

A FUNCTIONS AND ITS GRAPH FUCTION

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ABSTRACT. For topological spaces X, Y and the function $f : X \rightarrow Y$, it induces a function $g_r(f) : X \rightarrow X \times Y$ defined as $g_r(f)(x) = (x, f(x))$, for every $x \in X$. It deals with some preliminary investigations relating to the behavior of functions and its graph functions. It has also been found that continuous functions are homotopic if and only if their graph functions are homotopic.

1. INTRODUCTION

In literature, the mutual relationships between functions and their graph functions have appeared here and there in the context of some particular type of functions, *e. g.*, Kohli [6] and Levine [7]. Therefore, it is being attempted to examine closely several variants between functions and their graph functions systematically. A function $f : X \rightarrow Y$ induces a function $g_r(f) : X \rightarrow X \times Y$ defined as $g_r(f)(x) = (x, f(x))$, for every $x \in X$ and is called its *graph function*. We list and investigate some definitions, terms and terminologies used throughout this paper to deal with some properties relating to the behavior of functions and its graph functions.

Throughout this paper X, Y and Z will denote topological spaces. Let $A \subset X$. $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure and interior of A respectively. And $\tau(X)$, $\tau(X, x)$ and $\tau(X, A)$ denote a class of all open sets of X , a class of all open sets of X containing $x \in X$ and a class of all open sets of X containing $A \subset X$ respectively.

Definition 1.1. $f : X \rightarrow Y$ is called *almost continuous* (*cf.* Husain [4]) if for neighborhood of $f(x)$, $\overline{(f^{-1}(V))}$ is a neighborhood of x .

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It is known that $f : X \rightarrow Y$ is almost continuous iff for each $x \in X$ and each $V \in \tau(Y, f(x))$, there is a $U \in \tau(X, x)$ such that $f^{-1}(V)$ is dense in U .

Definition 1.2. $f : X \rightarrow Y$ is called *c-continuous* (cf. Long & Carnahan [10]) if for each point $x \in X$ and each $V \in \tau(Y, f(x))$ such that $Y - V$ is compact, there is a $U \in \tau(X, x)$ such that $f(U) \subset V$.

It is known in Long & Carnahan [10] that $f : X \rightarrow Y$ is *c-continuous* iff if for each $V \in \tau(Y)$ and having compact complement, $f^{-1}(V) \in \tau(X)$.

Definition 1.3. $f : X \rightarrow Y$ is called *s-continuous* (cf. Kohli [6]) if for each point $x \in X$ and each $V \in \tau(Y, f(x))$ such that $Y - V$ is connected, there is a $U \in \tau(X, x)$ such that $f(U) \subset V$.

It is known that $f : X \rightarrow Y$ is *s-continuous* iff for each $V \in \tau(Y)$ and having connected complement, $f^{-1}(V) \in \tau(X)$.

Definition 1.4. $f : X \rightarrow Y$ is called *weakly continuous* (cf. Noiri [11]) if for each point $x \in X$ and each $V \in \tau(Y, f(x))$, there is a $U \in \tau(X, x)$ such that $f(U) \subset \text{Cl}(V)$.

It is known that $f : X \rightarrow Y$ is weakly continuous iff for each

$$V \in \tau(Y), f^{-1}(V) \subset \text{Int}(f^{-1}(\text{Cl}(V))).$$

Definition 1.5. $f : X \rightarrow Y$ is said to be *irresolute* (cf. Crossley & Hilderbrand [2]) if for any semi-open set S of Y , $f^{-1}(S)$ is semi-open in X . It may be noted that a set A is said to be *semi-open* in Levine [7] if there is an $O \in \tau(X)$ such that $O \subset A \subset \text{Cl}(O)$.

It is easy to prove that $f : X \rightarrow Y$ is irresolute iff for each $x \in X$ and each $V \in \text{SO}(Y, f(x))$, there is a $U \in \text{SO}(X, x)$ such that $f(U) \subset V$.

2. GRAPH FUNCTIONS

We begin with defining closeness of graphs of functions and their graph functions, and then investigate properties of functions that are transferred to their graph functions and vice-versa.

Let $f : X \rightarrow Y$ be a given function. Then $G_f = \{(x, f(x)) : x \in X\}$ is called the *graph of f* . f is said to have *closed graph* if G_f is closed in the product space $X \times Y$. Equivalently f is said to have closed graph (cf. Long [8]) if for each pair $y \neq f(x)$, there exist a $U \in \tau(X, x)$ and a $V \in \tau(Y, y)$ such that $f(U) \cap V = \emptyset$.

Likewise, for a function $f : X \rightarrow Y$ we define that its corresponding graph function $g_r(f) : X \rightarrow X \times Y$ has closed graph.

Definition 2.1. The graph function $g_r(f) : X \rightarrow X \times Y$ is said to have *closed graph* if for each pair $(x, y) \neq g_r(f)(t)$, there exist a $U_t \in \tau(X, t)$ and a $U_x \in \tau(X, x)$ respectively and a $V_y \in \tau(Y, y)$ such that

$$g_r(f)[U_t] \cap (U_x \times V_y) = \emptyset$$

where

$$g_r(f)[U_t] = \{(u, f(u)) : u \in U_t\}.$$

Proposition 2.1. $f : X \rightarrow Y$ is continuous if and only if its graph function $g_r(f) : X \rightarrow X \times Y$ is continuous.

Proof. The proofs are obtained immediately from Dugundji [3, Theorem 2.2], because $P_Y \circ g_r(f) = f$ where projection $P_Y : X \times Y \rightarrow Y$ is continuous. \square

In Husain [4] it has been found that composition of almost continuous mappings need not be almost continuous (*cf.* Husain [5, Proposition 6]). For our purpose, in the positive direction, we state the following results, which is proved in Noiri [12].

Proposition 2.2. Let $f : X \rightarrow Y$ be almost continuous mapping and $g : Y \rightarrow Z$ be continuous. Then $g \circ f : X \rightarrow Z$ is almost continuous.

From Long [9, Theorem 2] which is proved by Proposition 2.2, we know that $f : X \rightarrow Y$ be almost continuous if and only if its graph function $g_r(f) : X \rightarrow X \times Y$ is almost continuous. And from Noiri [11, Theorem 1], we know that $f : X \rightarrow Y$ is weakly continuous if and only if $g_r(f) : X \rightarrow X \times Y$ is weakly continuous. Similar investigations led the referee in a paper of Kohli [6] pose a question concerning the reverse implication of his Theorem 2.7 stating that if $f : X \rightarrow Y$ is a function from a connected space X into a space Y such that graph function is s -continuous, then f is s -continuous. Similar is the cases with c -continuous and irresolute mappings. We could derive implication in one way only. The reverse implications in these cases are open questions.

Proposition 2.3. If $g_r(f) : X \rightarrow X \times Y$ is irresolute, then $f : X \rightarrow Y$ is irresolute.

Proof. Since the projection $P_Y : X \times Y \rightarrow Y$ is continuous and open, and is thus irresolute from Crossley & Hilderbbrand [2, Theorem 1.2], $P_Y \circ g_r(f) = f$ is irresolute from Crossley & Hilderbbrand [2, Theorem 1.7]. \square

Proposition 2.4. *Let $f : X \rightarrow Y$ be c -continuous. Then $g_r(f) : X \rightarrow X \times Y$ is c -continuous.*

Proof. Let $U \times V \in \tau(X \times Y)$ and let $K = (X \times Y) - (U \times V)$ be compact. Then $X \times (Y - V)$ is compact because $X \times (Y - V)$ is a closed subset of compact set

$$K = [(X - U) \times Y] \cup [X \times (Y - V)].$$

Hence

$$P_Y(X \times (Y - V)) = Y - V$$

is compact. Since f is c -continuous, $f^{-1}(V) \in \tau(X)$. Therefore,

$$[g_r(f)]^{-1}(U \times V) = U \cap f^{-1}(V) \in \tau(X).$$

Thus $g_r(f)$ is c -continuous. □

Converse implication of the above proposition could not be settled. However, it holds whenever X is assumed to be compact. For our purpose we state the following results which is proved in Long & Carnahan [10].

Proposition 2.5. *Let X be a compact space. Then $g_r(f) : X \rightarrow X \times Y$ is c -continuous if $f : X \rightarrow Y$ is c -continuous.*

Proposition 2.6. *Let the graph function $g_r(f) : X \rightarrow X \times Y$ have closed graph. Then the graph of $f : X \rightarrow Y$ is closed.*

Proof. Let $y \neq f(x)$. Then $(x, y) \notin g_r(f)(x)$. This means $(x, y) \neq (x, f(x))$. Thus $(x, y) \neq g_r(f)(x)$. Since $g_r(f)$ has closed graph, there exist a $U \in \tau(X, x)$ and a $V \in \tau(Y, y)$ such that $g_r(f)(U) \cap (U \times V) = \emptyset$. So we have

$$f(U) \cap V = \emptyset.$$

Therefore, f has closed graph. □

We disprove its converse by illustrating an example.

Example 2.1. Let X be an infinite set equipped with cofinite topology and $f : X \rightarrow X$ be the constant mapping defined as $f(x) = y$ for every $x \in X$, where y is a fixed element of X . Then f has closed graph from Chae, Singh & Misra [1, Proposition 3.2], but the graph of $g_r(f)$ is not closed. To this end, let $(x, y) \neq g_r(f)(y)$ for every $x \in X$, then $x \neq y$. Obviously for each $U \in \tau(X, x)$ and each $V \in \tau(X, y)$, we have

$$g_r(f)(V) \cap (U \times V) = (V \times \{y\}) \cap (U \times V) = (U \cap V) \times \{y\}.$$

Since topology on X is cofinite, $U \cap V \neq \emptyset$. So

$$g_r(f)(V) \cap (U \times V) \neq \emptyset.$$

Hence the graph of $g_r(f)$ is not closed.

In the context of this example, the following result may be stated.

Proposition 2.7. *Let X be a T_2 -space and $f : X \rightarrow Y$ be a function with closed graph. Then*

$$g_r(f) : X \rightarrow X \times Y$$

has closed graph.

Proof. We prove it analytically by making just two cases that suffice our purpose.

[Case 1] Let $t \neq x$ for any pair $(x, y) \neq g_r(f)(t)$. Then $(x, y) \neq (t, f(t))$. Since X is a T_2 -space, there exists $U \in \tau(X, x)$ and $U^* \in \tau(X, t)$ such that $U \cap U^* = \emptyset$. This implies that

$$g_r(f)(U^*) \cap (U \times V) = \emptyset$$

for any arbitrary $V \in \tau(Y, y)$.

[Case 2] Let $(x, y) \neq g_r(f)(x)$. Then $(x, y) \neq (x, f(x))$ and so $y \neq f(x)$. Since f has closed graph, there exist an $O \in \tau(X, x)$ and a $W \in \tau(Y, y)$ such that $f(O) \cap W = \emptyset$. Hence

$$g_r(f)(O) \cap (O \times W) = \emptyset.$$

Thus $g_r(f)$ has closed graph. □

Combining the above Proposition 2.6 and Proposition 2.7, we have the following result.

Proposition 2.8. *Let X be a T_2 -space. Then $f : X \rightarrow Y$ has closed graph if and only if $g_r(f)$ has closed graph.*

Proposition 2.9. *Let $g_r(f) : X \rightarrow X \times Y$ be a closed function, then $f : X \rightarrow Y$ is a closed function.*

Proof. Let F be any closed subset of X . To show $f(F) = \overline{f(F)}$, let $y \notin f(F)$. Then

$$(F \times \{y\}) \cap g_r(f)(F) = \emptyset.$$

So we obtain

$$[g_r(f)]^{-1}(F \times \{y\}) \subset X - F. \tag{a}$$

Since $g_r(f)$ is a closed function, $g_r(f)(F) = \{(x, f(x)) : x \in F\}$ is closed in the product space $X \times Y$. This means $g_r(f)$ has closed graph on F . Since $y \in f(F)$ implies $y \neq f(x)$ for every $x \in F$, there exist $U_x \in \tau(X, x)$ and $V \in \tau(Y, y)$ such that $g_r(f)(U_x) \cap V = \emptyset$. Putting

$$U = \bigcup_{x \in F} U_x, U \in \tau(X, F).$$

So we obtain

$$U \times V \in \tau(X \times Y, F \times \{y\}). \quad (\text{b})$$

Thus from (a), (b) and Crossley & Hilderbrand [2, Theorem 11.2], there exists a

$$U \times V \in \tau(X \times Y, F \times \{y\})$$

such that

$$[g_r(f)]^{-1}(F \times \{y\}) \subset [g_r(f)]^{-1}(U \times V) \subset X - F.$$

Thus

$$[g_r(f)](F) \cap (U \times V) = \emptyset$$

and so

$$V \cap f(F) = \emptyset.$$

Hence $y \notin \overline{f(F)}$ because of $V \in \tau(X \times Y, y)$ and so $f(F) = \overline{f(F)}$. Therefore, f is a closed function. \square

It is very easy to check that if $g_r(f) : X \rightarrow X \times Y$ is open, then $f : X \rightarrow Y$ is also open which follows directly from $f = P_Y \circ g_r(f)$, indicating that f is the composition of open mappings and hence open. Therefore, we obtain the following obvious conclusion, which is stated for the sake of completeness.

Proposition 2.10. *If $g_r(f) : X \rightarrow X \times Y$ is open(closed), then $f : X \rightarrow Y$ is open(closed).*

3. GRAPHICAL HOMOTOPY

The theory of homotopy plays a vital role in the study of algebraic topology, as you know. This theory is based on continuous deformation of continuous mappings. The concept of being homotopic has been introduced through continuous deformation of their graph functions in this section. After having formalized this notion, we

have examined its relationship with usual known concept of homotopy for the class $C(X, Y)$ of continuous mappings from topological space X into Y .

It is known in Dugundji [3] that for $f, f^* \in C(X, Y)$, f is said to be *homotopic* to f^* if there is a continuous map $F : X \times I \rightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f^*(x)$, for every $x \in X$, where $I = [0, 1]$ denotes the parameter space. We say that the function F is a homotopy between f and f^* . Likewise we define a kind of homopoty between graph functions and then investigate their properties.

Definition 3.1. Let $f, f^* \in C(X, Y)$. Then f is said to be *graphically homotopic* to f^* if there exists a continuous mapping $F : X \times I \rightarrow X \times Y$ such that $F(x, 0) = g_r(f)(x)$ for every $x \in X$ and $F(x, 1) = g_r(f^*)(x)$ for every $x \in X$ where $I = [0, 1]$ denotes the parameter space. Equivalently, f is to be *graphically homotopic* to f^* if and only if $g_r(f)$ is homotopic to $g_r(f^*)$. We say the function F is a graphical homotopy between $g_r(f)$ and $g_r(f^*)$.

Example 3.1. Let $f, f^* \in C(R, R)$. Define $F : R \times I \rightarrow R \times R$ as $F(x, t) = (1 - t)(x, f(x)) + t(x, f^*(x))$, i. e., $F(x, t) = (1 - t) g_r(f)(x) + t g_r(f^*)(x)$ for every $x \in X$ and $t \in [0, 1]$. Then $g_r(f)$ and $g_r(f^*)$ are continuous from Proposition 2.1 and their linear combination must also be continuous. Thus $F(x, t)$ is continuous. Moreover, we have $F(x, 0) = g_r(f)(x)$, for every $x \in X$ and $F(x, 1) = g_r(f^*)(x)$, for every $x \in X$. Hence F is the required graphical homotopy between $g_r(f)$ and $g_r(f^*)$.

Here arises a question as to what relationship between homotopy and graphical homotopy of two continuous functions exists.

Proposition 3.1. Let $f, f^* \in C(X, Y)$. Then f are *graphically homotopic* to f^* if and only if f is *homotopic* to f^* .

Proof. Assume that f is graphically homotopic to f^* . Then there exists a homotopy $F : X \times I \rightarrow X \times Y$ between $g_r(f)$ and $g_r(f^*)$. From a composition $X \times I \xrightarrow{F} X \times Y \xrightarrow{P_Y} Y$, let $H = P_Y \circ F$. Then H is continuous, for the homotopy F and P_Y are continuous. Moreover $H(x, 0) = P_Y(F(x, 0)) = P_Y(g_r(f)(x)) = f(x)$ for every $x \in X$. Similarly, we have $H(x, 1) = f^*(x)$ for every $x \in X$. So f is homotopic to f^* .

Conversely, let f be homotopic to f^* . Then there exists a continuous function $H : X \times I \rightarrow Y$ such that $H(x, 0) = f(x)$ for every $x \in X$ and $H(x, 1) = f^*(x)$ for

each $x \in X$. Define $F : X \times I \rightarrow X \times Y$ as $F(x, t) = (x, H(x, t))$ for each $x \in X$ and each $t \in I$. Then we have to show that F is a graphical homotopy between $g_r(F)$ and $g_r(f^*)$. To prove that F is continuous, let $U \times V \in \tau(X \times Y, (x, H(x, t)))$. Then for the $V \in \tau(Y)$, there exist a $W \in \tau(X, x)$ and a $O \in \tau(I, t)$ such that $H(W \times O) \subset V$, because H is continuous. Putting $N = U \cap W$, $N \in \tau(X, x)$ and so $(N \times O) \in \tau(X \times I, (x, t))$. Hence there is a $(N \times O) \in \tau(X \times I)$ such that $F(N \times O) \subset U \times V$.

Thus F is continuous. Moreover, $F(x, 0) = (x, H(x, 0)) = (x, f(x)) = g_r(f)(x)$ for every $x \in X$. Similarly we have $F(x, 1) = g_r(f^*)(x)$ for every $x \in X$. Therefore, f is graphically homotopic to f^* . \square

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