J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.
Volume 12, Number 1 (February 2005), Pages 57–63 ISSN 1226-0657

ON A CLASS OF MEROMORPHICALLY P-VALENT STARLIKE FUNCTIONS

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ABSTRACT. Let $\Sigma(p)(p \in N)$ be the class of functions $f(z) = z^{-p} + a_{1-p}z^{1-p} + a_{2-p}z^{2-p} + \cdots$ analytic in $0 < |z| < 1$ and let $M(p, \lambda, \mu)(0 < \lambda \leq 2$ and $2\lambda(\lambda - 1) \leq \mu \leq \lambda^2)$ denote the class of functions $f(z) \in \Sigma(p)$ which satisfy

$$\left(\operatorname{Re}\frac{zf'(z)}{pf(z)}\right)^2 + \mu > \left|\frac{zf'(z)}{pf(z)} + \lambda\right|^2 \quad (|z| < 1).$$

The object of the present paper is to derive some properties of functions in the class $M(p, \lambda, \mu)$.

1. INTRODUCTION

Let S be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the unit disk $E = \{z : |z| < 1\}$. Let S^* and K denote the usual subclasses of S consisting of starlike and convex functions, respectively. And let $UCV(\subset K)$ be the class of functions called uniformly convex and introduced by Goodman [3]. In Ronning [6], Ronning investigated the class S_p defined by

$$S_p = \{f(z) \in S^* : f(z) = zg'(z), g(z) \in UCV\}.$$

It was shown in Ronning [6] and Ma & Minda [4] that a function of the form (1.1) is in S_p if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \left|\frac{zf'(z)}{f(z)} - 1\right| \quad (z \in E).$$

Received by the editors May 26, 2004 and, in revised form, February 16, 2005.
2000 *Mathematics Subject Classification.* 30C45.
Key words and phrases. meromorphic, p-valent starlike, subordination.

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The uniformly convex and related functions have been studied by several authors (see, *e. g.*, Dixit & Misra [1], Ma & Minda [4], Owa [5], Ronning [6] and Srivastava & Mishra [7]).

Let $\sum(p)(p \in N = \{1, 2, 3, \ldots\})$ be the class of functions $f(z)$ of the form

$$f(z) = z^{-p} + \sum_{n=1}^{\infty} a_{n-p} z^{n-p} \tag{1.2}$$

which are analytic in the punctured unit disk $E_0 = \{z : 0 < |z| < 1\}$. A function $f(z) \in \sum(p)$ is said to be meromorphically p-valent starlike of order α if it satisfies

$$-\operatorname{Re} \frac{zf'(z)}{f(z)} > p\alpha \quad (z \in E) \tag{1.3}$$

for some $\alpha(0 \le \alpha < 1)$. We denote by $\sum^*(p, \alpha)(0 \le \alpha < 1)$ the subclass of $\sum(p)$ consisting of functions which are meromorphically p-valent starlike of order α.

In this paper we introduce and investigate the new subclass of $\sum(p)$ as follows.

A function $f(z) \in \sum(p)$ is said to be *in the class* $M(p, \lambda, \mu)$ if it satisfies the condition

$$\left(\operatorname{Re} \frac{zf'(z)}{pf(z)}\right)^2 + \mu > \left|\frac{zf'(z)}{pf(z)} + \lambda\right|^2 \quad (z \in E), \tag{1.4}$$

where λ and μ are real such that

$$0 < \lambda \le 2, \quad 2\lambda(\lambda - 1) \le \mu \le \lambda^2. \tag{1.5}$$

Note that for $\lambda = 1$ and $\mu = 0$,

$$M(p, 1, 0) = \left\{ f(z) \in \Sigma(p) : -\operatorname{Re} \frac{zf'(z)}{pf(z)} > \left|\frac{zf'(z)}{pf(z)} + 1\right| \quad (z \in E) \right\}.$$

Let $f(z)$ and $g(z)$ be analytic in E. Then we say that the function $f(z)$ *is subordinate* to $g(z)$ in E, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in E such that $|w(z)| \le |z|$ and $f(z) = g(w(z))$ for $z \in E$. If $g(z)$ is univalent in E, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

In proving our results, we need the following lemmas.

Lemma 1.1. *Let $f(z)$ be analytic in E with $f(0) = 0$ and let $g(z) \in S^*$. If $f(z) \prec g(z)$, then*

$$\int_0^z \frac{f(t)}{t} dt \prec \int_0^z \frac{g(t)}{t} dt.$$

Lemma 1.2. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \prec g(z)$$

and $g(z) \in K$. *Then* $|a_n| \leq 1 (n \in N)$.

Lemma 1.1 is due to Suffridge [8] and Lemma 1.2 can be found in (*cf.* Duren [2, p. 195]).

2. Properties of the class $M(p, \lambda, \mu)$

Theorem 2.1. *A function* $f(z)$ *in* $\Sigma(p)$ *belongs to* $M(p, \lambda, \mu)$ *if and only if*

$$-\frac{zf'(z)}{pf(z)} \prec h(z), \tag{2.1}$$

where

$$h(z) = \lambda - \frac{\mu}{2\lambda} + \frac{2\lambda}{\pi^2}\left(\log \frac{1+\sqrt{(z+\beta)/(1+\beta z)}}{1-\sqrt{(z+\beta)/(1+\beta z)}}\right)^2, \tag{2.2}$$

$$\beta = \left(\frac{e^b - 1}{e^b + 1}\right)^2 \quad \text{and} \quad b = \frac{\pi}{2\lambda}\sqrt{2\lambda(1-\lambda)+\mu}.$$

Proof. Define the function $w(z) = u + iv$ by

$$w(z) = -\frac{zf'(z)}{pf(z)}.$$

Then the inequality (1.4) can be rewritten as $u^2 + \mu > (\lambda - u)^2 + v^2$, which, in view of $\lambda > 0$, is equivalent to

$$u > \frac{1}{2\lambda}\left(v^2 + \lambda^2 - \mu\right). \tag{2.3}$$

Thus the domain of values of $-zf'(z)/(pf(z)) (z \in E)$ is contained in the region

$$D = \{w = u + iv : u \text{ and } v \text{ satisfy (2.3)}\}.$$

It follows from (2.2) that

$$h(0) = \lambda - \frac{\mu}{2\lambda} + \frac{2\lambda}{\pi^2}\left(\log \frac{1+\sqrt{\beta}}{1-\sqrt{\beta}}\right)^2 = 1.$$

In order to prove our theorem, it suffices to show that the transformation $w = h(z)$ defined by (2.2) maps E conformally onto the region D.

From (1.5) we have

$$0 \leq \frac{\lambda^2 - \mu}{2\lambda} < \min\left\{\lambda - \frac{\mu}{2\lambda}, 1\right\}.$$

Consider the transformations

$$w_1 = \sqrt{w - \left(\lambda - \frac{\mu}{2\lambda}\right)}, \quad w_2 = \exp\left(\pi w_1 \sqrt{\frac{2}{\lambda}}\right), \quad t = \frac{1}{2}\left(w_2 + \frac{1}{w_2}\right).$$

It can be verified that the composite function

$$t = \operatorname{ch}\left(\frac{\pi}{\lambda}\sqrt{2\lambda w - (2\lambda^2 - \mu)}\right) \equiv \varphi(w)(\text{say})$$

maps $D^+ = D \cap \{w = u + iv : v > 0\}$ conformally onto the upper half plane $\operatorname{Im} t > 0$ so that $w = \frac{\lambda^2 - \mu}{2\lambda}$ corresponds to $t = -1$ and $w = \lambda - \frac{\mu}{2\lambda}$ to $t = 1$. With the help of the symmetry principle, the function $t = \varphi(w)$ maps D conformally onto the region $G = \{t : |\arg(t+1)| < \pi\}$.

Since

$$t = 2\left(\frac{1+\zeta}{1-\zeta}\right)^2 - 1$$

maps the unit disk $|\zeta| < 1$ onto G, we see that

$$\begin{aligned} w = \varphi^{-1}(t) &= \lambda - \frac{\mu}{2\lambda} + \frac{\lambda}{2\pi^2}\left(\log(t + \sqrt{t^2 - 1})\right)^2 \\ &= \lambda - \frac{\mu}{2\lambda} + \frac{2\lambda}{\pi^2}\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2 \end{aligned}$$

maps $|\zeta| < 1$ conformally onto D so that $\zeta = \beta \in (-1, 1)$ corresponds to $w = 1$. Now we easily know that the function $w = h(z)$ maps E conformally onto the region D. Hence the proof of the theorem is completed. □

Corollary 2.1. *If $f(z) \in M(p, \lambda, \mu)$, then $f(z) \in \Sigma^*(p, \frac{\lambda^2 - \mu}{2\lambda})$. The result is sharp with the extremal function*

$$f_0(z) = z^{-p}\exp\left(-p\int_0^z \frac{h(t) - 1}{t}dt\right) \in M(p, \lambda, \mu), \tag{2.4}$$

where $h(z)$ is given by (2.2).

Proof. Using the inequality (2.3) in the proof of Theorem 2.1, we have

$$-\operatorname{Re}\frac{zf'(z)}{pf(z)} > \frac{\lambda^2 - \mu}{2\lambda} \geq 0 \quad (z \in E),$$

that is, $f(z)$ is meromorphically p-valent starlike of order $\frac{\lambda^2 - \mu}{2\lambda}$. Noting that

$$-\operatorname{Re}\frac{zf_0'(z)}{pf_0(z)} = \operatorname{Re} h(z) \to \lambda - \frac{\mu}{2\lambda} + \frac{2\lambda}{\pi^2}\left(\log\frac{1+i}{1-i}\right)^2 = \frac{\lambda}{2} - \frac{\mu}{2\lambda}$$

as $z \to -1$, the proof is complete. □

Corollary 2.2. *If $f(z) \in M(p, \lambda, \mu)$ with $0 < \lambda \leq 2$ and $2\lambda(\lambda - 1) \leq \mu < \lambda^2$, then*

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right| < \arctan\left(\frac{\lambda}{\sqrt{\lambda^2 - \mu}}\right) \quad (z \in E). \tag{2.5}$$

The bound in (2.5) *is sharp with the extremal function $f_0(z)$ given by* (2.4).

Proof. Let $h(z)$ be given by (2.2). Then $h(E) = D$ and a straightforward calculation yields

$$\min\{\theta : |\arg h(z)| < \theta \ (z \in E)\} = \arctan\left(\frac{\lambda}{\sqrt{\lambda^2 - \mu}}\right)$$

for $0 < \lambda \leq 2$ and $2\lambda(\lambda - 1) \leq \mu < \lambda^2$. Therefore the corollary follows immediately from Theorem 2.1. □

Theorem 2.2. *Let $f(z) \in M(p, \lambda, \mu)$ and $h(z)$ be defined by* (2.2). *Then*

$$\exp\left(-p\int_0^1 \frac{h(\rho) - 1}{\rho} d\rho\right) < |z^p f(z)| < \exp\left(-p\int_0^1 \frac{h(-\rho) - 1}{\rho} d\rho\right) \tag{2.6}$$

for $z \in E$. The bounds in (2.6) *are sharp with the extremal function $f_0(z)$ given by* (2.4).

Proof. Since the analytic function $h(z) - 1$ is univalent and starlike with respect to the origin, it follows from Theorem 2.1 and Lemma 1.1 that

$$-\int_0^z \left(\frac{f'(t)}{f(t)} + \frac{p}{t}\right) dt \prec p\int_0^z \frac{h(t) - 1}{t} dt,$$

that is,

$$\log(z^p f(z)) \prec -p\int_0^1 \frac{h(\rho z) - 1}{\rho} d\rho. \tag{2.7}$$

Noting that the univalent function $h(z)$ maps the disk $|z| < \rho (0 < \rho \leq 1)$ onto a region which is convex and symmetric with respect to the real axis, we have

$$\int_0^1 \frac{h(-\rho) - 1}{\rho} d\rho < \mathrm{Re}\left(\int_0^1 \frac{h(\rho z) - 1}{\rho} d\rho\right) < \int_0^1 \frac{h(\rho) - 1}{\rho} d\rho \quad (z \in E).$$

Consequently, the subordination (2.7) leads to

$$-p\int_0^1 \frac{h(\rho) - 1}{\rho} d\rho < \log|z^p f(z)| < -p\int_0^1 \frac{h(-\rho) - 1}{\rho} d\rho \quad (z \in E),$$

which implies (2.6). Sharpness can be verified easily. □

Theorem 2.3. *Let $f(z)$ given by* (1.2) *be in the class $M(p, \lambda, \mu)$. Then*

$$|a_{1-p}| \leq \frac{8p\lambda(1 + \beta)}{\pi^2}\left|1 + \sum_{n=1}^{\infty} \frac{\beta^n}{2n + 1}\right|, \tag{2.8}$$

where β is given by (2.2). *The result is sharp.*

Proof. By using the expansion

$$\left(\log\frac{1+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right)^2 = 4\zeta\left(\sum_{n=1}^{\infty}\frac{\zeta^{n-1}}{2n-1}\right)^2 = 4\sum_{n=1}^{\infty}\left(\sum_{m=1}^{n}\frac{1}{2m-1}\right)\frac{\zeta^n}{n} \quad (|\zeta|<1),$$

it follows from (2.2) in Theorem 2.1 that

$$\begin{aligned} h(z) &= h(0) + h'(0)z + \cdots \\ &= 1 + \frac{8\lambda(1-\beta^2)}{\pi^2}\left(1+\sum_{n=1}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{2m+1}\right)\beta^n\right)z + \cdots \\ &= 1 + \frac{8\lambda(1+\beta)}{\pi^2}\left(1+\sum_{n=1}^{\infty}\frac{\beta^n}{2n+1}\right)z + \cdots \end{aligned} \tag{2.9}$$

for $z \in E$. On the other hand, it is easy to see that

$$-\frac{zf'(z)}{pf(z)} = 1 - \frac{a_{1-p}}{p}z + \cdots \tag{2.10}$$

for $f(z) = z^{-p} + a_{1-p}z^{1-p} + \cdots \in M(p, \lambda, \mu)$.

In view of $h(z)$ is analytic, convex and univalent in E, from (2.9), (2.10), Theorem 2.1 and Lemma 1.2, we conclude that

$$\left|-\frac{a_{1-p}}{p}\right| \le \frac{8\lambda(1+\beta)}{\pi^2}\left|1+\sum_{n=1}^{\infty}\frac{\beta^n}{2n+1}\right|.$$

This proves (2.8).

Obviously, the equality in (2.8) is attained for the function $f_0(z)$ given by (2.4). The proof of the theorem is complete. □

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