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THE HÁJECK-RÈNYI INEQUALITY FOR AQSI RANDOM VARIABLES AND ITS APPLICATION

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ABSTRACT

In this paper we establish the Hájeck-Rènyi type inequality for asymptotically quadrant sub-independent random variables and derive the strong law of large numbers by this inequality.

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1. INTRODUCTION

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on $(\Omega, \mathcal{F}, P)$. We start with definitions.

Lehmann(1996) introduced the notion of positive quadrant dependence: A sequence $\{X_n, n \geq 1\}$ is said to be *pairwise positive quadrant dependent* if, for $s, t \in \mathbf{R}$,

$$P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\} \geq 0, \tag{1.1}$$

or

$$P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \geq 0. \tag{1.2}$$

Dropping the assumption of positive dependence, but using the magnitude of the left hand side in (1.1) and (1.2) as a measure of dependence, Birkel(1992) introduced the notion of asymptotic quadrant independence: A sequence $\{X_n\}$

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of random variables is called *asymptotically quadrant independent(AQI)* if there exists a nonnegative sequence $\{q(m)\}$ such that, for all $i \neq j$ and $s, t \in \mathbf{R}$,

$$|P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\}| \leq q(|i-j|)\alpha_{ij}(s,t), \tag{1.3}$$

$$|P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\}| \leq q(|i-j|)\beta_{ij}(s,t), \tag{1.4}$$

where $q(m) \to 0$ and $\alpha_{ij}(s,t) \geq 0, \beta_{ij}(s,t) \geq 0$.

Chandra and Ghosal(1996) considered a dependence condition which is a useful weakening of this definition of AQI proposed by Birkel(1992): A sequence $\{X_n, n \geq 1\}$ of random variables is said to be *asymptotically quadrant sub-independent(AQSI)* if there exists a nonnegative sequence $\{q(m)\}$ such that $q(m) \to 0$, and for all $i \neq j$,

$$P\{X_i > s, X_j > t\} - P\{X_i > s\}P\{X_j > t\} \leq q(|i-j|)\alpha_{ij}(s,t), \ s,t > 0, \tag{1.5}$$

$$P\{X_i < s, X_j < t\} - P\{X_i < s\}P\{X_j < t\} \leq q(|i-j|)\beta_{ij}(s,t), \ s,t < 0, \tag{1.6}$$

where $\alpha_{ij}(s,t)$ and $\beta_{ij}(s,t)$ are nonnegative numbers. This AQSI condition is satisfied by asymptotically quadrant independent sequence as well as by pairwise m-dependent and pairwise negative quadrant dependent sequences.

There are two well-known results; namely, the Kolmogorov strong law of large numbers and the Rademacher-Mensov strong law of large numbers(e.g. Rao(1973, page 114), Hall and Heyde (1980, page 22)). Chandra and Ghosal(1996) proved the strong law of large numbers for sum of AQSI sequences by using an extension of the well-known Rademacher-Mensov inequality which was derived by Chandra and Ghosal(1993).

Hájeck and Rènyi(1955) proved the following important inequality: If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with $EX_n = 0$ and $EX_n^2 < \infty$, $n \geq 1$, and $\{b_n, n \geq 1\}$ is a positive nondecreasing real sequence, then for any $\epsilon > 0$, any positive integer $m < n$,

$$P\left(\max_{m \leq k \leq n} \left|\frac{\sum_{j=1}^{k} X_j}{b_n}\right| \geq \epsilon \right) \leq \epsilon^{-2} \left(\sum_{j=m+1}^{n} \frac{EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{EX_j^2}{b_m^2} \right) \tag{1.7}$$

In this paper we derive the Hájeck-Rènyi inequality for the AQSI random variables and also use this inequality to obtain the strong law of large numbers and integrability of supremum for AQSI sequence.

2. THE HáJECK-RèNYI INEQUALITY FOR AQSI SEQUENCES

Note that the following lemma is an extension of the well-known Rademacher-Mensov inequality.

LEMMA 2.1 (Chandra, Ghosal(1993)). *Let $X_1, \cdots, X_n$ be square integrable random variables and let there exist $a_1^2, \cdots, a_n^2$ satisfying*

$$E(X_{m+1} + \cdots + X_{m+p})^2 \leq a_{m+1}^2 + \cdots + a_{m+p}^2 \tag{2.1}$$

for all $m, p \geq 1,\ m + p \leq n$. Then, we have

$$E(\max_{1 \leq k \leq n} (\sum_{i=1}^{k} X_i)^2) \leq ((\log n / \log 3) + 2)^2 \sum_{i=1}^{n} a_i^2. \tag{2.2}$$

PROOF OF LEMMA 2.1. See the proof of Theorem 10 in Chandra and Ghosal (1993). □

From the definition of AQSI random variable we obtain easily the following property:

LEMMA 2.2 (Chandra, Ghosal(1996)). *If $\{X_n, n \geq 1\}$ is a sequence of AQSI and $\{f_n, n \geq 1\}$ is a sequence of nondecreasing(nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is also a sequence of AQSI random variables.*

Aplying Lemma 2.1 we also have the following Hàjeck-Rényi inequality for AQSI random variables:

THEOREM 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables such that $EX_n^2 < \infty, n \geq 1,\ \sum_{m=1}^{\infty} q(m) < \infty$, and for all $i \neq j$*

$$\int_0^\infty \int_0^\infty \alpha_{ij}(s,t)dsdt \leq D(1 + EX_i^2 + EX_j^2), \tag{2.3}$$

$$\int_0^\infty \int_0^\infty \beta_{ij}(s,t)dsdt \leq D(1 + EX_i^2 + EX_j^2), \tag{2.4}$$

where $\alpha_{ij}(s,t) \geq 0$ *and* $\beta_{ij}(s,t) \geq 0$ *and* D *is a positive constant. Let* $\{b_n, n \geq 1\}$ *be a positive sequence of nondecreasing real numbers. Then, for* $\epsilon > 0$ *we have*

$$P\{\max_{1\leq k\leq n} |\frac{\sum_{i=1}^{k}(X_i - EX_i)}{b_k}| \geq \epsilon \log n\}$$
$$\leq C(\epsilon \log n)^{-2}((\log n/\log 3) + 2)^2 \sum_{i=1}^{n} \frac{1+EX_i^2}{b_i^2}. \tag{2.5}$$

PROOF. Let $X_n^+ = \max\{X_n, 0\}$ and $X_n^- = \max\{-X_n, 0\}$. Clearly $\{X_n^+\}$ and $\{X_n^-\}$ form AQSI sequences by Lemma 2.2. By Lemma 2 of Lehmann(1966)

$$Cov(X_i^+, X_j^+) \leq Dq(|i-j|)(1 + EX_i^2 + EX_j^2).$$

So

$$Var(\sum_{i=1}^{n} b_i^{-1}X_i^+) \leq C\sum_{i=1}^{n} \frac{1+EX_i^2}{b_i^2} \quad for\ all\ n \geq 1.$$

Similarly,

$$Var(\sum_{i=1}^{n} b_i^{-1}X_i^-) \leq C\sum_{i=1}^{n} \frac{1+EX_i^2}{b_i^2} \quad for\ all\ n \geq 1.$$

Thus

$$Var(\sum_{i=1}^{n} b_i^{-1}X_i) \leq 2Var(\sum_{i=1}^{n} b_i^{-1}X_i^+) + 2Var(\sum_{i=1}^{n} b_i^{-1}X_i^-)$$
$$\leq C\sum_{i=1}^{n} \frac{1+EX_i^2}{b_i^2} \quad for\ all\ n \geq 1. \tag{2.6}$$

Let $S_n = \sum_{j=1}^{n}(X_j - EX_j), n \geq 1$. Without loss of generality, setting $b_0 = 0$, we have

$$\begin{aligned} S_k &= \sum_{j=1}^{k} b_j \frac{(X_j - EX_j)}{b_j} \\ &= \sum_{j=1}^{k}(\sum_{i=1}^{j}(b_i - b_{i-1})\frac{(X_j - EX_j)}{b_j}) \\ &= \sum_{i=1}^{k}(b_i - b_{i-1}) \sum_{i\leq j\leq k} \frac{(X_j - EX_j)}{b_j}. \end{aligned}$$

Note that $(1/b_k)\sum_{j=1}^{k}(b_j - b_{j-1}) = 1$. So

$$\{|\frac{S_k}{b_k}| \geq \epsilon \log n\} \subset \{\max_{1\leq i\leq k} |\sum_{i\leq j\leq k} \frac{X_j - EX_j}{b_j}| \geq \epsilon \log n\}.$$

Therefore,

$$\begin{aligned}\{\max_{1\leq k\leq n} |\frac{S_k}{b_k}| &\geq \epsilon \log n\} \subset \{\max_{1\leq k\leq n}\max_{1\leq i\leq k} |\sum_{i\leq j\leq k} \frac{X_j - EX_j}{b_j}| \geq \epsilon \log n\} \\ &= \{\max_{1\leq i\leq k\leq n} |\sum_{j\leq k} \frac{X_j - EX_j}{b_j} - \sum_{j<i} \frac{X_i - EX_j}{b_j}| \geq \epsilon \log n\} \\ &\subset \{\max_{1\leq i\leq n} |\sum_{j=1}^{i} \frac{X_j - EX_j}{b_j}| \geq \frac{\epsilon}{2} \log n\}.\end{aligned}$$

By Lemma 2.1 and (2.6) we obtain the following result

$$P\{\max_{1\leq k\leq n} |\frac{S_k}{b_k}| \geq \epsilon \log n\} \leq C(\epsilon \log n)^{-2}((\log n/\log 3) + 2)^2 \sum_{i=1}^{n} \frac{1 + EX_i^2}{b_i^2}.$$

□

From Theorem 2.3 we can get the following more generalized Hájeck-Rènyi inequality.

THEOREM 2.4. *Let $\{b_n, n \geq 1\}$ be a positive sequence of nondecreasing real numbers. Let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables with $EX_n^2 < \infty, n \geq 1, \sum_{m=1}^{\infty} q(m) < \infty$ and satisfying (2.3) and (2.4), then $\epsilon > 0$ and for any positive integer $m < n$,*

$$\begin{aligned}&P\{\max_{m\leq k\leq n} |\frac{\sum_{i=1}^{k}(X_i - EX_i)}{b_k}| \geq \epsilon \log n\} \\ &\leq C(\epsilon \log n)^{-2}((\log n/\log 3) + 2)^2 (\sum_{j=m+1}^{n} \frac{1 + EX_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{1 + EX_j^2}{b_m^2}).\end{aligned}$$

PROOF. By Theorem 2.3 we have

$$
\begin{aligned}
&P\{\max_{m\le k\le n}|\frac{\sum_{i=1}^{k}(X_i-EX_i)}{b_k}|\ge \epsilon\log n\}\\
&\le P\{|\frac{\sum_{i=1}^{m}(X_i-EX_i)}{b_m}|\ge \frac{\epsilon}{2}\log n\}\\
&\quad +P\{\max_{m+1\le k\le n}|\frac{\sum_{i=m+1}^{k}(X_i-EX_i)}{b_k}|\ge \frac{\epsilon}{2}\log n\}\\
&\le P\{\max_{1\le k\le m}|\frac{\sum_{i=1}^{k}(X_i-EX_i)}{b_m}|\ge \frac{\epsilon}{2}\log n\}\\
&\quad +P\{\max_{m+1\le k\le n}|\frac{\sum_{i=m+1}^{k}(X_i-EX_i)}{b_k}|\ge \frac{\epsilon}{2}\log n\}\\
&\le C(\epsilon\log n)^{-2}((\log n/\log 3)+2)^2(\sum_{j=m+1}^{n}\frac{1+EX_j^2}{b_j^2}+\sum_{j=1}^{m}\frac{1+EX_j^2}{b_m^2}).
\end{aligned}
$$

□

3. THE SLLN AND COMPLETE CONVERGENCE FOR AQSI SEQUENCE

THEOREM 3.1. *Let $\{X_n, n\ge 1\}$ be a sequence of AQSI random variables with $EX_n^2<\infty, n\ge 1$, $\sum_{m=1}^{\infty}q(m)<\infty$, and satisfying (2.3) and (2.4) and let $\{b_n, n\ge 1\}$ be a positive sequence of nondecreasing real numbers. If*

$$
\sum_{n=1}^{\infty}\frac{1+\sigma_n^2}{b_n^2}<\infty \tag{3.1}
$$

holds, then $(b_n\log n)^{-1}S_n\to 0$ a.s as $n\to\infty$, where $\sigma_n^2=EX_n^2, n\ge 1$ and $S_n=\sum_{i=1}^{n}(X_i-EX_i)$.

PROOF. By Theorem 2.4 we have

$$
\begin{aligned}
&P(\max_{m\le k\le n}|\frac{1}{b_n}\sum_{i=1}^{k}(X_i-EX_i)|\ge\epsilon\log n)\\
&\le C(\epsilon\log n)^{-2}((\log n/\log 3)+2)^2(\sum_{j=m+1}^{n}\frac{1+\sigma_j^2}{b_j^2}+\sum_{j=1}^{m}\frac{1+\sigma_j^2}{b_m^2}).
\end{aligned}
$$

But

$$
\begin{aligned}
&P(\sup_n |\frac{1}{b_n}\sum_{i=1}^{n}(X_i - EX_i)| \geq \epsilon \log n) \\
&\leq \lim_{n\to\infty} P(\max_{m\leq k\leq n} |\frac{1}{b_k}\sum_{i=1}^{k}(X_i - EX_i)| \geq \epsilon \log n) \\
&\leq C \lim_{n\to\infty} (\epsilon \log n)^{-2}((\frac{\log n}{\log 3}) + 2)^2 (\sum_{i=m+1}^{n} \frac{1+\sigma_i^2}{b_i^2} + \sum_{j=1}^{m} \frac{1+\sigma_j^2}{b_m^2}) \\
&\leq C(\sum_{j=m+1}^{\infty} \frac{1+\sigma_j^2}{b_j^2} + \sum_{j=1}^{m} \frac{1+\sigma_j^2}{b_m^2}).
\end{aligned}
$$

Hence, by the Kronecker Lemma and (3.1) we get

$$
\lim_{n\to\infty} P(\sup_n \frac{1}{b_n}\sum_{i=1}^{n}(X_i - EX_i) \geq \epsilon \log n) = 0,
$$

which completes the proof. □

From Theorem 3.1 we prove the following strong law of large numbers for square integrable AQSI random variables satisfying (2.3) and (2.4).

COROLLARY 3.2. *Let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables with $EX_n^2 < \infty$, $n \geq 1$, $\sum_{m=1}^{\infty} q(m) < \infty$ and satisfying (2.3) and (2.4). If $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$ where $\sigma_n^2 = EX_n^2$, $n \geq 1$, then for $0 < t < 2$, $n^{-\frac{1}{t}}(\log n)^{-1} S_n \to 0$ a.s. as $n \to \infty$ where $S_n = \sum_{i=1}^{n}(X_i - EX_i)$.*

COROLLARY 3.3. *Let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables with $\sup_n EX_n^2 < \infty, n \geq 1, \sum_{m=1}^{\infty} q(m) < \infty$, and satisfying (2.3) and (2.4). Then, for $0 < t < 2$ $n^{-\frac{1}{t}}(\log n)^{-1} S_n \to 0$ a.s., where $S_n = \sum_{i=1}^{n}(X_i - EX_i)$.*

THEOREM 3.4. *Let $\{b_n, n \geq 1\}$ be a sequence of positive nondecreasing real numbers and let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables with $EX_n^2 < \infty$, $n \geq 1$, $\sum_{m=1}^{\infty} q(m) < \infty$ and satisfying (2.3) and (2.4). If (3.1) holds then for any $0 < r < 2$,*

$$
E \sup_n (\frac{|S_n|}{b_n \log n})^r < \infty,
$$

where $S_n = \sum_{i=1}^{n}(X_i - EX_i)$, $n \geq 1$.

PROOF. Note that

$$E(\sup_n \frac{|S_n|}{b_n \log n})^r < \infty \Leftrightarrow \int_1^\infty P(\sup_n \frac{|S_n|}{b_n \log n} > t^{1/r})dt < \infty.$$

By Theorem 2.3,

$$\begin{aligned}
&\int_1^\infty P(\sup_n \frac{|S_n|}{b_n \log n} > t^{1/r})dt \\
&\leq C\int_1^\infty t^{-2/r} \lim_{n\to\infty} (\log n)^{-2}((\frac{\log n}{\log 3}) + 2)^2 \sum_{j=1}^n \frac{1+\sigma_j^2}{b_j^2} dt \\
&= C \lim_{n\to\infty} (\log n)^{-2}((\frac{\log n}{\log 3}) + 2)^2 \sum_{j=1}^n \frac{1+\sigma_j^2}{b_j^2} \int_1^\infty t^{-2/r} dt \\
&\leq C \sum_{n=1}^\infty \frac{1+\sigma_n^2}{b_n^2} \int_1^\infty t^{-2/r} dt < \infty.
\end{aligned}$$

□

COROLLARY 3.5. *Let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables with $EX_n = 0$, $EX_n^2 < \infty$, $n \geq 1$, $\sum_{m=1}^\infty q(m) < \infty$ and satisfying (2.3) and (2.4). If $\sup_n EX_n^2 < \infty$ holds, then for $0 < t < 2$, $m \geq 1$ and for all $\epsilon > 0$,*

$$\begin{aligned}
&P\{\sup_{n\geq m} |\sum_{i=1}^n X_i|/n^{\frac{1}{t}} \geq \epsilon \log n\} \\
&\qquad \leq C(\epsilon \log n)^{-2}((\log n/\log 3) + 2)^2 \frac{2}{2-t}(1 + \sup_n \sigma_n^2) m^{(t-2)/t}
\end{aligned}$$

where $\sigma_n^2 = Var(X_n)$.

COROLLARY 3.6. *Let $\{X_n, n \geq 1\}$ be a sequence of AQSI random variables satisfying the conditions of Corollary 3.3. Then for $0 < t < 2$ and any $0 < r < 2$ $E\sup_n (n^{-\frac{1}{t}} \log n^{-1} |S_n|)^r < \infty$ holds.*

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