thrust vs frequency relationship determined by the finite element method shows some deviation from linearity, which has been already reported in the previous study. On the whole, the application of this linear relationship to the design of the structure under dynamic loadings will lead to the conservative design. By measuring the reduced frequencies corresponding to different axial forces, one can estimate the buckling load and fundamental frequency of the structure. In this sense, this linear relationship can be applied to the structural safety assessment of existing structures.

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(접수일자 : 2004년 12월 31일)