International Journal of Fuzzy Logic and Intelligent Systems, vol. 5, no. 1, March 2005 pp. 29-34

Fuzzy Topology On Fuzzy Sets: Fuzzy γ-Continuity and Fuzzy γ- Retracts

I. M. Hanafy, F. S. Mahmoud and M. M. Khalaf.

Department of Mathematics, Faculty of Education, Suez Canal University, El-Arish, Egypt.
Department of Mathematics, Faculty of Science (82524), Sohag-Egypt.

Abstract

The aim of this paper is to introduce fuzzy γ-continuity and fuzzy γ -retracts in a fuzzy topology on fuzzy sets and establish some of their properties. Also, the relations between these new concepts are discussed.

Key words : Fuzzy topology, fuzzy γ -continuity, fuzzy γ -retracts, fuzzy proper function.

1. Introduction and Preliminaries

Chakrabarty and Ahsanullah [2] introduced the concept of a fuzzy topology on a fuzzy set and defined the category FUZZ-TOP where the objects are fuzzy topological spaces γ and the morphisms are fuzzy-continuous proper functions. In [8,11] the concepts of fuzzy retract and fuzzy neighborhood retract has been introduced. Also, the concept of a fuzzy retract in a fuzzy topology on fuzzy sets has been introduced in 2003 [9]. Hanafy in [6] introduced the concepts of fuzzy γ -open (γ -closed) sets and fuzzy γ -continuity. In the present paper, we introduce and study the concepts of fuzzy γ -open(γ -closed) sets and fuzzy γ -continuity in a fuzzy topology on fuzzy sets. Also, this paper is devoted to introduce various fuzzy retracts in a fuzzy topology on fuzzy sets and establish some of their properties. A comparison between these new concepts is of interest.

Throughout this paper, I will denote the closed unit interval, let X be a non-empty set. A fuzzy set of X is a function with domain X and values in I , that is, an element of I^X . A fuzzy point x_r is a fuzzy set whose support is the point x and its value r, 0<r ≤ 1. In this paper , we will shorten the word fuzzy as F [7] .The family of all F-points of λ will be denoted by Pt(λ)

Let $\lambda \in I^X$, $A_\lambda = \{\zeta \in I^X : \zeta \le \lambda\}$ and $B_\mu = \{\eta \in I^Y : \eta \le \mu\}$ .Then, $\forall \lambda \in I^X, A_\lambda$ is an F-lattice (completely distributive lattice) which has an order-reversing involution $': A_\lambda \to A_\lambda$ [7].

If $\rho \in A_\lambda$, the complement of ρ referred to λ , denoted by ρ'_λ is defined by $\rho'_\lambda(x) = \lambda(x) - \rho(x), \forall x \in X$ and ρ is said to be maximal if $\forall x \in X, \rho(x) \ne 0 \Rightarrow \rho(x) = \lambda(x)$ [3].

An F-subset f of X×Y is said to be an F-proper function from $\forall\lambda(\in I^X)$ to $\forall\mu(\in I^Y)$ if

(i) $f(x, y) \le \lambda(x) \wedge \mu(y), \quad \forall (x, y) \in X \times Y$.

(ii) $\forall x \in X, \exists y_0 \in Y$ such that $f(x, y_0) = \lambda(x)$ and $f(x, y) = 0$ if $y \ne y_0$.

Let $f: \lambda \to \mu$ be an F-proper function from λ to μ Define(adopting Rodabaugh's symbols [12])

(i) $f^{\to}(\rho)(y) = \vee\{f(x, y) \wedge \rho(x) : x \in X\}, \forall \rho \in A_\lambda, \forall y \in Y$

(ii) $f^{\to}(\sigma)(y) = \vee\{f(x, y) \wedge \sigma(x) : x \in X\}, \forall \sigma \in B_\mu, \forall x \in X$ [5].

Let $\rho \in A_\lambda$. Then $f|\rho$ defined by $(f|\rho)(x, y) = f(x, y) \wedge \rho(x) \forall (x, y) \in X \times Y$, is said to be the restriction of f to ρ [3].

An F-proper function Let $f: \lambda \to \mu$ is said to be

(i) injective if $f(x_1, y) = \lambda(x_1)(\ne 0), f(x_2, y) = \lambda(x_2)(\ne 0) \Rightarrow x_1 = x_2 \forall x_1 x_2 \in X, y \in Y$,

(ii) surjective if $\forall y \in Y$ with $\mu(y) \ne 0, \exists x \in X$ such that $f(x, y) = \lambda(x)$,

(iii) bijective if f is both injective and surjective.

The F-proper function $id_\lambda : \lambda \to \lambda$ defined by $id_\lambda(x, y) = \lambda(x)$ or 0 accordingly as $x = y$ or $y \ne x$ is said to be the identity F-proper function on λ [3].

If $f : \lambda \to \mu$ and $g : \mu \to v (v \in I^Z)$ are F-proper functions, then the F-proper function $gf : \lambda \to v$ is defined by

$gf(x, z) = \{^{\lambda(x) \text{ if } \exists y \in Y \text{ such that } f(x,y)=\lambda(x), g(y,z)=\mu(y),}_{0 \text{ otherwise}}$

or $gf(x, z) = \vee_{y \in Y} f(x, y) \wedge g(y, z)$..

A collection δ of F-subsets of λ i.e., $\delta \subset A_\lambda$ is said to be an F-topology on λ if

(i) $\underline{0}, \lambda \in \delta$,

(ii) $\rho_j \in \delta \forall j \in J \Rightarrow \vee_{j \in J} \rho_j \in \delta$,

(iii) $\rho, \sigma \in \delta \Rightarrow \rho \wedge \sigma \in \delta$.

(λ, δ) is said to be an F-topological space (briefly , F-ts). The members of δ are said to be F-open (F-o) sets of λ We denote δ the family of F-closed (F-c) sets of λ , that is, $\rho \in \delta$ iff $\lambda - \rho$. If $\rho \in A_\lambda$, then $\delta_\rho = \{\rho \wedge \zeta : \zeta \in \delta\}$ is an fuzzy topology on ρ and (ρ, δ_ρ) is called a subspace of (λ, δ) [3].

Manuscript received Jan. 14, 2005; revised Feb. 22, 2005

Let (λ,δ) be an F-ts and $\rho \in A_\lambda$. Then the interior (closure) of ρ is defined by

(i) $\operatorname{int}\rho = \vee\{\zeta : \zeta \in \delta, \zeta \le \rho\}$ [12].

(ii) $cl\rho = \wedge\{\eta : \eta \in \delta', \rho \le \eta\}$ [2].

Let $f:(\lambda,\delta) \to (\mu,\delta^*)$ be an F-proper function from an F-ts (λ,δ) into an F-ts (μ,δ^*). Then f is called F-continuous if $f^{\leftarrow}(\sigma) \in \delta, \forall \sigma \in \delta^*$) [3].

The graph function $g:\lambda \to \lambda\times\mu$ of f is defined by : $g(z,(x,y)) = f(x,y)$ if $z = x$ and $g(z,(x,y)) = 0$ if $z \ne x$ $\forall z \in X, \forall (x,y) \in X\times Y$ [5].

If $f_i:\lambda_i \to \mu_i (i=1,2)$ are two F-proper functions, then the function $f_1\times f_2 : \lambda_1\times\lambda_2 \to \mu_1\times\mu_2$ defined by : $(f_1\times f_2)((x_1,x_2),(y_1,y_2)) = f_1(x_1,y_1)\wedge f_2(x_2,y_2) \forall (x_1,x_2) \in X_1\times X_2, \forall (y_1,y_2) \in Y_1\times Y_2$, is called the F-product function of f_1 and f_2. One can easily prove that $f_1\times f_2$ is an F-proper function [5].

Let (λ,δ) and (μ,δ^*) be two F-topological spaces. The collection $\Re = \{\zeta\times\eta : \zeta \in \delta, \eta \in \delta^*\}$ forms an open base of an F-topology in $\lambda\times\mu$. The F-topology in $\lambda\times\mu$ induced by $\Re$ is called the product F-topology of δ and δ^* and is denoted by $\delta\times\delta^*$. The F-topological space $(\lambda\times\mu, \delta\times\delta^*)$ is called the product of the F-topological spaces (λ,δ) and (μ,δ^*) [4].

The following results are fundamental for the next sections.

Proposition 1.1 [3]. If $\rho \in A_\lambda$, then $\forall \sigma \in B_\mu$, $(f|\rho)^{\leftarrow}(\sigma) = \rho \wedge f^{\leftarrow}(\sigma)$.

Proposition 1.2 [3]. If $\sigma \in B_\mu$ is maximal ,then $f^{\leftarrow}(\sigma'_\mu) = (f^{\leftarrow}(\sigma))'_\lambda$.

Proposition 1.3 [3]. For an F-proper function $f:\lambda \to \mu$ Then

(i) $f^{\rightarrow}(f^{\leftarrow}(\sigma)) \le \sigma, \forall \sigma \in B_\mu$.

(ii) $f^{\leftarrow}(f^{\rightarrow}(\rho)) \ge \rho, \forall \rho \in A_\lambda$

(iii) $f^{\leftarrow}(\rho \vee \sigma)) = f^{\leftarrow}(\rho) \vee f^{\leftarrow}(\sigma)$ and in general $f^{\leftarrow}(\vee_{j\in J}\sigma_j)) = \vee_{j\in J} f^{\leftarrow}(\sigma_j) \forall \rho,\sigma,\sigma_j \in B_\mu$.

(iv) $f^{\leftarrow}(\rho \wedge \sigma)) = f^{\leftarrow}(\rho) \wedge f^{\leftarrow}(\sigma)$, $\forall \sigma,\rho \in B_\mu$.

Remark 1.4. An F-topology on a F-set can not be extended to the L-fuzzy setting. For this, see Remark 3.1.7 in [7].

Proposition 1.5 [14]. Let ρ and σ be F-subsets of an F-ts (λ,δ). Then $\operatorname{int}(\rho'_\lambda) = (cl(\rho))'_\lambda$ and $cl(\sigma'_\lambda) = (\operatorname{int}(\sigma))'_\lambda$.

Theorem 1.6 [3]. If $f:(\lambda,\delta) \to (\mu,\delta^*)$ is F-continuous and $\rho \in A_\lambda$, then $f|\rho$: $(\rho, \delta_\rho) \to (f^{\rightarrow}(\rho), \delta^*_{f^{\rightarrow}(\rho)})$. is F-continuous.

Proposition 1.7 Let $f_1:\lambda_1 \to \mu_1$, $f_2:\lambda_2 \to \mu_2$ and $f_1\times f_2 : \lambda_1\times\lambda_2 \to \mu_1\times\mu_2$ be F- proper functions . Then

(i) if $\rho \le \lambda_1, \sigma \le \lambda_2$ then $(f_1\times f_2)^{\rightarrow}(\rho\times\sigma) = f_1^{\rightarrow}(\rho)\times f_2^{\rightarrow}(\sigma)$,

(ii) If $\zeta \le \mu_1, \eta \le \mu_2$ then $(f_1\times f_2)^{\rightarrow}(\zeta\times\eta) = f_1^{\rightarrow}(\zeta)\times f_2^{\rightarrow}(\eta)$.

Proposition 1.8 [5]. If g is the graph function of an F-proper function $f:\lambda \to \mu$, then $g^{\leftarrow}(\rho\times\sigma) = \rho \wedge f^{\leftarrow}(\sigma) \forall \rho \in A_\lambda, \sigma \in B_\mu$.

Corollary 1.9 [5]. If $f:\lambda \to \mu$ is an F-proper function and g its graph function, then $g^{\leftarrow}(\lambda\times\sigma) = f^{\leftarrow}(\sigma) \forall \sigma \in B_\mu$.

Theorem 1.10 [5]. An F-proper function $f:(\lambda,\delta) \to (\mu,\delta^*)$ is F-continuous iff its graph function $g:(\lambda,\delta) \to (\lambda\times\mu, \delta\times\delta^*)$ is F-continuous.

Theorem 1.11 [4]. Let (λ_i,δ_i) and (η_i,γ_i) be F-ts's and $f_i:(\lambda_i,\delta_i) \to (\mu_i,\gamma_i)$ be F-continuous proper functions for i = 1,2,.......,n. Then the F-proper function $f = \prod \lambda_i \to \prod \mu_i$ defined by: $f(x_1,......,x_n),(y_1,......,y_n) = \prod \lambda_i(x_1,......,x_n)$ or 0 accordingly $(y_1,......,y_n) = (y_{10},......,y_{n0})$ or $(y_1,......,y_n) \ne (y_{10},......,y_{n0})$ is also F-continuous.

For definitions and results not explained in this paper, we refer to [1, 9] assuming them to be well known.

Definition 1.1 [9]. Let (ρ,δ_ρ) be a maximal subspace of an F-ts (λ,δ). Then (ρ,δ_ρ) is called an F-retract (F-R, for short) of (λ,δ) there exists an F-continuous proper function $f:(\lambda,\delta) \to (\rho,\delta_\rho)$ such that $f|\rho = id_\rho i.e., f(x) = \rho(x) \forall x \in X$. In this case f is called an F-retraction.

Definition 1.2. Let η be an F-set of an F-ts (λ,δ) then, η is called :

(i) F-semiopen (briefly F-so) set of λ if, $\eta \le cl(\operatorname{int}\eta)$[1].

(ii) F-preopen (briefly F-po) set of λ if, $\eta \le \operatorname{int}(cl\ \eta)$[1].

(iii) F-strongly semiopen (briefly F-so)set of λ if, $\eta \le \operatorname{int}(cl(\operatorname{int}(\eta)))$[1]

(iv) F-semi-preopen (briefly F-spo) set of λ if, $\eta \le cl(\operatorname{int}(cl(\eta)))$[1].

Their complements are called F-semiclosed, F-preclosed, F-strongly semiclosed and F-semi-preclosed sets.(resp. F-sc, F-pc, F-ssc, F-spc, for short).

Definition 1.3 [2]. $\rho \in A_\lambda$ are said to be quasi-coincident to λ (written as $\rho q \sigma$) if there exists $x \in X$ such that $\rho(x) + \sigma(x) > \lambda(x)$. If ρand σ are not quasi-coincident referred to λ we denote for this $\rho \bar{q} \sigma$.

2. F-γ-open and F-γ -closed sets

Definition 2.1. Let η be an F-set on F-ts (λ,δ) Then, η is called a F-γ-open (resp. F γ-closed **[Trial mode]**, briefly, F-γo (resp., F- γ c) if

$\eta \le \operatorname{int}(cl\ \eta) \vee \eta \le cl(\operatorname{int}\ \eta)$ (resp., $\eta \le \operatorname{int}(cl\ \eta) \wedge \eta \le cl(\operatorname{int}\ \eta)$)

The family of all F-γ o (resp. F-γc) sets of X will be denoted by (FγO(λ) (resp., Fγ C(λ).

Remark 2.1. (i) It is clear that a F-γo set is weaker than the concepts of a F-so set or a F-po set and stronger than the concepts of a F-spo set.

(ii) The union of F-γo sets is a F-γo set.

Now from the above definition and some known types of a (F-o)(F-c) sets, we have the following diagram:

F-o(F-c)

⇓

F-sso (F-ssc)

⇓

F-po(F-pc) ⇓ **F-so (F-sc)** ⇓

F- γo (F- γo-c)

⇓

F-spo (F-spc)

The converse need not be true in general, as shown by the following examples.

Example 2.1. Let $X=\{x,y,z\},\lambda=\underline{0.6},\delta=\{\underline{0},x_{0.1}\vee y_{0.2}\vee z_{0.3},x_{0.2}\vee y_{0.3}\vee z_{0.4},\lambda\}$. It is clear that $\underline{0.4}$ is F-sso but not F-o, $\underline{0.3}$ is F-so but not F-sso and F-γo but not F-po and $\underline{0.21}$ is F-spo but not F-γo set.

Example 2.2. Let $X=\{x,y\},\lambda=\underline{0.2},\delta=\{\underline{0},\underline{0.1},x_{0.1}\vee y_{0.2},x_{0.1},\lambda\}$. It is clear that $y_{0.1}$ is F-γo but not F-so set and F-po but not F-sso set.

Remark 2.2. The intersection of two F-γo sets need not be F-γo as illustrated by the following example.

Example 2.3. Let $X=\{x,y\},\lambda=\underline{0.3},\delta=\{\underline{0},\underline{0.1},\underline{0.19},\lambda\}$ It is clear that $x_{0.3}\vee y_{0.1}$ is F-γo, $\underline{0.2}$ is F-γo, but $x_{0.2}\vee y_{0.1}$ not F-γo set.

Proposition 2.1. If υ is a F-γ o set and $\text{int}(cl(\upsilon))=\underline{0}$, then υ is F-so.

Corollary 2.1. If υ is a F- γ o set and $cl(\text{int}(\upsilon))=\underline{0}$, then υ is F-po.

Proposition 2.2. Each F- γo set which is F-c is F-so.

Corollary 2.2. Each F- γc set which is F-o is F-sc.

Proposition 2.3. Each F-spo set which is F-c is F- γo.

Remark 2.3. Let (λ,δ) and (μ,σ) be two F-ts's. Then the product $\zeta_1\times\zeta_2$ of a F-γ o set ζ_1 of λ and F-γo set ζ_2 of μ need not to be F-γ o set in the product space $(\lambda\times\mu,\delta\times\sigma)$.

Example 2.4. Let (λ,δ) be a F-ts where $,\lambda=\underline{0.6},\delta=\{\underline{0},\underline{0.1},x_{0.1}\vee y_{0.2},\lambda\}$. The F-set $\zeta_1=x_{0.5}\vee y_{0.1}$ is F-γo. Let (μ,σ) be a F-ts where $\mu=\underline{0.6}$, $\delta=\{\underline{0},x_{0.2}\vee y_{0.3},\mu\}$. The F-set $\zeta_2=x_{0.5}\vee y_{0.1}$ is F- γo, but $\zeta_1\times\zeta_2$ is not F- γo set.

Definition 2.2 Let η be an F-set of an F-ts (λ,δ) Then the γ -closure (γ- cl . for short) and γ-interior ($\gamma-cl$ for short) of η are defined as follows:

$\gamma-cl(\eta)=\wedge\{\upsilon:\upsilon$ is F- γ c and $\eta\le\upsilon\}$

$\gamma-\text{int}(\eta)=\vee\{\upsilon:\upsilon$ is F- γ o and $\upsilon\le\eta\}$.

Proposition 2.4. Let η be an F-set of an F-ts (λ,δ). Then,

(i) $\gamma-cl(\eta')=(\gamma-\text{int}(\mu))'$

(ii) $\gamma-\text{int}(\eta')=(\gamma-cl(\mu))'$.

Definition 2.3. Let η be an F-set of an F-ts (λ,δ). Then, η is called F- γ -nbd (F- γ q-nbd) of a F-point x_r if there exists a F-γo set υ such that $x\in\upsilon\le\eta(x_pq\upsilon$ and $\upsilon\le\eta$.

Proposition 2.5. A F-set η is F- γo iff for every F-point $x_pq\eta,\eta$ is a F- γ q- nbd of x_p.

Proposition 2.6. Let $x_p\in P_t(\lambda)$ and $\upsilon\in I^X$ .Then, x_pq $F-\gamma-cl(\upsilon)$ iff for every F- γ q-nbd η of $x_p,\eta q\upsilon$.

Proposition 2.7. If $\eta\in I^X$ and $\upsilon\in$ F- γo(λ) such that $\eta\bar{q}\upsilon$, then $F-\gamma-cl(\eta)\bar{q}\upsilon$.

3. F- γ-continuity

Definition.3.1. Let $f:(\lambda,\delta)\to(\mu,\delta)$ be an F-proper function from an F-ts (λ,δ) to another F-ts (μ,δ). Then, f is called:

F- γ-continuous (briefly F-sc) mapping if, $f^{\leftarrow}(\upsilon)$ is F- γo set of $\lambda\forall\upsilon\in\delta$.

Theorem 3.1. Let $f:(\lambda,\delta)\to(\mu,\delta)$ be a F-proper function from a F-ts (λ,δ) to another F-ts (μ,δ) If the graph $g:(\lambda,\delta)\to(\lambda\times\mu,\delta\times\delta)$ of f is F-γ c, then $f$is F- γc mapping.

Proof. Let υ be a F-o set of μ, then $f^{\leftarrow}(\upsilon)=\lambda\wedge f^{\leftarrow}(\upsilon)=g^{\leftarrow}(\lambda\times\upsilon)$. Since g is F- γc and $\lambda\times\upsilon$ is F-o set of $\lambda\times\mu$, then $f^{\leftarrow}(\upsilon)$ is a F-γo set of λ Hence f is a F- γc mapping.

Theorem 3.2. Let $f:(\lambda,\delta)\to(\mu,\delta)$ be a F-proper function from a F-ts (λ,δ) to another F-ts (μ,δ). Then the following statements are equivalent :

(i) f is a F- γc .

(ii) For each F-point $x_p \in P_t(\lambda)$ and each F-nbd ζ of μ containing $f(x_p)$ there exists a F- γ-nbd η of λ containing x_p such that $f(\eta) \le \zeta$.

(iii) For each F-point $x_p \in P_t(\lambda)$ and each F-q-nbd ζ of $f(x_p)$ there exists a F- γ -q-nbd of x_p such that $f(\eta) \le \zeta$.

(iv) The inverse image of each F-c set in μ is a F- γc set in λ.

(v) $f(\gamma - cl(\theta)) \le cl(f(\theta)), \forall \theta \in \lambda$.

(vi) $\gamma - cl(f^{\leftarrow}(\beta)) \le f^{\leftarrow}(cl(\beta)), \forall \beta \in \mu$.

(vii) $cl(\text{int}(f^{\leftarrow}(\beta))) \wedge \text{int}(cl(f^{\leftarrow}(\beta))) \le f^{\leftarrow}(cl(\beta))), \forall \beta \in \mu$.

Proof. $(i) \Leftrightarrow (ii) \Leftrightarrow (iii), (i) \Leftrightarrow (iv)$ It is obvious.

$(iv) \Leftrightarrow (v)$: Let $\theta \in \lambda$, then $cl(f(\theta)) \in \mu$. By (iv) $f^{\leftarrow}(cl(f(\theta)))$ is F-γc set in λ. Since $\theta \le f^{\leftarrow}(f(\theta))$ we have, $\gamma - cl(\theta) \le \gamma - clf^{\leftarrow}(f(\theta)) \le \gamma - clf^{\leftarrow}(clf(\theta))) = f^{\leftarrow}(clf(\theta)))$, then $f(\gamma - cl(\theta)) \le clf(\theta)$.

$(v) \Rightarrow (vi)$:Let $\beta \in \mu$. By (v) we have $f(\gamma - cl(f^{\leftarrow}(\beta))) \le cl(\beta) \Rightarrow \gamma - cl(f^{\leftarrow}(\beta)) \le f^{\leftarrow} cl(\beta))$.

$(vi) \Rightarrow (iv)$: Let $\beta \in \mu$. By (vi) $\gamma - cl(f^{\leftarrow}(\beta)) \le f^{\leftarrow} cl(\beta)) = f^{\leftarrow}(\beta))$. Then $f^{\leftarrow}(\beta)$ is a F-γc set in λ.

$(ii) \Rightarrow (vii)$:Let $\beta \in \mu$. Then, $cl(\beta) \in \mu$, by (ii) $f^{\leftarrow}(cl(\beta))$ is a F-γc set in λ.. Hence $f^{\leftarrow}(cl(\beta)) \ge cl(\delta_i - \text{int}(f^{\leftarrow}(cl(\beta)))) \wedge \text{int}(cl(f^{\leftarrow}(cl(\beta)))) \ge cl(\text{int}(f^{\leftarrow}(\beta))) \wedge \text{int}(cl(f^{\leftarrow}(\beta)))$.

$(vii) \Rightarrow (ii)$:Let $\beta \in \mu$. By(vii)

$cl(\text{int}(f^{\leftarrow}(\beta))) \wedge \le \text{int}(cl(f^{\leftarrow}(\beta))) \le f^{\leftarrow}(cl(\beta))) = f^{\leftarrow}(\beta)$. Hence $f^{\leftarrow}(\beta)$ is a F-γc set in μ.

Corollary 3.1. Let $f:(\lambda,\delta) \to (\mu,\delta^{\circ})$ be an F-proper function and F-γc. Then $f^{\leftarrow}(\text{int}\,\beta) \le \text{int}(cl(f^{\leftarrow}(\beta)) \vee cl(\text{int}(f^{\leftarrow}(\beta))) \forall \beta \in \mu$.

Now from the above theorem it is clear that each F-pc mapping is a F-γ c and each F-sc mapping is also F-γc and the following diagram summarizes the above discussion:

F-c
⇓
F-ssc
⇓

F-pc ⇓   F-sc ⇓

F- γ-c
⇓
F-spc

The converse need not be true in general, as shown by the following examples.

Example 3.1. Let $X = \{x,y\}, \lambda = \underline{0.4}, Y = \{a\}$, $\mu = \underline{0.4}\delta = \{\underline{0}, x_{0.2} \vee y_{0.2}, x_{0.1}, \lambda\}$, $\delta^{\circ}\{\underline{0}, \underline{0.2}, \mu\}$ .Let $f:(\lambda,\delta) \to (\mu,\delta^{\circ})$ defined by $f(x,a) = f(y,a) = 0.4,$. It is clear that f is F-ssc but not F-c.

Example 3.2.

Let $X = \{x,y\}, \lambda = \underline{0.4}, Y = \{a,b\}, \mu = \underline{0.4}\delta = \{\underline{0}, x_{0.2} \vee y_{0.12}, x_{0.29} \vee y_{0.28}, \lambda\}$, $\delta^{\circ}\{\underline{0}, \underline{0.2}, \mu\}$ .Let $f:(\lambda,\delta) \to (\mu,\delta^{\circ})$ defined by $f(x,a) = f(y,a) = \underline{0.4}, f(y,b) = 0$. It is clear that f is F-pc but not F-ssc. Also, f is F-γc but not F-sc.

Example 3.3.

Let $X = \{x,y,z\}, \lambda = \underline{0.6}, Y = \{a,b,c\}, \mu = \underline{0.6}, \delta = \{\underline{0}, x_{0.2} \vee y_{0.2} \vee z_{0.3}, x_{0.2} \vee y_{0.3} \vee z_{0.4}, \lambda\}$, . $\delta^{\circ}\{\underline{0}, \underline{0.21}, \mu\}$ .Let $f:(\lambda,\delta) \to (\mu,\delta^{\circ})$ defined by $f(x,a) = f(y,a) = f(z,a) = 0.6$. It is clear that f is F-spc but not F-γc.

Example 3.4. Let $X = \{x,y\}, \lambda = \underline{0.4}, Y = \{a,b\}, \mu = \underline{0.4}\delta = \{\underline{0}, \underline{0.1}, \underline{0.2}, \lambda\}$, $\delta^{\circ}\{\underline{0}, \underline{0.3}, \mu\}$ .Let $f:(\lambda,\delta) \to (\mu,\delta^{\circ})$ defined by $f(x,a) = f(y,a) = 0.6, f(y,b) = 0., f(x,b) = f(y,b) = 0$. It is clear that f is F-sc but not F-ssc. Also, f is F-γc but not F-pc.

Remark 3.1. The composition of two F-γc mappings need not be Fγc, as shown by the following example.

Example 3.5. Consider the F-ts 's $(\lambda,\delta), (\mu,\delta^{\circ})$ and (λ,σ) where $\delta = \{\underline{0}, \underline{0.1}, x_{0.1}, y_{0.19}, \lambda\}$, and $\delta^{\circ\circ} = \{\underline{0}, \underline{0.31}, \lambda\}$ .Then the F-proper function $f:(\lambda,\delta) \to (\mu,\delta^{\circ})$ defined by $f(x,a) = f(y,a) = 0.5, f(x,b) = f(y,b) = f(y,b) = 0$. Is F- γc, also, the F-proper function $g:(\lambda,\delta^{\circ}) \to (\lambda,\delta)$ defined by $f(x,a) = f(y,a) = 0.5, f(x,b) = f(y,b) = f(y,b) = 0$. Is F- γc, but gf is not F- γc .

Remark 3.2. Let $f_1:(\lambda_1,\delta) \to (\mu_1,\delta^{\circ})$ and $f_2:(\lambda_2,\delta) \to (\mu_2,\delta^{\circ\circ})$ be F-proper functions. Then if f_1 and f_2 are F-γcontinuous, then $f_1 \times f_2$ may not be Fγ-continuous.

Example 3.6.

Let $X_1 = \{x,y\}, Y_1 = \{x,y\}, X_2 = \{a,b\}, Y_2 = \{a,b\}, \lambda_1 = \underline{0.6}, \mu_1 = \underline{0.6}, \lambda_2 = \underline{0.41}, \mu_2 = \underline{0.41}$ $, \delta = \{\underline{0}, \underline{0.1}, x_{0.21} \vee y_{0.1}, \lambda\}$, $\delta^{\circ}\{\underline{0}, \underline{0.2}, \mu_1\}$. Let $f_1:(\lambda_1,\delta) \to (\mu_1,\delta^{\circ})$ defined by $f_1(x,x) = 0.6, f(y,x) = 0.6, f_1(x,y) = f_1(y,y) = 0$. is F-γc. Also. Also $\sigma = \{\underline{0}, \underline{0.3}, \lambda_2\}$, $\sigma^{\circ} = \{\underline{0}, \underline{0.3}, \mu_1\}$, $f_2:(\lambda_2\delta) \to (\mu_2,\delta^{\circ})$ defined by $f_2(a,a) = f(b,a) = 0.41, f_2(a,b) = f_2(b,b) = 0$. is F-γc. But $f_1 \times f_2$ is not F-γ continuous.

4. F-γ-retracts

Definition 4.1 [9]. Let (ρ,δ_ρ) be a maximal subspace of an F-ts (λ,δ) . Then (ρ,δ_ρ) is called an F-strongly semi-retract, F-semi retract, F-pre retract and F-semi pre retract of (λ,δ) ;(briefly F-SSR; F-SR,F-PR,F-SPR) if there exists an F-strongly semi-continuous, F-semi continuous, F-pre continuous and F-semi pre continuous (briefly, F-SSC; F-SC,F-PC,F-SPC)-proper function $f:(\lambda,\delta) \to (\rho,\delta_\rho)$ such that $f|\rho = id_\rho$ i.e., $f(x) = \rho(x)$ $\forall x \in X$. In this case f is called an F-strongly semi-retraction, F-semi retraction, F-pre retraction and F-semi pre retraction.

Definition 4.2. Let (ρ,δ_ρ) be a maximal subspace of an F-ts (λ,δ). Then (ρ,δ_ρ) is called an F- γ -retract of (λ,δ) ; briefly F- γ -R; if there exists an F- γ c proper function $f:(\lambda,\delta)\to(\rho,\delta_\rho)$ such that $f|\rho=id_\rho$ i.e., $f(x)=\rho(x)\ \forall x\in X$. In this case f is called an F- γ - retraction.

Remark 4.1. From the above definitions one may notice that:

F-R
⇓
F-SSR
⇓
F-PR ⇓ F-SR ⇓
F- γR
⇓
F-SPR

Example4.1. Let $X=\{x,y\},\lambda=\underline{0.6},,\delta=\{\underline{0},x_{0.2}\vee y_{0.3},\lambda\}$ and $\rho=x_{0.6}$ .One can easily verify that (ρ,δ_ρ) is an F-PR and F- γ --R of (λ,δ) but neither an F-SSR nor an F-SR of it.

Example 4.2. Let $X=\{x,y\},\lambda=\underline{0.4},,\delta=\{\underline{0},x_{0.1}\vee y_{0.09},\lambda\}$ and $\rho=x_{0.4}$ .One can easily verify that (ρ,δ_ρ) is an F-SR and F- γ –R of (λ,δ) but neither an F-SSR nor an F-PR of it.

Example 4.3. Let $X=\{x,y\},\lambda=\underline{0.7},,\delta=\{\underline{0},x_{0.2}\vee y_{0.1},\underline{0.3},\lambda\}$ and $\rho=x_{0.7}$ .One can easily verify that (ρ,δ_ρ) is an F-SSR of (λ,δ) but not F-R of it.

Example4.4. Let $X=\{x,y,z\},\lambda=\underline{0.6},\delta=\{\underline{0},x_{0.1}\vee y_{0.2}\vee z_{0.3},x_{0.21}\vee y_{0.3}\vee z_{0.4},\lambda\}$ and $\rho=x_{0.6}$ .One can easily verify that (ρ,δ_ρ) is an F-SPR of (λ,δ) but not F- γ -R of it.

Theorem 4.1. Let $f:(\lambda,\delta)\to(\rho,\delta_\rho)$ be a F-proper function such that $f|\rho=id_\rho$. If the graph $g:(\lambda,\delta)\to(\lambda\times\rho,\delta\times\delta_\rho)$ is F- γ -continuous, then *f* is a F- γ -R.

Remark 4.2. Let $f:(\lambda,\delta)\to(\rho,\delta_\rho)$ be a F-proper function such that $f|\rho=id_\rho$. If $f:(\lambda,\delta)\to(\rho,\delta_\rho)$ is F- γ-R, $g:(\rho,\delta_\rho)\to(\sigma,(\delta_\rho)_\sigma)$ is F- γ -R, $\sigma\le\rho$ then gf is need not be a F-- γ -R.

Example4.5. $X=\{x,y,z\},\lambda=\underline{0.7},\delta=\{\underline{0},x_{0.2}\vee y_{0.3}\vee z_{0.1},\lambda\}$, $\rho=x_{0.7}\vee y_{0.7}$ and $\sigma=x_{0.7}$. One can easily verify that (ρ,δ_ρ) is a F -γ-R of (λ,δ) and $(\sigma,(\delta_\rho)_\sigma)$ is a F- γ-R of (ρ,δ_ρ) but $(\sigma,(\delta_\rho)_\sigma)$ is not a F- γ-R of (λ,δ).

Remark 4.3. If (ρ,δ_ρ) is a F- γ R of (λ,δ) and (σ,δ_σ) is a F- γ -R of $(\mu,\delta\ )$ then $(\rho\times\sigma,\delta_\rho\times\delta_\sigma)$ need not be a F- γ - R of $(\lambda\times\mu,\delta\times\delta\ )$.

Example 4.6 Let $X=\{x,y\},\lambda=\underline{0.6},,\delta=\{\underline{0},\underline{0.1},x_{0.21}\vee y_{0.1},\lambda\}$ and $\rho=x_{0.6}$. ,$Y=\{a,b\}$, $\mu=\underline{0.41}$, ,$\delta^{*}=\{\underline{0},\underline{0.3},\lambda\}$ and $\sigma=a_{0.41}$. One can easily verify that (ρ,δ_ρ) is a F- γ-R of (λ,δ) and $(\sigma,\delta^{*}_\sigma)$ is a F- γ-R of (μ,δ^{*}) but $(\rho\times\sigma,\delta_\rho\times\delta^{*}_\sigma)$ is not a F- γ- R of $(\lambda\times\mu,\delta\times\delta\ )$.

References

[1] K. K. Azad, On fuzzy semi-continuity, fuzzy almost continuity and fuzzy weakly continuity, J. Math. Anal. Appl. 82(1981), 14-32.

[2] M. K. Chakrabarty and T. M. G. Ahsanullah, Fuzzy topology on fuzzy sets and tolerance topology, Fuzzy Sets and Systems 45(1992), 103-108.

[3] A.K. Chaudhuri and P. Das, Some results on fuzzy topology on fuzzy sets, Fuzzy Sets and System 56(1993), 331-336.

[4] P. Das, Fuzzy topology on fuzzy sets, product fuzzy topology and fuzzy topological groups, Fuzzy Sets and Systems 100(1998), 367-372.

[5] M. A. Fath Alla and F.S. Mahmoud, Fuzzy topology on fuzzy sets :functions with fuzzy closed graphs and strong fuzzy closed graphs, J. Fuzzy Math.9(3), (2001), 525-533.

[6] I. M. Hanafy, On fuzzy γ -open sets and fuzzy γ -continuity, J. fuzzy Math. 7(2)(1999).

[7] Liu Ying-Ming and Luo Mao-Kang, Fuzzy topology, World Scientific. Singapore (1997).

[8] F. S. Mahmoud, and M. A. Fath Alla, Strong-AE and Strong-ANE fuzzy topological spaces, J. fuzzy Math. 8(2).(2000), 677-690.

[9] F.S.Mahmoud , M.A.Fath Alla and S.M.Abd Ellah, Fuzz y topology on Fuzzy Sets: Fuzzy Retracts (submitted).

[10] Pu Pao-Ming and Bu Young-Lee, Fuzzy semi-preopen sets and fuzzy semi-precontinuous mappings, Fuzzy Sets and Systems, 67(1994),359-364.

[11] S. E. Rodabaugh, Suitability in fuzzy topological spaces, J. Math. Anal. Appl. 79(1981), 273-285.

[12] S. E. Rodabaugh, Categorical frameworks for stone representation theories, in" Applications of category theory to fuzzy subsets", 177-231, Ed by S. E. Rodabaugh et al. Kluwer Academic Publishers, (1992), Netherlands.

[13] S. Sampath Kumar, On fuzzy pairwise γ -continuity and fuzzy pairwise pre-continuity, Fuzzy Sets and Systems, 62 (1994), 231-238.

[14] S. Sampath Kumar, Semi-open sets, semi-continuity and semi-open mapping in fuzzy bitopological spaces, Fuzzy Sets and Systems 64 (1994), 421-426.

[15] B.S. Zhong, Fuzzy strongly semiopen sets and fuzzy strong semicontinuity, Fuzzy Sets and Systems, 52(1992),345-351.

Ibrahim Mohamed Hanafy

Degrees :

- B. SC. in Math. from Fac. Of Science, Assiut Univ. 1976.
- M. SC in Math. (numerical analysis) from Fac. Of Science, Assiut Univ. 1981.
- Ph. D. in Math. (Topology) from Fac. Of Science, Tanta Univ. 1990.

Academic Responsibilities:

- Assistant of the Dept. of Math. Fac. Of Science, Assiut Univ.
- Assistant lecturer of the Dept. of Math. Fac. Of Science, Assiut Univ.
- lecturer of the Dept. of Math. Fac. Of Education, Suez Canal University, El-Arish, Egypt.
- Assistant Prof. of the Dept. of Math. Fac. Of Education, Suez Canal University, El-Arish, Egypt.

E-Mail : ihanafy@hotmail.com

F. S. Mahmoud

Department of Mathematics, Faculty of Science (82524), Sohag-Egypt.

M. M. Khalaf

Department of Mathematics, Faculty of Science (82524), Sohag-Egypt.