*Journal of the Korean Statistical Society (2005), **34**: 1, pp 61–71*

BINARY RANDOM POWER APPROACH TO MODELING ASYMMETRIC CONDITIONAL HETEROSCEDASTICITY

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ABSTRACT

A class of asymmetric ARCH processes is proposed via binary random power transformations. This class accommodates traditional nonlinear models such as threshold ARCH (Rabemanjara and Zacoian (1993)) and Box-Cox type ARCH models(Higgins and Bera (1992)). Stationarity condition of the model is addressed. Iterative least squares(ILS) and pseudo maximum likelihood(PML) methods are discussed for estimating parameters and related algorithms are presented. Illustrative analysis for Korea Stock Prices Index (KOSPI) data is conducted.

AMS 2000 subject classifications. Primary 62M10.
Keywords. Asymmetric ARCH, binary random power, iterative least squares(ILS), KOSPI data, pseudo maximum likelihood(PML).

1. INTRODUCTION

It has been an usual practice in conditional heteroscedastic autoregressive context that the conditional variance(h_t, hereafter) exhibits typical symmetry as a linear combination of the squared residuals($cf.$ Engle (1982)). Recently there has been strong evidence particularly in financial time series that the conditional variance may be non-symmetric. See for instance Li and Li (1996) and references therein. In econometric time series, Rabemanjara and Zacoian (1993) proposed threshold-asymmetric ARCH for $\{Z_t\}$, so called TARCH model, given by

$$Z_t = \sqrt{h_t} \cdot e_t,$$

$$\sqrt{h_t} = \alpha_0 + \alpha_{11} Z_{t-1}^{+} + \alpha_{12} Z_{t-1}^{-}, \qquad (1.1)$$

Received August 2004; accepted October 2004.
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where and throughout the paper the notation

$$a_t^+ = max(a_t, 0) \ \ and \ \ a_t^- = max(-a_t, 0),$$

is used and $\{e_t\}$ is a sequence of *i.i.d.* random variables with mean zero and variance unity. Li and Li (1996) studied the following threshold asymmetric process which shares much in common with (1.1), with successful applications to Hong Kong Hang Seng Index(HSI);

$$h_t = \alpha_0 + \alpha_{11}(Z_{t-1}^+)^2 + \alpha_{12}(Z_{t-1}^-)^2. \tag{1.2}$$

Higgins and Bera (1992) introduced a class of power transformed ARCH processes which is appropriate for improving forecast accuracy and for achieving normality or symmetry of the density $f(\cdot)$ of e_t. The first order model is specified as

$$h_t^r = \alpha_0 + \alpha_1(Z_{t-1}^2)^r, \tag{1.3}$$

where the power r is positive and note that $r = 1$ gives standard Engle's ARCH(1). The conditional variance h_t is concave in Z_{t-1}^2 for $0 < r < 1$ and is convex for $r > 1$. Refer to Higgins and Bera (1992) for excellent remarks on an interpretation of power transformations in the context of econometrics. In this article we propose a new class of models possessing threshold-asymmetric conditional variances to which distinct power transformation parameters are applied according to the signs of Z_{t-1}, and therefore generalizing aforementioned ARCH type processes. The developed model is given by

$$\begin{aligned} h_t^{r_1} &= \alpha_0 + \alpha_{11}(Z_{t-1}^2)^{r_1} \ \ when \ \ Z_{t-1} \geq 0 \\ h_t^{r_2} &= \alpha_0 + \alpha_{12}(Z_{t-1}^2)^{r_2} \ \ when \ \ Z_{t-1} < 0, \end{aligned} \tag{1.4}$$

where $\alpha_0 > 0$; $\alpha_{11}, \alpha_{12} \geq 0$; $r_1, r_2 > 0$. Notice that the conditional variance h_t is continuous on the whole real line of Z_{t-1}. The main goal of the paper is to motivate the model to be an applicable class in asymmetric ARCH context and therefore we are mainly concerned with exploratory and practical modeling steps such as how to numerically estimate model parameters and in turn pursuing comparative study of (1.4) with traditional models in the literature. The paper is organized as follows. Section 2 briefly addresses the stationarity of the models and a set of conditions for the existence of stationary moments is presented. Parameter estimation is discussed in Section 3 where both the iterative least squares(ILS) and pseudo maximum likelihood(PML) algorithms are given. Section 4 reports a comparative data analysis for Korean Stock Prices Index(KOSPI), from which it is observed that (1.4) is better than traditional models.

2. THE PROPOSED MODEL

Define Z_{t-1} measurable (binary) random quantities

$$R_{t-1} = r_1 I[Z_{t-1} \geq 0] + r_2 I[Z_{t-1} < 0]$$

$$\alpha_{t-1} = \alpha_{11} I[Z_{t-1} \geq 0] + \alpha_{12} I[Z_{t-1} < 0].$$

Here $I[\cdot]$ stands for the standard indicator function. The proposed model in (1.4) can then be written in a compact way

$$Z_t = \sqrt{h_t} \cdot e_t, \quad h_t^{R_{t-1}} = \alpha_0 + \alpha_{t-1} \cdot (Z_{t-1}^2)^{R_{t-1}}. \tag{2.1}$$

Notice that (2.1) includes as special cases (1.1) and (1.2) according to $r_1 = r_2 = 1/2$ and $r_1 = r_2 = 1$, respectively. Define

$$r = max(r_1, r_2)$$

and consider the following subset region of parameter space, given by

$$\alpha_{11} E(e_t^+)^{2r} + \alpha_{12} E(e_t^-)^{2r} < 1. \tag{2.2}$$

(C1) . The common probability distribution of $\{e_t\}$ is absolutely continuous with respect to Lebesgue measure and has a support on the whole real line$(-\infty, \infty)$.

THEOREM 1. *Assume (C1) and (2.2). Then $\{Z_t\}$ defined in (2.1) satisfies the following.*

(1) $\{Z_t\}$ is geometrically ergodic and strictly stationary.

(2) $E|Z_t|^{2r}$ is finite where the expectation is taken under the stationary distribution.

REMARKS . The results (2) concerning finiteness of moments up to certain order will be needed in applying CLT and the ergodic theorem to various statistics. When $r_1 = r_2 = r$, (2.2) reduces to those in Hwang and Basawa (2004). In particular when $r_1 = r_2 = r$ and $\alpha_{11} = \alpha_{12} = \alpha_1$, the model (2.1) reduces to (1.3) and the stationarity condition (2.2) is identical with those in Higgins and Bera (1992).

PROOF. To provide outlines of the proof, note first that $\{Z_t\}$ in (2.1) is a Markov process and thus we may establish the geometric ergodicity and strict stationarity of $\{Z_t\}$ by employing various sets of conditions for Markovian time series. Refer to, for instance, Feigin and Tweedie (1985). According to Tweedie (1975), it suffices to verify three conditions for the geometric ergodicity of $\{Z_t\}$

(i) $\{Z_t\}$ is ϕ-irreducible for a σ-finite measure ϕ.

(ii) The transition probability is strongly continuous.

(iii) There exist constants b and c such that, for all $z > b$, $r_z \leq -c$ and for all z with $z \leq b$, r_z is bounded above where

$$r_z = E(Z_t \mid Z_{t-1} = z) - z. \tag{2.3}$$

The verification of (i), (ii) and (iii) are straightforward but rather tedious. Following the lines as in Hwang and Kim (2004), under the condition (C1) and (2.2), one can deduce (1). For details together with the proof for (2) regarding the existence of stationary moments, refer, for instance, to arguments as in Hwang and Basawa (2004). Details are omitted. □

3. NONLINEAR AUTO-REGRESSION AND PARAMETER ESTIMATION

It is well known that the conditional(on the data at hand) mean serves as MMSE(minimum mean squared error) point forecast for the time series and 95% forecast interval is of the form

$$\textit{conditional mean} \pm 1.96 \cdot \sqrt{h_t},$$

where 95% level coefficient 1.96 may be replaced by more conservative value, for instance, 2. We are now willing to incorporate nonlinear conditional mean function μ_{t-1}, say, into (2.1). Consider the observable time series $\{X_t\}$ generated by unobservable innovations $\{Z_t\}$ specified in (2.1). Formulate

$$X_t - \mu_{t-1}(\theta) = Z_t$$

$$Z_t = \sqrt{h_t} \cdot e_t, \quad h_t^{R_{t-1}} = \alpha_0 + \alpha_{t-1} \cdot (Z_{t-1}^2)^{R_{t-1}}, \tag{3.1}$$

where $\mu_{t-1}(\theta)$ denotes the conditional mean of X_t given the past observations. Of course the conditional variance of X_t given the past observations is denoted by h_t. With appropriate choices of $\mu_{t-1}(\theta)$, (3.1) accommodates diverse nonlinear autoregressive(AR) processes such as threshold-AR and exponential-AR models(*cf*. Tong (1990)). Note that the parameter (vector) θ is indexing the conditional mean. Collect parameters appearing in h_t as

$$\beta = (\alpha_0, \alpha_{11}, \alpha_{12}, r_1, r_2).$$

In order to emphasize the dependency of h_t both on θ and β, we will use $h_t(\theta, \beta)$ in stead of h_t. Denoting $\phi = (\theta, \beta)$ with the dimension k, we proceeds to discuss estimation methods which do not require the specification of the distribution $f(\cdot)$ of e_t. Let $X_1, X_2, \cdots, X_n$ be given data. First, taking into account heteroscedasticity, the weighted least squares estimator of θ and β is obtained by minimizing with respect to $\phi = (\theta, \beta)$

$$\sum_{t=1}^{n} [X_t - \mu_{t-1}(\theta)]^2 / h_t(\theta, \beta). \tag{3.2}$$

As indicated by Li and Li (1996), optimizing (3.2) entails complications in that parameter θ appears simultaneously in the mean function $\mu_{t-1}(\theta)$ and the variance function $h_t(\theta, \beta)$. Moreover for the model (3.1) with a complex nonlinear $\mu_{t-1}(\theta)$ and the random power R_{t-1} the problem would become much more complicated. To circumvent, ILS(iterative least squares, *cf*. Li and Li (1996) ; Hwang and Basawa (2003)) method can be adopted. ILS consists of the following steps.

(ILS-1) Minimize $\Sigma(X_t - \mu_{t-1}(\theta))^2$ with respect to θ to obtain $\tilde{\theta}$.
(ILS-2) Minimize $\Sigma(X_t - \mu_{t-1}(\tilde{\theta}))^2 / h_t(\tilde{\theta}, \beta)$ with respect to β to obtain $\tilde{\beta}$.
(ILS-3) Minimize $\Sigma(X_t - \mu_{t-1}(\tilde{\theta}))^2 / h_t(\tilde{\theta}, \tilde{\beta})$ with respect to θ to obtain $\hat{\theta}$.
(ILS-4) Go to (ILS-2) with $\tilde{\theta}$ replaced by $\hat{\theta}$. Repeat (ILS-2) to (ILS-4) until obtaining satisfactory convergence between consecutive iterations.

The resulting estimators from the steps above are called ILS-estimators for θ and β. Under some regularity conditions(see for instance Hwang and Basawa (2003)), ILS estimators can be shown to be asymptotically equivalent to the weighted least squares estimators and therefore they are shown to be asymptotically normally distributed. We now present

PROPOSITION 1. As the sample size n goes to infinity, ILS estimator $\widehat{\phi_{ILS}} = (\widehat{\theta_{ILS}}, \widehat{\beta_{ILS}})$ for $\phi = (\theta, \beta)$ satisfies the following.

$$\sqrt{n}(\widehat{\phi_{ILS}} - \phi) \xrightarrow{d} N(0, A^{-1} B A^{-1}),$$

where

$$A = plim[n^{-1} \cdot \sum_{t=1}^{n} \partial^2 \psi_t(\phi)/\partial\phi^2] \ : k \times k \ matrix,$$

and

$$B = E[(\frac{\partial\psi_t(\phi)}{\partial\phi})(\frac{\partial\psi_t(\phi)}{\partial\phi})^T] \ : k \times k \ matrix,$$

with

$$\psi_t(\phi) = [X_t - \mu_{t-1}(\theta)]^2/h_t(\theta,\beta).$$

Second, we are interested in discussing pseudo-maximum likelihood estimation of parameters. Assume for the moment that $\{e_t\}$ is Gaussian. The log-likelihood $l_n(\phi)$ (conditional on the initial value) of the data $X_1, X_2, \cdots, X_n$ is given by, apart from constants,

$$l_n(\phi) = \sum_{t=1}^{n} l_t(\phi), \tag{3.3}$$

where

$$l_t(\phi) = -n \log h_t(\phi)/2 \ - \ Z_t(\theta)^2/2h_t(\phi), \tag{3.4}$$

with

$$Z_t(\theta) = X_t - \mu_{t-1}(\theta).$$

The value $\widehat{\phi_{PML}}$ obtained by maximizing $l_n(\phi)$ is referred to as pseudo-maximum likelihood estimator. Properties of $\widehat{\phi_{PML}}$ depends upon the true distribution $f(\cdot)$ of e_t. Denote first order partial derivative(*w.r.t.*ϕ) $k \times 1$ vector and second order partial derivatives matrix : $k \times k$ matrix, by $D[\cdot]$ and $D^2[\cdot]$ respectively. Define two matrices

$$I_f(\phi) = -E_f(D^2[l_t(\phi)]) \text{ and } J_f(\phi) = E_f(D[l_t(\phi)] \cdot D[l_t(\phi)]^T), \tag{3.5}$$

where $E_f(\cdot)$ indicates expectation under the true distribution $f(\cdot)$ of e_t. Under regularity conditions, it can be shown that $\widehat{\phi_{PML}}$ is asymptotically normal. Refer to Gouerioux (1997, ch.4) for details.

PROPOSITION 2. Under the true distribution $f(\cdot)$ of e_t, we have

$$\sqrt{n}(\widehat{\phi_{PML}} - \phi) \xrightarrow{d} N(0, I^{-1}JI^{-1}).$$

In particular when $f(\cdot)$ is normal $I = J$ holds and thus

$$\sqrt{n}(\widehat{\phi_{PML}} - \phi) \xrightarrow{d} N(0, J^{-1}).$$

One-step scoring algorithm(*cf*. Engle (1982) in ARCH context) can be used for obtaining pseudo MLE $\widehat{\phi_{PML}}$.

(PML-1) Take initial value $\phi(0)$ for ϕ.
(PML-2) Update estimate via

$$\phi(1) = \phi(0) + [F_n(\phi(0))]^{-1} G_n(\phi(0)) \tag{3.6}$$

where

$$F_n(\phi) = \sum_{t=1}^{n} D^2[l_t(\phi)] \quad : k \times k \ matrix,$$

and

$$G_n(\phi) = \sum_{t=1}^{n} D[l_t(\phi)] \quad : k \times 1 \ vector.$$

(PML-3) Repeat (PML-2) with $\phi(1)$ replacing $\phi(0)$ on RHS of (3.6) until convergence.

Then, the resulting value $\phi(1)$ becomes $\widehat{\phi_{PML}}$. It is recommended that the initial value $\phi(0)$ be $\sqrt{n}$ - consistent estimate of ϕ for facilitating the desired convergence in (PML-3). One can choose $\phi(0) = \widehat{\phi_{ILS}}$ in view of Proposition 1.

4. EMPIRICAL STUDY

This section is concerned with an application of the proposed model (3.1) to Korean Stock Prices Index(KOSPI) data. The size of the daily data is 1096 ranging from January, 3, 1995 to December, 28, 2002. First, we transform the data using the log differences of the successive data to ensure the stationarity. Let r_t be original current index of the stock market at time t and then we denote

$$X_t = 100 \times \left\{ \log \frac{r_t}{r_{t-1}} - \frac{1}{n} \sum_{t=1}^{n} \log \frac{r_t}{r_{t-1}} \right\}.$$

The transformed X_t is usually referred as the return rate. The original data r_t and the transformed data X_t are depicted in Figures 1 and 2, respectively.

It is reasonable to say that diverse mean functions $\mu_{t-1}(\theta)$ may be consistent with the data and the final choice of $\mu_{t-1}(\theta)$ relies heavily on subject considerations. In order to employ a threshold asymmetry in the mean function, we postuate

$$\mu_{t-1}(\theta) = \theta_1 X_{t-1}^{+} + \theta_2 X_{t-1}^{-}. \tag{4.1}$$

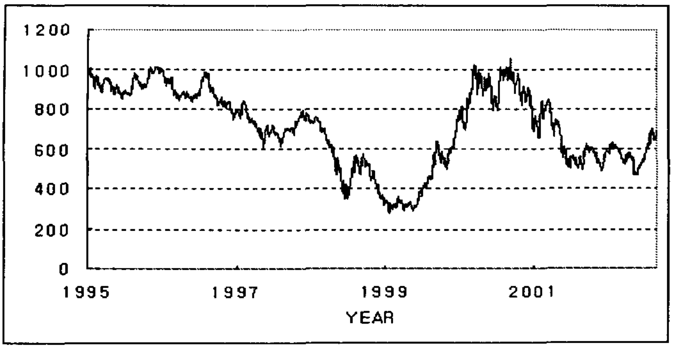

FIGURE 4.1 *Korean Stock Price Indices(KOSPI) data*

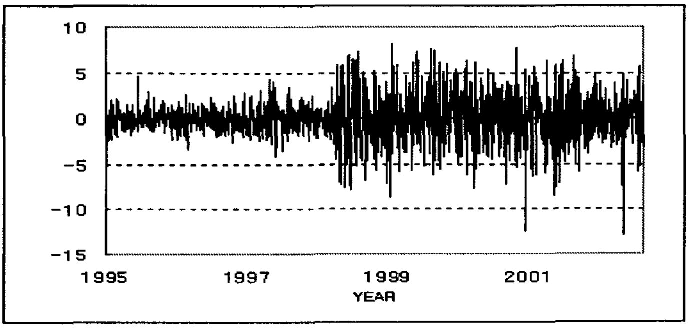

FIGURE 4.2 *Transformed data* (X_t)

For modeling heteroscedasticity, standard Engle's ARCH is denoted by [M1]. TARCH model in (1.1) is employed as [M2]. Li and Li (1996) and references therein argued that the conditional mean and conditional variance in financial time series are probably asymmetric and thus they have chosen threshold models simultaneously for $\mu_{t-1}(\theta)$ and h_t in order to analyze Hong Kong Hang Seng Index(HSI) data. Their model is referred to as [M3] below. These models however assume intrinsically symmetric conditional variance h_t. Our model in (3.1) admits asymmetric power transformation on the conditional variance both for improving forecast accuracy and for achieving normality and/or symmetry of the density $f(\cdot)$ of e_t. The sample period for daily KOSPI data includes one bear market period(Aug. 97- May 98) and one bull market period(Mar. 99 - Dec. 99). Bear market period is due to IMF(International Monetary Fund) crisis in Korea.

TABLE 4.1 *Estimates for the parameters*

Model	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{r}_1$	$\hat{r}_2$
M1	0.192	−0.195	2.180	0.199			
M2	0.195	−0.196	1.911	0.235	0.265		
M3	0.139	−0.242	2.280	0.223	0.309		
M	0.194	−0.198	2.037	0.203	0.331	0.967	0.980

Accordingly threshold-asymmetries both in conditional mean and in conditional variance may be expected to be prominent in the data. The proposed model (3.1) is abbreviated as [M].

[M]

$$X_t - \theta_1 X_{t-1}^+ - \theta_2 X_{t-1}^- = Z_t$$

$$Z_t = \sqrt{h_t} \cdot e_t, \quad h_t^{R_{t-1}} = \alpha_0 + \alpha_{t-1} \cdot (Z_{t-1}^2)^{R_{t-1}}$$

Three competing models explained above are given as follows.

[M1]

$$X_t - \theta_1 X_{t-1}^+ - \theta_2 X_{t-1}^- = Z_t$$

$$Z_t = \sqrt{h_t} \cdot e_t, \quad h_t = \alpha_0 + \alpha_1 Z_{t-1}^2$$

[M2]

$$X_t - \theta_1 X_{t-1}^+ - \theta_2 X_{t-1}^- = Z_t$$

$$Z_t = \sqrt{h_t} \cdot e_t, \quad \sqrt{h_t} = \alpha_0 + \alpha_{11} Z_{t-1}^+ + \alpha_{12} Z_{t-1}^-$$

[M3]

$$X_t - \theta_1 X_{t-1}^+ - \theta_2 X_{t-1}^- = Z_t$$

$$Z_t = \sqrt{h_t} \cdot e_t, \quad h_t = \alpha_0 + \alpha_{11}(Z_{t-1}^+)^2 + \alpha_{12}(Z_{t-1}^-)^2$$

We adopted the ILS(iterative least squares) method to calculate the model parameters.

For instance, [M1] is formulated as

$$X_t - (0.192 X_{t-1}^+ - 0.195 X_{t-1}^-) = Z_t$$

$$Z_t = \sqrt{h_t} \cdot e_t, \quad h_t = 2.180 + 0.199 Z_{t-1}^2.$$

The suggested model [M] is estimated by

$$X_t - (0.194 X_{t-1}^+ - 0.198 X_{t-1}^-) = Z_t$$

TABLE 4.2 *Average length of 95% PI(one step ahead prediction)*

Model	*Length of PI*
M1	7.340
M2	7.303
M3	7.331
M	7.222

$$Z_t = \sqrt{h_t} \cdot e_t, \quad h_t^{R_{t-1}} = 2.037 + \alpha_{t-1} \cdot (Z_{t-1}^2)^{R_{t-1}},$$

with

$$R_{t-1} = 0.967\, I[Z_{t-1} \geq 0] + 0.980\, I[Z_{t-1} < 0],$$

and

$$\alpha_{t-1} = 0.203\, I[Z_{t-1} \geq 0] + 0.331\, I[Z_{t-1} < 0].$$

In Table 4.1, it is noted that estimates in each model are not quite different from one another. Examining $\hat{\theta}_1$ and $\hat{\theta}_2$, the absolute values of $\hat{\theta}_2$ are greater than those of $\hat{\theta}_1$ across all models. Also, values of $\hat{\alpha}_2$ are considerably greater than those of $\hat{\alpha}_1$. This reflects asymmetric pattern of the volatilities of the stock market. Regarding [M], the estimates of r_1 and r_2 are close to 1. It indicates that the model may be quite similar to [M3].

For each model, one step ahead 95% prediction interval (PI) for X_t is given by

$$(\hat{\theta}_1 X_{t-1}^+ + \hat{\theta}_2 X_{t-1}^-) \;\pm\; 1.96\sqrt{\hat{h}_t},$$

and hence length of PI is defined as $2(1.96\sqrt{\hat{h}_t})$. Empirical average length of PI is computed for each models and is reported in Table 4.2. Our model [M] clearly provides the smallest length of 95% prediction intervals among the models. This indicates that [M] exhibits the most accurate prediction interval for the future values and thus [M] deserves much attention in analyzing KOSPI data in terms of asymmetric volatilities.

ACKNOWLEDGEMENTS

This work was supported by a grant from KOSEF(R01-2003-000-10589-0). We thank the referee for careful reading of the paper.

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