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# FUZZY CONVERGENCE THEORY - I

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ABSTRACT. The main objective of this paper is to introduce the gradation of neighbourhoodness in $L$-fuzzy topology and to introduce the fuzziness in the concept of convergence of $L$-fuzzy nets.

## 0. INTRODUCTION

The study of neighbourhood systems and convergence of nets and filters in a Chang fuzzy topological space (CFTS) was initiated by Pu & Liu [6] and Liu & Luo [13]. Later on Chang fuzzy topology was generalised by Höhle [5], Sostak [11]. Chattopadhyay, Hazra & Samanta [2] introduced gradation of openness and studied fuzzy topology. Side by side the study of graded neighbourhood system was also in progress. In Ying [12] introduced the degree to which a fuzzy point $x_\lambda$ belongs to a fuzzy subset $A$ of $X$ by $m(x_\lambda, A) = \min\big(1, 1-\lambda+A(x)\big)$ and gave the idea of graded neighbourhood on a CFTS. Using this concept of graded neighbourhood Ramadan, El Deeb & Abdel-Sattar [9] studied the convergence of a net in a smooth topological space (a smooth topological space is similar to fuzzzy topologys as defined by Höhle [5], Sostak [11] and Chattopadhyay, Hazra & Samanta [2] using crisp points as well as fuzzy points.

Apart from Ying [12], Demirci [4] introduced the idea of graded neighbourhood in smooth topological space in a different approach but restriçted himself to the $I$-valued fuzzy sets where $I = [0,1]$.

In this paper we have generalised the idea of graded neighbourhood system for $L$-fuzzy sets, where $L$ is a $F$-lattice. Also we take the definition of neighbourhood. system slightly different from those of Demirci [4].

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In section 2 of the present paper we have studied the characteristic properties of graded neighbourhood system named as "gradation of q-neighbourhoodness" in $L$-fuzzy setting.

In section 3 we have used this concept of graded neighbourhood to develop the concept of graded convergence of a fuzzy net. Relations between graded closure of a fuzzy set and graded convergence of a fuzzy net have also been studied.

## 1. NOTATION AND PRELIMINARIES

In this paper $X$ denotes a nonempty set; unless otherwise mentioned, $L$ denotes a completely distributive order dense complete lattice with an order reversing involution $'$ whereas $L_0 = L - \{0\}$. Let 0 and 1 denote the least and the greatest elements of $L$. $L^X$ denotes the collection of all $L$-fuzzy subsets of $X$; $\mathrm{Pt}(L^X)$ denotes the set of all $L$-fuzzy points of $X$. By $\tilde{0}$ and $\tilde{1}$ we denote the constant $L$-fuzzy subsets of $X$ taking values 0 and 1 respectively. For $p_x \in \mathrm{Pt}(L^X)$ and $A, B \in L^X$ we say

$$p_x \,\mathrm{q}\, A \text{ if } p_x \notin A'$$

and

$$A \,\mathrm{q}\, B \text{ if } A \not\subseteq B'.$$

For other notations we follow Liu & Luo [13].

**Definition 1.1** (Sostak [11]). A function $\tau : L^X \to L$ is called an *L-fuzzy topology on X* if it satisfies the following conditions:

(O1) $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$

(O2) $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$, for $A_1, A_2 \in L^X$

(O3) $\tau(\vee_{i\in\Delta} A_i) \geq \wedge_{i\in\Delta} \tau(A_i)$ for any $\{A_i\}_{i\in\Delta} \subset L^X$.

The pair $(X, \tau)$ is called an *L-fuzzy topological space* and $\tau$ is called *a gradation of openness* (GO) on $X$.

**Definition 1.2** (Sostak [11]). A function $\mathcal{F} : L^X \to L$ is called an *L-fuzzy co-topology of X* if it satisfies the following conditions:

(C1) $\mathcal{F}(\tilde{0}) = \mathcal{F}(\tilde{1}) = 1$

(C2) $\mathcal{F}(A_1 \vee A_2) \geq \mathcal{F}(A_1) \wedge \mathcal{F}(A_2)$, for $A_1, A_2 \in L^X$

(C3) $\mathcal{F}\Big(\bigwedge_{i\in\Delta} A_i\Big) \geq \bigwedge_{i\in\Delta} \mathcal{F}(A_i)$ for any $\{A_i\}_{i\in\Delta} \subset L^X$.

The pair $(X, \mathcal{F})$ is called an *L-fuzzy co-topological space* and $\mathcal{F}$ is called a *gradation of closedness* (GC) on $X$.

## 2. Some results on graded neighbourhood system

**Proposition 2.1.** *Let $(X, \tau)$ be an L-fuzzy topological space with $\tau$ as a gradation of openness on $X$ and let $\tau_r = \{U \in L^X; \tau(U) \geq r\}$ then*

(1) *$\tau_r$ is a Chang L-fuzzy topology for every $r \in L_0$,*

(2) *$\tau_r \subseteq \tau_s$ if $r \geq s; r, s \in L_0$, and*

(3) $\bigcap_{i\in\Delta} \tau_{r_i} = \tau_{\vee_{i\in\Delta} r_i}$.

*Proof.* (1) We have $\tau(\tilde{0}) = \tau(\tilde{1}) = 1$ so $\tilde{0}, \tilde{1} \in \tau_r \forall\ r \in L_0$.

For any $U_1, U_2 \in L^X, U_1, U_2 \in \tau_r \Rightarrow \tau(U_1), \tau(U_2) \geq r$

$$\Rightarrow \tau(U_1 \wedge U_2) \geq \tau(U_1) \wedge \tau(U_2), \quad \text{by (O2)}$$
$$\geq r.$$

Finally, $U_i \in \tau_r \forall\ i \in \Delta \Rightarrow \tau(U_i) \geq r, \forall\ i \in \Delta$.

So,

$$\tau\Big( \vee_{i\in\Delta} U_i\Big) \geq \wedge_{i\in\Delta}\tau(U_i), \quad \text{by (O3)}$$
$$\geq r.$$

Hence $\tau_r$ is a Chang $L$-fuzzy topology for every $r \in L_0$.

(2) and (3) are straightforward. □

**Proposition 2.2.** *Let $\{T_r\}_{r\in L_0}$ be a collection of fuzzy subsets of $X$ satisfying*

(1) *$T_r$ is a Chang L-fuzzy topology on $X$ for each $r \in L_0$,*

(2) *$T_r \subseteq T_s$ if $r \geq s$; $r, s \in L_0$ then the mapping $\tau : L^X \to L$ defined by $\tau(A) = \vee\{r;\ A \in T_r\}$ is an L-fuzzy topology on $X$ and if further $T_r$ satisfies,*

(3) *$\bigcap_{i\in\Delta} T_{r_i} = T_{\vee_{i\in\Delta} r_i}$ then the collection $\tau_r = \{U \in L^X;\ \tau(U) \geq r\}$ is identical with $T_r$ for every $r \in L_0$.*

*Proof.* (i) $\tilde{0}, \tilde{1} \in T_r\ \forall\ r \in L_0 \ \Rightarrow\ \tau(\tilde{0}) = \tau(\tilde{1}) = 1$.

(ii) $A_1 \in T_{r_1},\ A_2 \in T_{r_2} \ \Rightarrow\ A_1, A_2 \in T_{r_1 \wedge r_2}$, by (2) $\Rightarrow A_1 \wedge A_2 \in T_{r_1\wedge r_2}$, by (1) $\Rightarrow \tau(A_1 \wedge A_2) \geq r_1 \wedge r_2$, by the definition of $\tau$. As $L$ is completely distributive so $\tau(A_1 \wedge A_2) \geq \tau(A_1) \wedge \tau(A_2)$.

(iii) Let $A_i \in T_{r_i};\ i \in \Delta$.

If $\bigwedge_{i\in\Delta} r_i = 0$ then obviously $\tau\big(\bigvee_{i\in\Delta} A_i\big) \geq \wedge_{i\in\Delta} r_i$.

So let $\bigwedge_{i\in\Delta} r_i \neq 0$ then $A_i \in T_{\wedge_{i\in\Delta} r_i}$, by (2) $\forall\ i \in \Delta \Rightarrow \bigvee_{i\in\Delta} A_i \in T_{\wedge_{i\in\Delta} r_i}$, by (1) $\Rightarrow \tau\big(\bigvee_{i\in\Delta} A_i\big) \geq \bigwedge_{i\in\Delta} r_i$, by the definition of $\tau$. Again as $L$ is completely distributive so $\tau\big(\bigvee_{i\in\Delta} A_i\big) \geq \wedge_{i\in\Delta}\tau(A_i)$.

Next we want to show that $T_r = \tau_r$ for $r \in L_0$. $A \in T_r \Rightarrow \bigvee\{k; A \in T_k\} \geq r \Rightarrow \tau(A) \geq r \Rightarrow A \in \tau_r$ for $r \in L_0$. So $T_r \subseteq \tau_r \forall\, r \in L_0$. Again $B \in \tau_r \Rightarrow \tau(B) \geq r \Rightarrow \vee\{k \in L_0; B \in T_k\} \geq r$.

Let $S = \{k \in L_0; B \in T_k\}$. Then $B \in T_k$ for $k \in S \Rightarrow B \in \cap_{k \in S} T_k = T_{\vee_{k \in S}} = T_s$ and $s \geq r$.

So, $B \in T_r \Rightarrow \tau_r \subseteq T_r$ for $r \in L_0$. □

**Definition 2.3.** Let $(X, \tau)$ be $L$-fuzzy topological space and let $Q : \mathrm{Pt}(L^X) \times L^X \to L$ be a mapping defined by $Q(p_x, A) = \bigvee\{\tau(U);\ p_x \,\mathrm{q}\, U \subset A\}$. Then $Q$ is said to be a *gradation of* q*-neighbourhoodness*.

**Proposition 2.4.** *Let $Q$ be a gradation of* q*-neighbourhoodness in an $L$-fuzzy topological space $(X, \tau)$. Then*

(QN1): $Q(p_x, \tilde{1}) = 1$, $Q(p_x, \tilde{0}) = 0$ *for* $p_x \in \mathrm{Pt}(L^X)$,

(QN2): $Q(p_x, A) \leq Q(p_x, B)$ *if* $A, B \in L^X$, $A \subseteq B$,

(QN3): $Q(p_x, A \wedge B) = Q(p_x, A) \wedge Q(p_x, B)$ *for* $p_x \in M(L^X)$ *and* $A, B \in L^X$.

(QN4): $Q(p_x, A) \not\leq k \Rightarrow \exists B_p \in L^X$ *such that* $p_x \,\mathrm{q}\, B_p \subseteq A$ *and* $\wedge\{Q(r_y, B_p);\ r_y \in \mathrm{Pt}(L^X);\ r_y \,\mathrm{q}\, B_p\} \not\leq k$.

*Proof.* The proof of (QN1)–(QN2) is straightforward.

(QN3): $Q(p_x, A \wedge B) \leq Q(p_x, A) \wedge Q(p_x, B)$ is obvious from (QN2).
Next let $p_x \,\mathrm{q}\, U \subset A$ and $p_x \,\mathrm{q}\, V \subset B$ then as $p_x \in M(L^X)$ so,

$$p_x \,\mathrm{q}(U \wedge V) \subseteq A \wedge B \Rightarrow Q(p_x, A \wedge B) \geq \tau(U \wedge V) \geq \tau(U) \wedge \tau(V).$$

Since $L$ is completely distributive so $Q(p_x, A \wedge B) \geq Q(p_x, A) \wedge Q(p_x, B)$.
So, $Q(p_x, A \wedge B) = Q(p_x, A) \wedge Q(p_x, B)$.

$$\text{(QN4)}: \quad Q(p_x, A) \not\leq k \Rightarrow \vee\{\tau(U);\ p_x \,\mathrm{q}\, U \subset A\} \not\leq k$$
$$\Rightarrow \exists U_1 \in L^X \text{ such that } p_x \,\mathrm{q}\, U_1 \subset A \text{ and } \tau(U_1) \not\leq k.$$

Taking $B_p = U_1$ we have $\forall\, r_y \,\mathrm{q}\, B_p$, $r_y \,\mathrm{q}\, B_p \subset B_p$ and $\tau(B_p) \not\leq k$.
Now

$$Q(r_y, B_p) = \vee\{\tau(U); r_y \,\mathrm{q}\, U \subset B_p\} \geq \tau(B_p) \text{ for } r_y \,\mathrm{q}\, B_p$$
$$\Rightarrow \wedge_{(r_y \,\mathrm{q}\, B_p)} Q(r_y, B_p) \geq \tau(B_p).$$

So, $\tau(B_p) \not\leq k \Rightarrow \wedge_{(r_y \,\mathrm{q}\, B_p)} Q(r_y, B_p) \not\leq k$. □

**Proposition 2.5.** *Let $Q : \mathrm{Pt}(L^X) \times L^X \to L$ be a mapping satisfying* (QN1)–(QN3) *of Proposition* 2.4. *Let $\bar{\tau} : L^X \to L$ be defined by*

$$\bar{\tau}(A) = \wedge\{Q(p_x, A);\ p_x \in M(L^X) \text{ and } p_x \,\mathrm{q}\, A\}.$$

*Then $(X, \bar{\tau})$ forms an $L$-fuzzy topological space. If further the condition* (QN4) *of Proposition* 2.4 *is satisfied by $Q$ then the mapping $\bar{Q} : \mathrm{Pt}(L^X) \times L^X \to L$ defined by $\bar{Q}(p_x, A) = \vee\{\bar{\tau}(U); p_x \,\mathrm{q}\, U \subset A\}$ is identical with $Q$.*

*Proof.* Verification of (O1) is straightforward.

(O2):
$$\begin{aligned}\bar{\tau}(A \wedge B) &= \wedge\{Q(p_x, A \wedge B);\ p_x \in M(L^X);\ p_x \,\mathrm{q}(A \wedge B)\}\\ &= \wedge\{Q(p_x, A) \wedge Q(p_x, B);\ p_x \in M(L^X);\ p_x \,\mathrm{q}(A \wedge B)\}, \text{ by (QN3)}\\ &= \{\wedge [Q(p_x, A);\ p_x \in M(L^X);\ p_x \,\mathrm{q}(A \wedge B)]\}\\ &\qquad \wedge\{\wedge [Q(p_x, B);\ p_x \in M(L^X);\ p_x \,\mathrm{q}(A \wedge B)]\}\\ &\geq \{\wedge [Q(p_x, A);\ p_x \in M(L^X);\ p_x \,\mathrm{q}\, A]\}\\ &\qquad \wedge\{\wedge [Q(p_x, B);\ p_x \in M(L^X);\ p_x \,\mathrm{q}\, B]\}\\ &= \bar{\tau}(A) \wedge \bar{\tau}(B).\end{aligned}$$

(O3): For any $p_x \in M(L^X)$, $p_x \,\mathrm{q} \left(\vee_{i\in\Delta} A_i\right) \Rightarrow p_x \,\mathrm{q}\, A_j$ for some $j \in \Delta$.

Then $Q(p_x, \vee_{i\in\Delta} A_i) \geq Q(p_x, A_j)$, by (QN2),
$$\begin{aligned}&\geq \wedge_{(r_y \,\mathrm{q}\, A_j)} Q(r_y, A_j),\ r_y \in M(L^X) \text{ as } p_x \,\mathrm{q}\, A_j\\ &= \bar{\tau}(A_j) \geq \wedge_{i\in\Delta} \bar{\tau}(A_j).\end{aligned}$$

Since this is true for all $p_x \in M(L^X)$ with $p_x \,\mathrm{q} \left(\vee_{i\in\Delta} A_i\right)$ so

$$\wedge\{Q(p_x, \vee_{i\in\Delta} A_i);\ p_x \in M(L^X);\ p_x \,\mathrm{q}(\vee_{i\in\Delta} A_i)\} \geq \wedge_{i\in\Delta} \bar{\tau}(A_i),$$

*i. e.*, $\bar{\tau}\left(\vee_{i\in\Delta} A_i\right) \geq \wedge_{i\in\Delta} \bar{\tau}(A_i)$

To prove the last part of the proposition let us suppose that Q satisfies the condition (QN4) of Proposition 2.4. If $\bar{Q}(p_x, A) = 0$ then obviously $\bar{Q}(p_x, A) \leq Q(p_x, A)$.

So let us suppose $\bar{Q}(p_x, A) > 0$.

Then

$$\begin{aligned}\bar{Q}(p_x, A) \not\leq m &\Rightarrow \vee\{\bar{\tau}(U);\ p_x \,\mathrm{q}\, U \subset A\} \not\leq m\\ &\Rightarrow \exists\, U_p \in L^X \text{ such that } p_x \,\mathrm{q}\, U_p \subset A \text{ and } \bar{\tau}(U_p) \not\leq m.\end{aligned}$$

Now

$$\begin{aligned}\bar{\tau}(U_p) \not\leq m &\Rightarrow \bigwedge \left\{Q(r_y, U_p);\ r_y \in M(L^X);\ r_y \,\mathrm{q}\, U_p\right\} \not\leq m\\ &\Rightarrow Q(r_y, U_p) \not\leq m \text{ and } r_y \in M(L^X) \text{ with } r_y \,\mathrm{q}\, U_p.\end{aligned}$$

Again as

$$\begin{aligned}
p_x \,q\, U_p \Rightarrow\ & p \not\leq U_p'(x) \\
\Rightarrow\ & \exists\, s \in M(L) \text{ such that } s \leq p \text{ but } s \not\leq U_p'(x) \\
& \qquad\qquad [\text{since } M(L) \text{ is join generating subset of } L] \\
\Rightarrow\ & s_x \,q\, U_p \text{ and } s_x \in M(L^X) \\
\Rightarrow\ & Q(s_x, U_p) \not\leq m \\
\Rightarrow\ & Q(p_x, U_p) \not\leq m \ [\text{since } p_x \geq s_x \ \Rightarrow\ Q(p_x, U_p) \geq Q(s_x, U_p)] \\
\Rightarrow\ & Q(p_x, A) \not\leq m \ [\text{since } U_p \subset A].
\end{aligned}$$

Hence

$$\bar{Q}(p_x, A) \leq Q(p_x, A) \qquad (*)$$

Again

$$\begin{aligned}
& Q(p_x, A) \not\leq \alpha \\
& \quad \Rightarrow \exists\, B_p \in L^X \text{ such that } p_x \,q\, B_p \subset A \text{ and} \\
& \qquad\qquad\qquad \wedge \left\{Q(r_y, B_p);\ r_y \in \mathrm{Pt}(L^X);\ r_y \,q\, B_p\right\} \not\leq \alpha. \\
& \quad \Rightarrow \exists\, B_p \in L^X \text{ such that } p_x \,q\, B_p \subset A \text{ and } \bar{\tau}(B_p) \not\leq \alpha \\
& \quad \Rightarrow \bigvee\{\bar{\tau}(U);\ p_x \,q\, U \subset A\} \not\leq \alpha \\
& \quad \Rightarrow \bar{Q}(p_x, A) \not\leq \alpha.
\end{aligned}$$

Hence

$$Q(p_x, A) \leq \bar{Q}(p_x, A). \qquad (**)$$

So, $Q(p_x, A) = \bar{Q}(p_x, A)$ for $p_x \in \mathrm{Pt}(L^X)$ and $\forall\ A \in L^X$. □

**Lemma 2.6.** *If for every $p_x \in M(L^X)$ with $p_x \,q\, A$ we choose any $U_{p_x} \in L^X$ with $p_x \,q\, U_{p_x} \subseteq A$ then $A = \bigvee\{U_{p_x};\ p_x \in M(L^X)$ and $p_x \,q\, A\}$.*

*Proof.* $\bigvee\{U_{p_x};\ p_x \in M(L^X)$ and $p_x \,q\, A\} \subseteq A$ is obvious.

If possible let $\vee\{U_{p_x};\ p_x \in M(L^X)$ and $p_x \,q\, A\}$ be a proper subset of $A$ then $\exists\ z \in X$ such that

$$\begin{aligned}
& A(z) > \left(\vee \{U_{p_x};\ p_x \in M(L^X) \text{ and } p_x \,q\, A\}\right)(z) \\
& \qquad\qquad \Rightarrow\ A'(z) < \left(\vee \{U_{p_x};\ p_x \in M(L^X) \text{ and } p_x \,q\, A\}\right)'(z).
\end{aligned}$$

As $M(L)$ is a join generating set of $L$ so

$$\exists\, k \in M(L) \text{ such that } k \not\leq A'(z) \text{ but } k \leq \big(\vee\{U_{p_x};\ p_x \in M(L^X) \text{ and } p_x \,\mathrm{q}\, A\}\big)'(z)$$
$$\Rightarrow k_z \,\mathrm{q}\, A \text{ but } k_z \,\not{\mathrm{q}}\big(\vee\{U_{p_x};\ p_x \in M(L^X) \text{ and } p_x \,\mathrm{q}\, A\}\big)$$
$$\Rightarrow k_z \,\mathrm{q}\, A \text{ but } k_z \,\not{\mathrm{q}}\, U_{p_x}\ \forall\, p_x \in M(L^X) \text{ with } p_x \,\mathrm{q}\, A,$$

which is a contradiction to the given condition. □

**Proposition 2.7.** *Let $Q$ be a gradation of* q*-neighbourhoodness in an $L$-fuzzy topological space $(X,\tau)$ and $\bar{\tau} : L^X \to L$ be defined by*

$$\bar{\tau}(A) = \wedge\{Q(p_x, A);\ p_x \in M(L^X);\ p_x \,\mathrm{q}\, A\}$$

*then $\bar{\tau}$ is an $L$-fuzzy topology on $X$ and $\bar{\tau} = \tau$.*

*Proof.* As $Q$ is a gradation of q-neighbourhoodness in $(X,\tau)$, so all the conditions of Proposition 2.4 are satisfied by $Q$. So, by Proposition 2.5 we can say that $\bar{\tau}$ is an $L$-fuzzy topology on $X$.
Also $Q(p_x, A) = \vee\{\tau(U);\ p_x \,\mathrm{q}\, U \subset A\} \geq \tau(A)\ \forall\, p_x \in \mathrm{Pt}(L^X)$ with $p_x \,\mathrm{q}\, A$
$\Rightarrow \wedge\{Q(p_x, A);\ p_x \in M(L^X) \text{ and } p_x \,\mathrm{q}\, A\} \geq \tau(A) \Rightarrow \bar{\tau}(A) \geq \tau(A)\ \forall\, A \in L^X.$
$\Rightarrow \bar{\tau} \geq \tau.$
Next, if $A = \tilde{0}$ then $\exists$ no $p_x \in M(L^X)$ such that $p_x q \tilde{0}$ so $\bar{\tau}(\tilde{0}) = 1 = \tau(\tilde{0})$.

If $A \neq \tilde{0}$ then for each $p_x \in M(L^X)$ with $p_x \,\mathrm{q}\, A$ if we take any $U_{p_x}$ satisfying $p_x \,\mathrm{q}\, U_{p_x} \subseteq A$ then by Lemma 2.6 $A = \cup\{U_{p_x};\ p_x \in M(L^X) \text{ and } p_x \,\mathrm{q}\, A\}$.
So,

$$\tau(A) = \tau\big(\cup\{U_{p_x};\ p_x \in M(L^X) \text{ and } p_x \,\mathrm{q}\, A\}\big) \geq \wedge\{\tau(U_{p_x});\ p_x \in M(L^X);\ p_x \,\mathrm{q}\, A\}. \tag{1}$$

Again as $L$ is completely distributive and the relation (1) is true for any $U_{p_x}$ satisfying $p_x \,\mathrm{q}\, U_{p_x} \subseteq A$ it follows that $\tau(A) \geq \wedge\{Q(p_x, A);\ p_x \in M(L^X) \text{ and } p_x \,\mathrm{q}\, A\}$, *i. e.*, $\tau(A) \geq \bar{\tau}(A)$. As $A \in L^X$ is arbitrary, $\tau \geq \bar{\tau}$. □

*Remark* 2.8. It may be noted that for $r \in L_0$ and $e \in \mathrm{Pt}(L^X)$, $Q_r(e) = \{A \in L^X;\ Q(e, A) \geq r\}$ is not necessarily a q-neighbourhood system of $e$ with respect to the Chang fuzzy topology $\tau_r$ which is shown by the following example.

*Example* 2.9. Let $X = \{0, 1, 2, 3, \ldots\}$, $L = \mathcal{I} = \{(r, s) \in I \times I;\ r + s \leq 1\}$ and the P.O relation '$\leq$' in $\mathcal{I}$ is defined as $(r_1, s_1) \leq (r_2, s_2) \Leftrightarrow r_1 \leq r_2$ and $s_1 \geq s_2$, '$\vee$' and '$\wedge$' are defined by

$$(r_1, s_1) \vee (r_2, s_2) = (r_1 \vee r_2, s_1 \wedge s_2) \text{ and } (r_1, s_1) \wedge (r_2, s_2) = (r_1 \wedge r_2, s_1 \vee s_2)$$

respectively.

Let

$$A_n(x) = \begin{cases} \left(0.1 + \frac{1}{n+2}, 0.9 - \frac{1}{n+2}\right) & \text{if } x \geq n \text{ and } n = 1, 2, 3, \ldots \\ \left(\frac{1}{2}, \frac{1}{2}\right) \text{ if } x = 0 \\ (0, 1) \text{ elsewhere.} \end{cases}$$

Then $\{A_n\}$ is a monotone decreasing and $\{A'_n\}$ is a monotone increasing sequence of $L$-fuzzy subsets of $X$.

Let $B \in L^X$ be defined by

$$B(x) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2}\right) & \text{if } x = 0 \\ (1, 0) & \text{if } x > 0. \end{cases}$$

And let $\tau : L^X \to L$ be defined by

$$\tau(A_n) = \left(0.5 - \frac{1}{n+2}, 0.5 + \frac{1}{n+2}\right),$$

for $n = 1, 2, 3, \ldots$ and

$$\tau(A'_n) = \left(0.5 + \frac{1}{n+2}, 0.5 - \frac{1}{n-2}\right),$$

for $n = 1, 2, 3, \ldots$.

$\tau(\tilde{0}) = \tau(\tilde{1}) = (1, 0)$, $\tau(B) = (0.6, 0.4)$ and $\tau(A) = (0, 1)$ for any other $L$-fuzzy subsets $A$ of $X$. Then $\tau$ is an $L$-fuzzy topology on $X$. $Q\big((0.6, 0.4)_0, A_1\big) = (0.5, 0.5)$, *i. e.*, $A_1 \in Q_{(0.5,0.5)}, \big((0.6, 0.4)_0\big)$ but $\exists$ no $U \in \tau_{(0.5,0.5)}$ such that $(0.6, 0.4)_0 \, q \, U \subset A_1$.

Hence $Q_{(0.5,0.5)}\big((0.6, 0.4)_0\big)$ is not a q-neighbourhood system of $(0.6, 0.4)_0$ with respect to the Chang fuzzy topology $\tau_{(0.5,0.5)}$.

*Remark* 2.10. We shall denote the q-neighbourhood system of $e$ with respect to the Chang fuzzy topology $\tau_r$ by $\tilde{Q}_r(e)$, *i. e.*, $\tilde{Q}_r(e) = \{U \in L^X;\ \exists\ V \in \tau_r$ satisfying $e \, q \, V \subset U\}$.

**Definition 2.11.** Let $(X, \mathcal{F})$ be an $L$-fuzzy co-topological space with $\mathcal{F}$ as a GC on $X$. For each $r \in L_0$ and for each $A \in L^X$ we define

$$\text{cl}(A, r) = \wedge\{D \in L^X;\ A \subseteq D;\ D \in \mathcal{F}_r\} \text{ where } \mathcal{F}_r = \{C \in L^X;\ \mathcal{F}(C) \geq r\}.$$

cl is said to be $L$-fuzzy closure operator in $(X, \mathcal{F})$.

**Proposition 2.12.** *Let $(X, \mathcal{F})$ be an $L$-fuzzy co-topological space with $\mathcal{F}$ as a GC on $X$ and let* $\text{cl} : L^X \times L_0 \to L^X$ *be the $L$-fuzzy closure operator in $(X, \mathcal{F})$ where $L$ is a completely distributive order dense and complete lattice with an order reversing involution $'$. Then*

(CO1): $\mathrm{cl}(\tilde{0}, r) = \tilde{0}$; $\mathrm{cl}(\tilde{1}, r) = \tilde{1}\ \forall\ r \in L_0$.
(CO2): $\mathrm{cl}(A, r) \supseteq A,\ \forall\ A \in L^X$ *and* $\forall\ r \in L_0$.
(CO3): $\mathrm{cl}(A, r) \subseteq \mathrm{cl}(A, s$ *if* $r \leq s$.
(CO4): $\mathrm{cl}(A_1 \vee A_2, r) = \mathrm{cl}(A_1, r) \vee \mathrm{cl}(A_2, r),\ \forall\ r \in L_0$.
(CO5): $\mathrm{cl}\big(\mathrm{cl}(A, r), r\big) = \mathrm{cl}(A, r),\ \forall\ r \in L_0$.
(CO6): *If* $l = \vee\{r \in L_0;\ \mathrm{cl}(A, r) = A\}$ *then* $\mathrm{cl}(A, l) = A$.

The proof is straightforward.

**Proposition 2.13.** *Let $L$ be a completely distributive order dense and complete lattice with an order reversing involution $'$ and* $\mathrm{cl}:\ L^X \times L_0 \to L^X$ *be a mapping satisfying* (CO1)–(CO4) *of Proposition* 2.12. *Let $\bar{\mathcal{F}} : L^X \to L$ be a mapping defined by $\bar{\mathcal{F}}(A) = \vee\{r \in L_0;\ \mathrm{cl}(A, r) = A\}\ \forall\ A \in L^X$ then $\bar{\mathcal{F}}$ is a GC on $X$ and* $\mathrm{cl} = \mathrm{cl}_{\bar{\mathcal{F}}}$ *if and only if* (CO5) *and* (CO6) *are satisfied by* cl.

*Proof.* Obviously $\bar{\mathcal{F}}(\tilde{0}) = \bar{\mathcal{F}}(\tilde{1}) = 1$. Let $\mathrm{cl}(A_1, r_1) = A_1$ and $\mathrm{cl}(A_2, r_2) = A_2$ then

$$\begin{aligned}\mathrm{cl}(A_1 \vee A_2, r_1 \wedge r_2) = \mathrm{cl}(A_1, r_1 \wedge r_2) \vee \mathrm{cl}(A_2, r_1 \wedge r_2) &\leq \mathrm{cl}(A_1, r_1) \vee \mathrm{cl}(A_2, r_2)\\ &= A_1 \vee A_2 \Rightarrow \bar{\mathcal{F}}(A_1 \vee A_2) \geq r_1 \wedge r_2.\end{aligned}$$

As $L$ is completely distributive so $\bar{\mathcal{F}}(A_1 \vee A_2) \geq \bar{\mathcal{F}}(A_1) \wedge \bar{\mathcal{F}}(A_2)$ .

Let $\mathrm{cl}(A_i, r_i) = A_i\ \forall i \in \Delta$ then

$$\begin{aligned}\mathrm{cl}\left(\wedge_{i\in\Delta} A_i, \wedge_{i\in\Delta} r_i\right) &\leq \mathrm{cl}\left(A_i, \wedge_{i\in\Delta} r_i\right),\ \forall\ i \in \Delta,\ \text{by (CO4)}\\ &\leq \mathrm{cl}(A_i, r_i),\ \forall\ i \in \Delta,\ \text{by (CO3)}\\ &= A_i,\ \forall\ i \in \Delta\end{aligned}$$

$\Rightarrow\ \mathrm{cl}\left(\wedge_{i\in\Delta} A_i, \wedge_{i\in\Delta} r_i\right) \leq \wedge_{i\in\Delta} A_i$
$\Rightarrow\ \bar{\mathcal{F}}\left(\wedge_{i\in\Delta} A_i\right) \geq \wedge_{i\in\Delta} r_i$.
As $L$ is completely distributive so $\bar{\mathcal{F}}\left(\wedge_{i\in\Delta} A_i\right) \geq \wedge_{i\in\Delta} \bar{\mathcal{F}}(A_i)$.
Now to prove the second part, suppose cl satisfies conditions (CO5) and (CO6) in addition to the conditions (CO1)–(CO4) of Proposition 2.12.
First we shall prove $\bar{\mathcal{F}}(D) \geq l \iff \mathrm{cl}(D, l) = D\ \forall\ l \in L_0$. $\bar{\mathcal{F}}(D) \geq l$ implies $\vee\{r \in L_0;\ \mathrm{cl}(D, r) = D\} \geq l$.

Let $S = \{r \in L_0;\ \mathrm{cl}(D, r) = D\}$ then $\vee\{r;\ r \in S\} \geq l$.
Again $\mathrm{cl}(D, \vee_{r\in S} r) = D$, by (CO6), $\Rightarrow$ $\mathrm{cl}(D, l) = D$, by (CO3).
Obviously $\mathrm{cl}(D, l) = D\ \Rightarrow\ \bar{\mathcal{F}}(D) \geq l$,

Thus $\bar{\mathcal{F}}(D) \geq l \iff \mathrm{cl}(D, l) = D$.

Now $\mathrm{cl}_{\bar{\mathcal{F}}}(A,l) = \wedge\{D \supseteq A;\ D \in \bar{\mathcal{F}}_l\} = \wedge\{D \supseteq A;\ \mathrm{cl}(D,l) = D\} \leq \mathrm{cl}(A,l)$.
(as : $\mathrm{cl}(A,l) \supseteq A$ and $\mathrm{cl}(\mathrm{cl}(A,l),l) = \mathrm{cl}(A,l)$, by (CO5) ).
Again $\mathrm{cl}(D,l) = D \supseteq A \Rightarrow \mathrm{cl}(D,l) \geq \mathrm{cl}(A,l)$, by (CO4)
So, $\mathrm{cl}_{\bar{\mathcal{F}}}(A,l) = \wedge\{D \supseteq A;\ \mathrm{cl}(D,l) = D\} = \wedge\{\mathrm{cl}(D,l) = D \supseteq A\} \geq \mathrm{cl}(A,l)$.
Hence $\mathrm{cl}_{\bar{\mathcal{F}}}(A,l) = \mathrm{cl}(A,l)$. Converse is obvious. □

**Proposition 2.14.** *A fuzzy point* $p_x \in \mathrm{cl}(A,m) \iff \forall\ U \in L^X$ *satisfying* $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A$ *implies* $\tau(U) \not\geq m$.

*Proof.* Let $p_x \in \mathrm{cl}(A,m)$. If possible let $\exists\ U \in L^X$ such that $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A$ and $\tau(U) \geq m$.

Then $p_x \notin U^c$ and $A \subset U^c, U^c \in \mathcal{F}_m \Rightarrow \mathrm{cl}(A,m) \subset U^c$. But $p_x \notin U^c \supset \mathrm{cl}(A,m)$ is a contradiction.

Conversely, let the given condition be satisfied. Put $\mathrm{cl}(A,m) = B$, then clearly $B \in \mathcal{F}_m$. If possible let $p_x \notin B$ then $p_x \,\mathrm{q}\, B^c \,\not\mathrm{q}\, B$. So taking $B^c = U$ we see that $\tau(U) \geq m$ and $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A$ (since $A \subset B$) which is a contradiction to our assumption.

So $p_x \in \mathrm{cl}(A,m)$. □

**Corollary 2.15.** $p_x \notin \mathrm{cl}(A,m) \iff \exists$ *at least one* $U \in \tilde{Q}_m(p_x)$ *such that* $U \,\not\mathrm{q}\, A$.

**Proposition 2.16.** *Let* $(X,\tau)$ *be an L-fuzzy topological space with L as an order dense chain and* cl *be the closure operator on X. Then* $p_x \in \mathrm{cl}(A,m) \iff \forall U$ *satisfying* $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A \exists$ *at least one L-fuzzy point.* $r_y \,\mathrm{q}\, U$ *such that* $Q(r_y, U) < m$.

*Proof.* $p_x \in \mathrm{cl}(A,m)$

$\iff \forall\ U$ satisfying $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A$ implies $\tau(U) \not\geq m$

$\iff \forall\ U$ satisfying $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A$ implies $\tau(U) < m$ (as $L$ is a chain)

$\iff \forall\ U$ satisfying $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A$, $\wedge_{(r_y \,\mathrm{q}\, U)} Q(r_y, U) < m$

$\iff \forall\ U$ satisfying $p_x \,\mathrm{q}\, U \,\not\mathrm{q}\, A \exists$ at least one $r_y \,\mathrm{q}\, U$ such that $Q(r_y, U) < m$. □

**Proposition 2.17.** *In an L-fuzzy topological space* $(X,\tau), p_x \in \mathrm{cl}(A,m) \iff \forall\ U \in \tau_m$, $p_x \,\mathrm{q}\, U \Rightarrow U \,\mathrm{q}\, A$.

*Proof.* Let $p_x \in \mathrm{cl}(A,m)$ and let $\exists\ U \in \tau_m$ such that $p_x \,\mathrm{q}\, U$ but $U \,\not\mathrm{q}\, A$ then $U^c \in \mathcal{F}_m$ and $p_x \notin U^c$ but $A \subseteq U^c$. Now $A \subseteq U^c$ and $U^c \in \mathcal{F}_m \Rightarrow \mathrm{cl}(A,m) \subseteq U^c$. Hence $p_x \in \mathrm{cl}(A,m)$ but $p_x \notin U^c$ is a contradiction.

Conversely, let the given condition be satisfied and if possible let $p_x \notin \mathrm{cl}(A,m)$. Put $\mathrm{cl}(A,m) = B$ then $p_x \notin B \Rightarrow p_x \,\mathrm{q}\, B^c$ also $B \in \mathcal{F}_m \Rightarrow B^c \in \tau_m$. So $B^c \in \tau_m$ and $p_x \,\mathrm{q}\, B^c$ but $B^c \,\not\mathrm{q}\, A$ (since $A \subset B$) is a contradiction. □

**Proposition 2.18.** *In an L-fuzzy topological space* $(X, \tau)$ *the following statements are equivalent*

(i) $p_x \in \text{cl}(A, m)$

(ii) $\forall\, U \in \tau_m,\ p_x \,\mathrm{q}\, U \ \Rightarrow\ U \,\mathrm{q}\, A$

(iii) $U \in \tilde{Q}_m(p_x) \ \Rightarrow\ U \,\mathrm{q}\, A.$

*Proof.* (i) $\Longleftrightarrow$ (ii), by Proposition 2.17.

(ii) $\Longleftrightarrow$ (iii).

For, let (ii) hold and let $U \in \tilde{Q}_m(p_x)$ then $\exists\ V \in \tau_m$ such that $p_x \,\mathrm{q}\, V \subset U \Rightarrow\ V \,\mathrm{q}\, A$, by (ii) $\Rightarrow U \,\mathrm{q}\, A$ (since $V \subset U$), *i. e.*, (ii) $\Rightarrow$ (iii).

Conversely, let (iii) hold and let $U \in \tau_m$ with $p_x \,\mathrm{q}\, U$ then $U \in \tilde{Q}_m(p_x) \Rightarrow\ U \,\mathrm{q}\, A$, by (iii), *i. e.*, (iii) $\Rightarrow$ (ii). □

## 3. Fuzzy Net and its Convergence

**Definition 3.1.** Let $(X, \tau)$ be an $L$-fuzzy topological space where $L$ is completely distributive order dense complete lattice with an order reversing involution $'$ and $e \in \text{Pt}(L^X)$. Let $D$ be any directed set and $S : D \to \text{Pt}(L^X)$ be any fuzzy net. For $U \in L^X$ if $\exists\ m \in D$ such that $S(n) \,\mathrm{q}\, U\ \forall n \geq m$ holds then we say that $S \,\mathrm{q}\, U$ eventually; if for every $m \in D\ \exists\ n \in D$ such that $n \geq m$ and $S(n) \,\mathrm{q}\, U$ then we say $S \,\mathrm{q}\, U$ frequently. Call '$e$' a cluster point of upper grade $l$, denoted by $S \infty^l e$ and of lower grade $k$, denoted by $S \infty_k e$ of a fuzzy net $S : D \to \text{Pt}(L^X)$, if $l' = \wedge\{r \in L_0;\ \forall\ U \in \tilde{Q}_r(e),\ U \,\mathrm{q}\, S$ frequently$\}$ and $k' = \vee\{r \in L_0;\ \exists\ V \in \tilde{Q}_r(e)$ such that $V \not\mathrm{q}\, S$ eventually$\}$ respectively. Call '$e$' a limit point of upper grade $l$, denoted by $S \to^l e$ and lower grade $k$, denoted by $S \to_k e$, of $S$ if $l' = \wedge\{r \in L_0;\ \forall\ U \in \tilde{Q}_r(e),\ U \,\mathrm{q}\, S$ frequently$\}$ and $k' = \vee\{r \in L_0;\ \exists\ V . \in \tilde{Q}_r(e)$ such that $V \not\mathrm{q}\, S$ frequently$\}$.

**Proposition 3.2.** *For any fuzzy net* $S$ *in an L-fuzzy topological space* $(X, \tau)$,

(i) $S \to^l e\ \&\ S \to_k e\ \Rightarrow\ k \not> l$

(ii) $S \infty^l e\ \&\ S \infty_k e\ \Rightarrow\ k \not> l.$

*Proof.* (i) Let $\mathcal{U} = \{r \in L_0;\ \forall\, U \in \tilde{Q}_r(e), U \,\mathrm{q}\, S$ eventually$\}$ and $\mathcal{V} = \{r \in L_0;\ \exists\ V \in \tilde{Q}_r(e), V \not\mathrm{q}\, S$ frequently$\}$. Then obviously $\mathcal{U} \cap \mathcal{V} = \varnothing$ and $\mathcal{U} \cup \mathcal{V} = L_0$ Also from the definition of upper limit and lower limit we have $l' = \wedge\mathcal{U}$ and $k' = \vee\mathcal{V}$. If

$\wedge\mathcal{U} > \vee\mathcal{V}$ then $\exists\ m \in L_0$ such that $\wedge\mathcal{U} > m > \vee\mathcal{V} \Rightarrow m \notin \mathcal{U}$ & $m \notin \mathcal{V}$, which is a contradiction that $\mathcal{U} \cup \mathcal{V} = L_0$. So, $\wedge\mathcal{U} > \vee\mathcal{V}$ is not possible, *i. e.*, $l' \not> k' \Rightarrow k \not> l$.

(ii) Similar to (i). □

**Proposition 3.3.** *If in addition $L$ is a chain then in the $L$-fuzzy topological space $(X, \tau)$,*

*(i)* $S \to^l e$ *and* $S \to_k e \Rightarrow k = l$

*(ii)* $S \infty^l e$ *and* $S \infty_k e \Rightarrow k = l$.

*Proof.* (i) As in the above Proposition, if we consider the partitions $\mathcal{U}$ and $\mathcal{V}$ of $L_0$, and $l' = \wedge\mathcal{U}, k' = \vee\mathcal{V}$ then we have $k \leq l$. If possible let $k < l$ then $k' > l' \Rightarrow \exists m \in L_0$ such that $k' > m > l' \Rightarrow \vee\mathcal{V} > m > \wedge\mathcal{U} \Rightarrow m \in \mathcal{V}$ and $m \in \mathcal{U}$, which is a contradiction that $\mathcal{U} \cap \mathcal{V} = \varnothing$. Hence $k \not< l$.

(ii) Similar to (i). □

*Remark* 3.4. If $L$ be an order dense chain then in the $L$-fuzzy topological space $(X, \tau)$ then $S\infty^l e$ and $S\infty_l e$ together will be commonly denoted by $S\infty(l)e$ Similarly, $S \to^l e$ and $S \to_l e$ together will be commonly denoted by $S \to (l)e$.

**Proposition 3.5.** *Let $(X, \tau)$ be an $L$-fuzzy topological space $S = \{S(n);\ n \in D\}$ a fuzzy net in $X$ and $e, f \in \mathrm{Pt}(L^X)$. Then*

(i) $S \to^l e \Rightarrow S\infty^k e$ *for some* $k \geq l;\ k, l \in L$.

(ii) $S\infty^l e \geq f \Rightarrow S\infty^k f$ *for some* $k \geq l;\ k, l \in L$.

(iii) $S \to^l e \geq f \Rightarrow S \to^k f$ *for some* $k \geq l;\ k, l \in L$.

(iv) $S\infty_k e \Rightarrow S \to_l e$ *for some* $l \leq k;\ k, l \in L$.

(v) $S\infty_k e \leq f \Rightarrow S\infty_k f$ *for some* $l \leq k$ ; $k, l \in L$.

(vi) $S \to_k e \leq f \Rightarrow S \to_l f$ *for some* $l \leq k;\ k, l \in L$.

The proof is straightforward.

**Definition 3.6** (Liu & Luo [13]). Let $(X, \tau)$ be an $L$-fuzzy topological space and $S : D \to \mathrm{Pt}(L^X), T : E \to \mathrm{Pt}(L^X)$ be two fuzzy nets in $X$. Call $T$ a subnet of $S$ or call $S$ a parental net of $T$ if $\exists$ a mapping $N : E \to D$, called a cofinal selection on $S$, such that (i) $T = S \odot N$ (ii) for every $n_0 \in D\ \exists\ m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

**Proposition 3.7.** *Let $(X, \tau)$ be an $L$-fuzzy topological space, $S$ be a fuzzy net in $X$ and $T$ be a subnet of $S$, $e \in \mathrm{Pt}(L^X)$. Then*

(i) $S \to^l e \;\Rightarrow\; T \to^k e$ *for some* $k \geq l$; $l, k \in L$.
(ii) $T\infty^l e \;\Rightarrow\; S\infty^k e$ *for some* $k \geq l$; $l, k \in L$.
(iii) $T \to_k e \;\Rightarrow\; S \to_l e$ *for some* $l \leq k$; $k, l \in L$.
(iv) $S\infty_k e \;\Rightarrow\; T\infty_l e$ *for some* $l \leq k$ ; $k, l \in L$.

**Proposition 3.8.** *Let* $(X, \tau)$ *be an L-fuzzy topological space with* $\tau$ *as a GO on* $X$, $S$ *be a fuzzy net in* $X$, $\Delta$ *be the collection of all subnets of* $S$, $e \in \mathrm{Pt}(L^X)$. *Then*

(1) $S \to^l e \;\Rightarrow\; l = \wedge_{T\in\Delta}\{r \in L;\ T \to^r e\}$
(2) $S\infty^l e \;\Rightarrow\; l = \vee_{T\in\Delta}\{r \in L;\ T\infty^r e\}$.
(3) $S\infty(l)e \;\Rightarrow\; l = \vee_{T\in\Delta}\{r \in L;\ T \to (r)e\}$ *if* $L$ *is a chain.*
(4) $S\infty(l)e \;\Rightarrow\; \exists$ *a subnet* $T$ *of* $S$ *such that* $T \to (l)e$ *if* $L$ *is a chain.*
(5) $S \to_l e \;\Rightarrow\; l = \wedge_{T\in\Delta}\{r;\ T \to_r e\}$.
(6) $S\infty_l e \;\Rightarrow\; l = \vee_{T\in\Delta}\{r \in L;\ T\infty_r e\}$.

The proof is straightforward.

*Proof.* (1) For any $T \in \Delta, T \to^r e$ and $S \to^l e \;\Rightarrow\; r \geq l$. So,

$$l \leq \bigwedge_{T\in\Delta}\{r \in L;\ T \to^r e\}.$$

Again as a particular case taking $T = S$ we get $l \geq \wedge_{T\in\Delta}\{r \in L;\ T \to^r e\}$.

Hence the proof.

The proof of (2) is similar to that of (1).

(3) Let $T : E \to \mathrm{Pt}(L^X)$ be a subnet of $S$ such that $T \to^r e$ and $N : E \to D$ be the function given in the definition of subnet.

Then for every $s > r'$, $U \in \tilde{Q}_s(e) \Rightarrow U \,\mathrm{q}\, T$ eventually. Let $m_0 \in D$, $s(> r')$ and $U \in \tilde{Q}_s(e)$ be given. Then $\exists\ m_1 \in E$ such that $\forall\ m \in E,\ m \geq m_1 \Rightarrow N(m) \geq m_0$. Also because $T \to^r e\ \exists\ m_2 \in E$ such that $\forall\ m \in E,\ m \geq m_2 \;\Rightarrow\; T(m) \,\mathrm{q}\, U$, *i.e.*, $S(N(m)) \,\mathrm{q}\, U$. Now choose $m \in E$ such that $m \geq m_1$ and $m \geq m_2$ and let $n = N(m)$. Then $n \geq m_0$ and $S(n) \,\mathrm{q}\, U$. As $s(> r')$, $m_0$ and $U$ were arbitrary it follows that $S\infty^l e$ for some $l \geq r$.

Again as $T \in \Delta$ is arbitrary so $S\infty^l e \;\Rightarrow\; l \geq \bigvee_{T\in\Delta}\{r \in L;\ T \to^r e\}$.

Conversely, let $S\infty^l e$ in $(X, \tau)$. We construct a subnet $T$ of $S$ as follows:

Let $E = \{(n, U) \in D \times \bigcup_{m>l'} \tilde{Q}_m(e);\ S(n) \,\mathrm{q}\, U\}$. For $(n, U)$, $(m, V) \in E$ we let $(n, U) \geq (m, V) \iff n \geq m$ in $D$ and $U \leq V$ in $\cup_{m>l'}\tilde{Q}_m(e)$. It is easy to show that the binary relation '$\geq$' directs the set $E$. Now define $T : E \to \mathrm{Pt}(L^X)$ by $T(n, U) = S(n)$ for $(n, U) \in E$. Then $T$ is a fuzzy net in $X$ and actually it is a subnet of $S$, because if we define $N : E \to D$ by $N(n,U) = n$, we see that both the conditions of definition of a subnet are satisfied. It only remains to verify that$T \to^r e$ for some $r \geq l$.

For this let $G \in \tilde{Q}_m(e)$ be given where $m > l'$ is arbitrary. Since $S\infty^l e$ so $S \mathrel{q} G$ frequently. In particular fix any $n \in D$ such that $S(n) \mathrel{q} G$. Then $(n,G) \in E$.
Now for any $(p,U) \in E$ with $(p,U) \geq (n,G)$, $T(p,U) \mathrel{q} U$(since $T(p) = S(p)$)
$\Rightarrow$ $T(p,U) \mathrel{q} G$(since $U \subseteq G$).
Thus $T \to^r e$ for some $r \geq l$. So, $l \leq \vee_{T\in\Delta}\{r \in L;\ T \to^r e\}$.

Hence the proof of (3). The proof of (4)–(6) can be obtained similarly. □

**Proposition 3.9.** *Let $(X,\tau)$ be an L-fuzzy topological space with $\tau$ as a GO on X, $A \in L^X$. Then $\forall\ e \in M(L^X)$, $e \in \mathrm{cl}(A,k' \Rightarrow \exists$ a fuzzy net $S$ in $A$ such that $S \to^l e$ for some $l \geq k$.*

*Proof.* $e \in \mathrm{cl}(A,k') \Rightarrow$ for every $U \in \tilde{Q}_{k'}(e)$, $U \mathrel{q} A$ (by Proposition 2.17).

As $e \in M(L^X)$ so $\tilde{Q}_{k'}(e)$ is a directed set with respect to the relation '$\geq$' defined by

$$U \geq V \iff U \subseteq V \text{ for } U,V \in \tilde{Q}_{k'}(e).$$

So, we define a fuzzy net $S : \tilde{Q}_{k'}(e) \to A$ by $S(U) =$ a fuzzy point having support at where $U \mathrel{q} A$ (if $U \mathrel{q} A$ at many points then take any one among them as a support) and grade equal to the grade of $A$ at this support. Then $S$ is a fuzzy net in $A$ and as $\forall\ U \in \tilde{Q}_{k'}(e)$, $U \mathrel{q} A$ so $\forall\ U \in \tilde{Q}_{k'}(e)$ $U \mathrel{q} S$ eventually, which implies $\wedge\{s \in L_0;\ \forall\ U \in \tilde{Q}_s(e),\ U \mathrel{q} S \text{ eventually}\} \leq k' \Rightarrow\ S \to^l e$ for some $l \geq k$. □

But converse of this proposition has some problem which can be shown by the following example.

*Example* 3.10. Let $X$ be any nonempty set and $L = \mathcal{I} = \{(r,s) \in I \times I;\ r + s \leq 1\}$, the set of all instuitionistic pairs. $A \in L^X$ be defined by $A(x) = (0.6, 0.4)$ for $x \in X$.

We define a mapping

$$\mathcal{F} : L^X \to L \text{ by } \mathcal{F}(\tilde{1}) = \mathcal{F}(\tilde{0}) = (1,0),\ \mathcal{F}(A) = (0.4, 0.6),$$
$$\mathcal{F}(A') = (0.6, 0.4) \text{ and } \mathcal{F}(B) = (0,1)$$

for any other $B \in L^X$. Then $\mathcal{F}$ is a GC on $X$. Let $S : N \to \mathrm{Pt}(L^X)$ be a fuzzy net defined by

$$S(n) = \Big(0.6 - \frac{1}{n+3}, 0.4 + \frac{1}{n+3}\Big)_\xi$$

for some $\xi \in X$, where $N$ is the set of all natural numbers. Now we observe that $S(n) \notin A'$ eventually, so $S \, q \, A$ eventually and $S(n) \in A$ eventually, so $S \not{q} A'$ eventually. Again the fuzzy point $(0.7, 0.3)_\xi \, q \, A$ as well as $(0.7, 0.3)_\xi \, q \, A'$ so

$\wedge\{(r,s) \in \mathcal{I}_0;\ \forall\, U \in \tilde{Q}_{(r,s)}(e),\ S \, q \, U \text{ eventually}\} \leq (0.4, 0.6) \ \Rightarrow\ S \to^{(\alpha,\beta)} (0.7, 0.3)_\xi$

for some $(\alpha, \beta) \geq (0.6, 0.4)$ with respect to the GO $\mathcal{F}_\tau$ but $\mathrm{cl}(A, (0.4, 0.6)) = A \ \Rightarrow\ (0.7, 0.3)_\xi \notin \mathrm{cl}(A, (0.4, 0.6))$ but $(0.7, 0.3)_\xi \in M(L^X)$.

**Proposition 3.11.** *In an L-fuzzy topological space* $(X, \tau)$, $e \notin \mathrm{cl}(A, k') \Rightarrow$ *for any fuzzy net* $S$ *in* $A$ *if* $S \to_l e$ *then* $l \leq k$.

*Proof.* We have, by Corollary 2.15, $e \notin \mathrm{cl}(A, k') \iff \exists$ at least one $U \in \tilde{Q}_{k'}(e)$ such that $U \not{q} A$. So, for any fuzzy net $S$ in $A$, $S \not{q} U$ at all. This means if $S \to_l e$ then $l' \geq k'$ and hence $l \leq k$. □

**Corollary 3.12.** *If* $\exists$ *a fuzzy net* $S$ *in* $A$ *such that* $S \to_l e$ *and* $l > k \in L_0$ *then* $e \in \mathrm{cl}(A, k')$.

**Proposition 3.13.** *Let* $f : (X, \tau) \to (Y, \delta)$ *be a gp map where* $(X, \tau)$ *and* $(Y, \delta)$ *be any two L-fuzzy topological spaces,* $S$ *be any fuzzy net in* $X$*. Then* $S \to^k e$ *in* $(X, \tau)$ *for some* $k \in L \ \Rightarrow\ f \odot S \to^l f(e)$ *in* $(Y, \delta)$ *for some* $l \geq k$.

*Proof.* Let $\tilde{Q}_r(e)$ and $\tilde{Q}''_r(f(e))$ be the q-nbd systems of $e$ and $f(e)$ with respect to the Chang fuzzy topology $\tau_r$ and $\delta_r$ respectively.
As $f$ is a gp-map so $V \in \tilde{Q}''_r(f(e)) \Rightarrow f^{-1}(V) \in \tilde{Q}_r(e) \forall\, r \in L_0$.

Again if $S \, q \, f^{-1}(V)$ eventually then $f(S) \, q \, V$ eventually. From these two facts we can conclude that if $\forall\, U \in \tilde{Q}_r(e)$, $U \, q \, S$ eventually then $\forall\, V \in \tilde{Q}''_r(f(e))$, $V \, q \, f(S)$ eventually, *i. e.*, $S \to^k e \Rightarrow f \odot S \to^l f(e)$ for some $l \geq k$. □

**Proposition 3.14.** *Let* $f : (X, \tau) \to (Y, \delta)$ *be a mapping where* $(X, \tau)$ and $(Y, \delta)$ *be any two L-fuzzy topological spaces. If for any L-fuzzy net* $S$, $S \to^k e \Rightarrow f \odot S \to_l f(e)$ *for some* $l \geq k$ *and for* $e \in M(L^X)$ *then* $f$ *is a gp-map.*

*Proof.* If possible let $f$ be not a gp-map, then $\exists\, V \in L^Y$ such that

$$\tau(f^{-1}(V)) \not\geq \delta(V).$$

Then from the order dense property of $L$ we can get $k_1, k_2 \in L$ such that

$$\tau(f^{-1}(V)) \not\geq k_1 < k_2 < \delta(V).$$

Now $\tau(f^{-1}(V)) \not\geq k_1 \Rightarrow \wedge_{e \, q \, f^{-1}(V)} Q(e, f^{-1}(V)) \not\geq k_1, e \in M(L^X)$

$\Rightarrow\ \exists\, e^0 \in M(L^X)$ such that $e^0\, q\, f^{-1}(V)$ and $Q(e^0, f^{-1}(V)) \not\geq k_1$

$\Rightarrow\ \vee\{\tau(U);\ e^0\, q\, U \subset f^{-1}(V)\} \not\geq k_1$

$\Rightarrow\ \forall\, U \in L^X$ with $\tau(U) \geq k_1$ and $e^0\, q\, U,\ U \not\subseteq f^{-1}(V)$.

If we take $\mathcal{D} = \tilde{Q}_{k_1}(e^0)$ then as $e^0 \in M(L^X)$ so $\mathcal{D}$ is a directed set with respect to the binary relation '$\subseteq$' and $\forall\, U \in \mathcal{D},\ U \not\subseteq f^{-1}(V)$, *i. e.*, $U\, q\{f^{-1}(V)\}'$.

Let us now define a fuzzy net $S : \mathcal{D} \to \mathrm{Pt}(L^X)$ by the following rule: $S(U) =$ the $L$-fuzzy point having the support at where $U\, q\{f^{-1}(V)\}'$ (if more than one such support exist then take any one of them) and grade equal to the grade of $\{f^{-1}(V)\}'$ at this support. Then

$$S(U) \in \{f^{-1}(V)\}'\ \forall U \in \mathcal{D}, \textit{i. e.}, S(U)\, \not{q}\, f^{-1}(V)\ \forall U \in \mathcal{D} \Rightarrow f(S(U))\, \not{q}\, V\ \forall U \in \mathcal{D}.$$

Also $e^0\, q\, f^{-1}(V) \Rightarrow f(e^0)\, q\, V$ where $\delta(V) > k_2$. Therefore, $\delta(V) > k_2$ and $f(e^0)\, q\, V$ but $f(S(U))\, \not{q}\, V$ for $U \in \mathcal{D}$ imply that if $f \odot S \to_l f(e^0)$ then $l' > k_2$. But from the construction of $S$, if $U \in \tilde{Q}_{k_1}(e^0)$ then $\forall V \geq U, S(V)\, q\, U$(since $V \geq U$ means $V \subseteq U$ and from the construction of $S$ we have $S(V)\, q\, V\ \forall\ V \in \mathcal{D}$ so $S(V)\, q\, U$), *i. e.*, $S\, q\, U$ eventually.

Hence if $S \to^k e^0$ then $k' \leq k_1$.

So, $k' \leq k_1 < k_2 < l' \ \Rightarrow\ k' < l' \ \Rightarrow\ k > l$. □

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