Mismatching Problem between Generic Pole-assignabilities by Static Output Feedback and Dynamic Output Feedback in Linear Systems

Su-Woon Kim

Abstract: In this paper, it is clearly shown that the two well-known necessary and sufficient conditions $mp \geq n$ as *generic* static output feedback pole-assignment and $mp + d(m+p) \geq n+d$ as *generic* minimum d-th order dynamic output feedback pole-assignment on complex field $\mathbb{C}$, unbelievably, *do not match up* each other in strictly proper linear systems. For the analysis, a diagram analysis is newly created (which is defined by the analysis of "convoluted rectangular/dot diagrams" constructed via node-branch conversion of the signal flow graphs of output feedback gain loops). Under this diagram analysis, it is proved that the minimum d-th order dynamic output feedback compensator for pole-assignment in m-input, p-output, n-th order systems is quantitatively decomposed into static output feedback compensator and its associated d number of arbitrary 1st order dynamic elements in *augmented* $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order systems. Total configuration of the mismatched data is presented in a Table.

Keywords: Decomposition of dynamic output feedback in augmented static output feedback systems, generic pole-assignment, Grassmann invariant of static output feedback linear systems.

1. INTRODUCTION

The previous and current studies on the static and dynamic output feedback (O. F.) pole-assignment problems in linear (finite-dimensional time-invariant) systems have heavily depended upon pure mathematic power (like algebraic geometry, algebraic topology, exterior algebra, and so on)[1-5]. It is due to intrinsic high nonlinearity of the O. F. problems which is hard to handle by traditional methodology and technical algorithms available in classical and modern system theory. However even in these pure mathematical approaches, major outcomes on pole-assignment conditions have been obtained *within incomplete forms* which can lose some essential control engineering attributes [6-8]. The goal of this paper is to check *in what extend* the incomplete mathematical outcomes, so-called *generic* static and dynamic O. F. pole-assignment conditions are valid or invalid, comparing with the *complete* (i.e., exact) static and dynamic O. F. pole-assignment conditions. As an effective tool for this goal, the author creates a diagram analysis, as a simplified signal flow graph analysis of O. F. gain loops (see Section 2 and Section 3), and which is named by "a lattice diagram analysis" in this paper.

This lattice diagram analysis is defined by the analysis of "convoluted rectangular/dot diagrams" constructed *via node-branch conversion* of the signal flow graphs (SFGs) of O. F. gain loops. Through the lattice diagram analysis, some invalid mismatching problem is revealed as follows:

> "The two well-known necessary and sufficient (simply, N and S) conditions $mp \geq n$ as generic static O. F. pole-assignment on complex field $\mathbb{C}$ by Hermann and Martin [1, Theorem 6.1] and $mp + d(m+p) \geq n+d$ as *generic* minimum d-th order dynamic O. F. pole-assignment on complex field $\mathbb{C}$ by Rosenthal [4, Theorem 5.11], do *not* match up each other in strictly proper linear (controllable and observable) systems."

Following rationale checks this mismatching problem. At first, through the lattice diagram analysis, it is shown that *structural quantitative relationship* between static O. F. compensation and minimum d-th order dynamic O. F. compensation for pole-assignment is derived within simple numerical relation as following:

> "The minimum d-th order dynamic O. F. compensator for pole-assignment in original m-input, p-output, n-th order linear systems *is decomposed into* static O. F. compensator and its associated d

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number of arbitrary 1st order dynamic (transfer function) elements in *maximally augmented* $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order linear systems." (1)

(The quantitative relationship was roughly expected by Kimura in [6, p.2105].) Secondly, for investigation of the invalidity of the mathematical incomplete outcomes (of generic ...), the structural quantitative relationship of (1) is applied to the well-known N and S condition of *generic* static O. F. pole-assignment on complex field $\mathbb{C}$ in strictly proper systems [1]

$$mp \geq n \tag{2}$$

then in the case of *strictly proper* dynamic elements, we can induce directly a N and S condition of *generic* minimum d-th order dynamic O. F. pole-assignment on $\mathbb{C}$ by

$$(m+d)(p+d) \geq n+d \tag{3a}$$

and in the case of *proper* dynamic elements, we can induce directly a N and S condition by

$$(m+d)(p+d) \geq n+d+1 \tag{3b}$$

(as it was known to be the N and S condition of generic *static* O. F. pole-assignment on $\mathbb{C}$ in $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order *proper* systems [12, Theorem 5.3(c)]). But unbelievably, the induced outcomes of (3a) and (3b) are dissimilar with the well-known N and S condition of *generic* minimum d-th order dynamic O. F. pole-assignment on $\mathbb{C}$ in strictly proper systems [4]

$$mp+d(m+p) \geq n+d, \tag{4}$$

which is right and which is wrong ?

Finally, for analyzing this mismatching problem, the rank of "dynamic Grassmann invariant" (defined by (static) Grassmann invariant in *augmented* static O. F. system, symbolizing L^{aug} in this paper [11,12]) is numerically calculated to check whether a necessary condition of *complete* static O. F. pole-assignment, full-rank of *Plücker submatrix* of L^{aug} is satisfied or not [12, Proposition 5.1]. Under the rank test, the mismatching problem is answered as follows:

"In CASE I ($\min\{m, p\} = 1$ systems with $d = 1$): The induced outcome (3a) provides *no intersections* for pole-assignment (i.e., none pole-assignment) in the marginal area of $mp + d(m+p) \sim (m+d)(p+d)$. Meanwhile, the original one (4) provides complex or real intersections for pole-assignment.

In CASE II ($\min\{m, p\} = 1$ systems with $d \geq 2$): The original one (4) *is conservative* by amount of d^2-1 in general. Meanwhile, the induced outcome (3a) provide complex or real intersections for pole-assignment in non-conservative way (except $m = p = 1, d = 2$).

In CASE III (MIMO system with $d \geq 1$): The original one (4) *is conservative* by amount of d^2 (when $d = 1$) and d^2-1 (when $d \geq 2$). Meanwhile, the induced ones (3a) and (3b) provide complex or real intersections for pole-assignment in non-conservative way." (5)

(For details, see the Table 1 in Section 6.) From (5), we can immediately deduce following facts.

Deduced fact-1 (*insufficient triples* (m, n, p) *for complete static O. F. pole-assignability*): It is clear that the N and S conditions of *complete* static O. F. pole-assignment conditions on $\mathbb{C}$ in strictly proper systems *cannot be* obtained only by terms of input number (m), output number (p) and system order (n). See the CASE I of $\min\{m, p\} = 1$ systems with $d = 1$, the induced outcome (3a) provides *no intersections* for pole-assignment in the marginal area of $mp + d(m+p) \sim (m+d)(p+d)$. Thus, the "degenerate" static O. F. $\min\{m, p\} = 1$ systems with input number $m' = m+d = m+1$, output number $p' = p+d = p+1$, and system order $n' = n+d = n+1$, have no intersections, i.e., none pole-assignable on $\mathbb{C}$ in the marginal area, even though the static O. F. systems in the CASE II and CASE III, are pole-assignable in the marginal area.

Deduced fact-2 (*conservatism in generic minimum order dynamic O. F. pole-assignability*): If the order of dynamic O. F. compensator for pole-assignment is somewhat high like $d \gg 1$ (see the CASE II and CASE III), then the currently well-known generic outcome $mp + d(m+p) \geq n+d$ *is too much conservative*. For instance, when $d = 10$, the conservatism of the generic outcome reaches to 10^2 -1 = 99th order.

The earlier major consequence in (1) is also significant in following sense:

"In (1), a *fixed* quantitative relationship between minimum d-th order dynamic O. F. compensation and static O. F. compensation for pole-assignment is derived. Thus any outcomes (like generic or complete necessary and/or sufficient conditions, invariants, canonical forms, etc.) regarding to minimum d-th order dynamic O. F. pole-assignment can be *induced directly* from the pre-known outcomes regarding to static O. F. pole-assignment, or *vice versa*." (6)

From the Deduced fact-1, Deduced fact-2 and (6), following questions naturally arise as future theoretical issues.

Question-1: Is the well-known sufficient condition of *generic* static O. F. pole-assignment on $\mathbb{C}$ in strictly proper systems, $mp > n$ [3], to be the sufficient condition of *complete* static O. F. pole-assignment on $\mathbb{C}$, combining with full-rank of *Plücker submatrix* of L

(as a necessary condition of *complete* static O. F. pole-assignment) ? — vice versa, is the $(m+d)(p+d) > n+d$ or $(m+d)(p+d) > n+d+1$ to be the sufficient condition of *complete* minimum d-th order dynamic O. F. pole-assignment on $\mathbb{R}$ in strictly proper systems, combining with full-rank of *Plücker submatrix* of L^{aug} ?

Question-2: Is the Grassmann invariant L to be *a complete invariant* (or a canonical form) under static O. F. group action for system poles ? — vice versa, is the dynamic Grassmann invariant L^{aug} to be *a complete dynamic invariant* (or a canonical form) under (minimum order) dynamic O. F. group action for system poles [13,14] ?

These two important questions seem to be deeply related in each other. So if one question is completely solved, then it is expected that the other question shall be also easily solved. So if these questions are partially or totally solved, we can *systematically apply* them to compute and parameterize the static O. F. compensator under Grassmannian-oriented para-meter L — vice versa, we can systematically apply them to compute and parameterize minimum order dynamic O. F. compensator under Grassmannian-oriented parameter L^{aug}).

Recall that the definitions of complete static O. F. pole-assignment and generic static O. F. pole-assignment in the linear system of transfer function matrix $G(s)$ are described as follows.

Definition 1 (complete static O. F. pole-assignment): In the closed-loop characteristic polynomial $det\ [D_L(s) + N_L(s)K] = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$ of irreducible strictly proper (or proper) transfer function matrix $G(s) = D_L(s)^{-1}N_L(s)$, if there exist real O. F. matrices $K \in \mathbb{R}^{m\times p}$ for all arbitrary real coefficients $(a_1, \dots, a_n) \in \mathbb{R}^n$ (or $(1, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$)), then it is called that the linear system $G(s)$ is completely pole-assignable by static O. F. (of real O. F. gain matrix).

Definition 2 (generic static O. F. pole-assignment): In the closed-loop characteristic polynomial $det\ [D_L(s) + N_L(s)K] = s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n$ of irreducible strictly proper (or proper) transfer function matrix $G(s) = D_L(s)^{-1}N_L(s)$, if there exist open dense sets of real coefficients $(a_1, \dots, a_n) \in \mathbb{R}^n$ (or $(1, a_1, \dots, a_n) \in \mathbb{R}^{n+1}$)) over all real O. F. matrices $K \in \mathbb{R}^{m\times p}$, then it is called that the linear system $G(s)$ is generically pole-assignable by static O. F. (of real O. F. gain matrix).

2. ONE-TO-ONE DIAGRAM REPRESENTATIONS OF ALL O. F. LOOPS

2.1. Alternative connectivity

In m-input, p-output static O. F. linear systems, all the O. F. gain loops (simply, O. F. loops) have a distinctive specific structure — forward transfer functions and feedback gains are always *alternatively connected* between input nodes and output nodes in same number. See the SFGs of single forward (transfer function) path loop, multi forward 2 (transfer function) path loop, and multi forward 3 (transfer function) path loop in Fig. 1, and see the 2 types of nontouching loops (of nonmaximal number of nontouching loops and maximal number of nontouching loops) in Fig. 2. The Fig. 1(a) shows a single forward path loop of $-G_{ij}(s)k_{ji}$, and the Fig. 1(b) shows a multi forward 2 path loop of $-G_{ij}(s)k_{js}G_{st}(s)k_{ti}$, and the Fig. 1(c) shows a multi forward 3 (transfer function) path loop of $-G_{ij}(s)k_{jv}G_{vw}(s)k_{ws}G_{st}(s)k_{ti}$ (where $1 \le i, s, v \le m$ indicate input nodes, and $1 \le j, t, w \le p$ indicate output nodes). The Fig. 2(a) exhibits nonmaximal 2-nontouching loops of $-G_{ab}(s)k_{ba}$, and $-G_{st}(s)k_{tv}\ G_{vw}(s)k_{ws}$, and the Fig. 2(b) does maximal m-nontouching loops of $-G_{1,1+e}(s)\ k_{1+e,1}$, $-G_{2,2+e}(s)k_{2+e,2}$, … , and $-G_{m,m+e}(s)k_{m+e,m}$ (where $0 \le e \le p\text{-}m$ in $m \le p$ systems).

Remark 1: In this paper, for convenient SFG analysis of forward (transfer function) path P_{ij} from

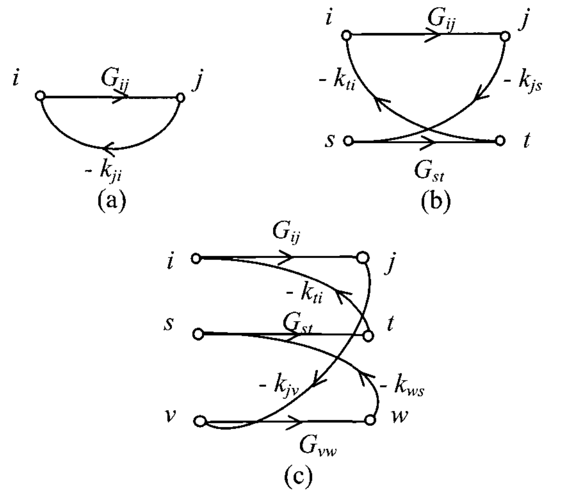

Fig. 1. SFGs of some O. F. loops.

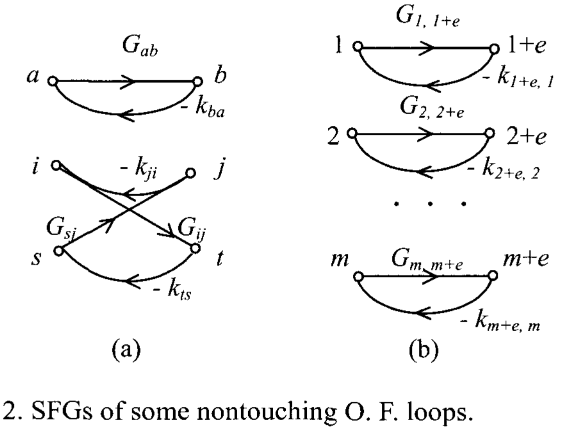

Fig. 2. SFGs of some nontouching O. F. loops.

node (i) to node (j) in coincidence with matrix coordinate $M(i, j)$, the author describes a transfer function matrix in $m\times p$ matrix setting with symbol $G(s)^{SFG} \in \mathbb{R}(s)^{m\times p}$ (rather than in traditional $p\times m$ matrix setting, $G(s) \in \mathbb{R}(s)^{p\times m}$).

2.2. Matrix coordination of O. F. loops

From the alternative connectivity of transfer functions and feedback gains between input nodes and output nodes of O. F. loops, one can simply transform the classical SFGs of O. F. loops *into certain coordinated diagrams* within the $m\times p$ matrix coordinates. Consider a loop transformation via "node-branch conversion" — where the branches of forward (transfer function) paths and (O. F.) feedbacks *are just shrunk to* be new nodes, and the input and output nodes *are just lengthened to* be new branches with direction of signal flow.

Then the SFGs of Fig. 1 and Fig. 2 are transformed to coordinated diagrams like Fig. 3 and Fig. 4, respectively. We shall call them by "coordinated rectangular/dot diagrams" (simply, rectangular/dot diagrams) of O. F. loops. In Fig. 3 and Fig. 4, "node with symbol (o)" in a coordinate ij indicates a forward transfer function $G_{ij}(s)$ and "node with symbol (Δ)" in a coordinate $(ij)^t$ as the transposed position of ij indicates a feedback gain k_{ji}. In Fig. 3(a), a single forward path loop is simply figured by "a twofold double node (◎)" without branches because the coordinate of transfer function $G_{ij}(s)$ and the transposed coordinate of feedback gain k_{ji} is overlapped in same position; see Fig. 5.

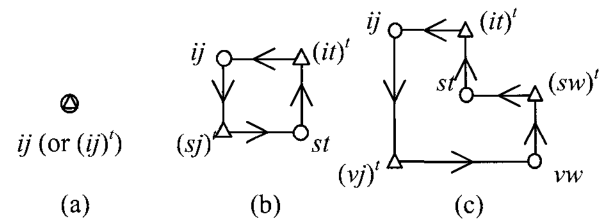

Fig. 3. Rectangular/dot diagrams of some O. F. loops.

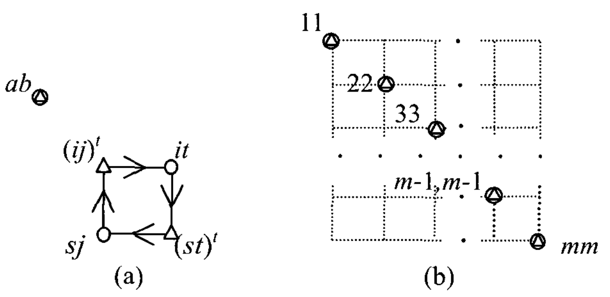

Fig. 4. Rectangular/dot diagrams of some nontouching O. F. loops.

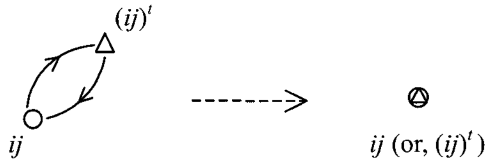

Fig. 5. One point coordination of a single forward path loop.

In Fig. 4(b), the m number of (double node) dot diagram shows maximal m-nontouching loops where $e = 0$. Now we need to check whether all the rectangular/dot diagrams obtained via node-branch conversion of O. F. loops are distinguishably one-to-one correspondent and is continuous over original SFGs of O. F. loops.

Theorem 1 (one-to-one diagram representations of all O. F. loops): In m-input, p-output static O. F. linear systems, the rectangular/dot diagrams of O. F. loops within the $m\times p$ matrix coordinates via node-branch conversion are one-to-one correspondent and continuous over all kinds of original SFGs of O. F. loops.

Proof: In all O. F. loops of static O. F. linear systems, the forward transfer functions and feedback gains are always *alternatively connected* between input nodes and output nodes in same number.

And as seen in Fig. 1, ... , Fig. 5, the node-branch conversion is defined by the loop transformation where the branches of forward (transfer function) paths and (output gain) feedbacks of original SFGs *are just shrunk to* be new nodes of (o) and (Δ), (or, doubly overlapped nodes (◎)) respectively, and the input and output nodes of original SFGs *are just lengthened to* be new branches with the same direction of signal flow of each O. F. loop.

Thus the loop transformation via node-branch conversion is continuous, and the rectangular diagrams with nodes of (o) and (Δ) and dot diagrams with doubly overlapped nodes (◎) within the $m\times p$ matrix coordinates are one-to-one correspondent to all kinds of original SFGs of O. F. loops.

3. LATTICE DIAGRAM — CONVOLUTED DIAGRAM OF RECTANGULAR/DOT DIAGRAMS

3.1. Lattice diagram expressions

When we focus locally on a 2×2 transfer function submatrix shown in Fig. 6(a), correspondent SFGs are figured like Fig. 6(b), where one can find 3 kinds of O. F. gain loops: *a multi 2-path loop* of $-G_{22}k_{23}G_{33}k_{32}$ (composed by thick signal flows), and *maximal 2-nontouching loops* of $(-G_{22}k_{22})$, $(-G_{33}k_{33})$, and *two single path loops* of $-G_{22}k_{22}$ and $-G_{33}k_{33}$.

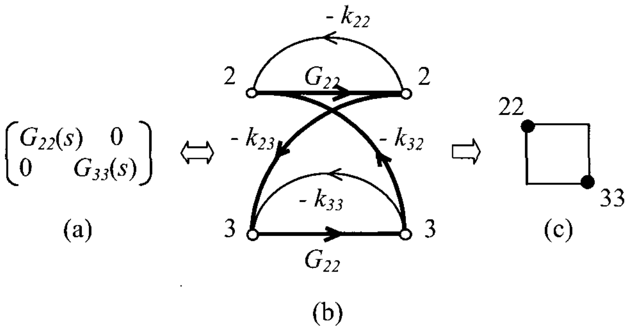

Fig. 6. SFG of a 2×2 sublattice.

We shall express all O. F. loops in a 2×2 (closed-loop) transfer function sub-matrix in a 2×2 lattice-form, replacing the overlapped transfer function nodes (o) with only black dots(•) without expression of the feedback gain nodes (Δ) because they are always fully connected. We shall call it by "2×2 sublattice".

In this way, we can express all O. F. loops in any $N \times N$ (closed-loop) transfer function submatrices by "$N \times N$ sublattices" with only black dots(•) (where $N \leq min\{m, p\}$). Main features of $N \times N$ sublattices are summarized as follows:

1) Any $N \times N$ sublattice of a MIMO system is decomposed into, at least, "3 kinds of" elementary O. F. loops of single path loops, maximal z-nontouching loops, and multi z-path loops (and multi i-path loops and nonmaximal (z-i)-nontouching loops) (where $z \leq N$ and $N = 2, \ldots, \min\{m, p\}$).

2) In any $N \times N$ sublattice of a MIMO system, there is, at least, "a product" of maximal N-nontouching loops, or "two products or greater than two products" of maximal w-nontouching loops (where $w < N$). The former is called *well-posed* $N \times N$ sublattices (or transfer function submatrices) and the latter is called *ill-posed* $N \times N$ sublattices (or transfer function submatrices). (7)

The diagram in Fig. 7(a) presents a well-posed 3×3 sublattice having a product of maximal 3-nontouching loops, $(-G_{13}k_{31})(-G_{22}k_{22})(-G_{31}k_{13})$. But the diagram in Fig. 7(b) presents an ill-posed 3×3 sublattice having no maximal 3-nontouching loops, but having 4 maximal 2-nontouching loops, $(-G_{13}k_{31})(-G_{21}k_{12})$, $(-G_{13}k_{31})(-G_{22}k_{22})$, $(-G_{21}k_{12})(-G_{33}k_{33})$, $(-G_{22}k_{22})(-G_{33}k_{33})$.

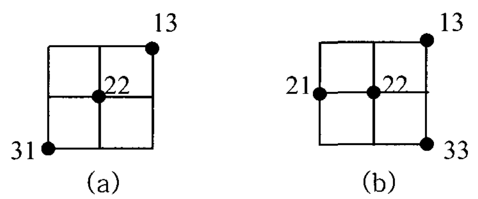

Fig. 7. A well-posed 3×3 sublattice and an ill-posed 3×3 sublattice.

By the same way, we can express all O. F. loops in the $m \times p$ (closed-loop) transfer function matrix by "a $m \times p$ lattice" with only black dots(•). For distinction with $N \times N$ sublattices, the $m \times p$ lattice is figured by the diagram with double outline. See the Fig. 8.

3.2. Minimal position of transfer functions

As seen in Fig. 7 and Fig. 4(b), the products of (transfer functions and O. F. gains in) maximal N-nontouching loops are always located in the crosses of any two different pairs of row (horizontal) lines and column (vertical) lines. Hence we can obtain following Corollary.

Corollary 1: In a well-posed $N \times N$ sublattices of transfer function submatrices, the minimum number of transfer functions is N.

Proof: By definition of well-posedness, there is at least a product of maximal N-nontouching loops which is located in the "N number of crosses" of different row (horizontal) lines and different column (vertical) lines. Since a product of maximal N-nontouching loops has only N number of transfer functions, the minimum number of transfer functions in well-posed $N \times N$ sublattices is to be N. ◻

From Corollary 1, we can define "a minimal position" which has minimum number of transfer functions as following.

$$G(s)^{SFG} = \begin{pmatrix} G_{11} & G_{12} & 0 & \ldots & 0 & G_{1p} \\ 0 & G_{22} & 0 & \ldots & G_{2,p-1} & 0 \\ 0 & 0 & G_{33} & \ldots & 0 & G_{3p} \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ G_{m1} & 0 & 0 & \ldots & G_{m,p-1} & 0 \end{pmatrix}$$

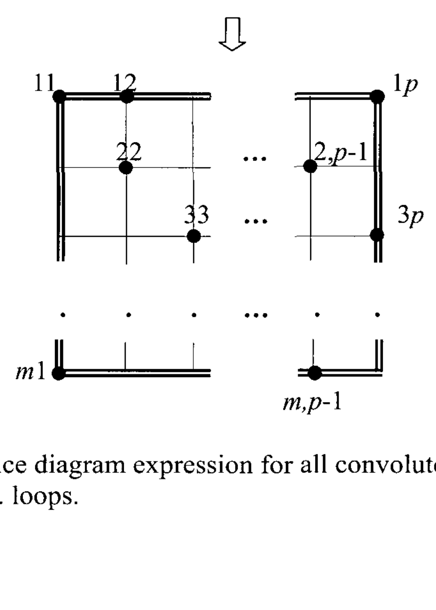

Fig. 8. Lattice diagram expression for all convoluted O. F. loops.

Definition 3 (minimal position of transfer functions): In a well-posed $N \times N$ sublattices of transfer function submatrices, any N-crossed position (of a product of maximal N-nontouching loops) is defined by "a minimal position" providing minimum number of transfer functions.

Remark 2: The author names the partial and total convolutions of rectangular/dot diagrams by "sub-lattices and lattice (diagram)" by just resemblance of their external appearances. But it is notable that these terminologies have utterly different meanings of the traditional terminologies, "lattice and sublattices" in lattice theory of graphic mathematics [15].

4. LATTICE DIAGRAM ANALYSIS—DECOMPOSITIVE ANALYSIS OF DYNAMIC O. F.

4.1. Decomposition mode

In SFG viewpoint, the dynamic O. F. compensator $H(s)$ between output $y(s)$ and input $r(s)$ of a SISO system can be decomposed into "static O. F. compensator $K^{SFG} \in {}^{(1+1)\times(1+1)}$ with elements k_{11}, k_{12}, k_{21}, k_{22}"and "its associated dynamic (transfer function) elements $G_{12}(s)$, $G_{22}(s)$, $G_{21}(s)$" of an augmented 2-input, 2-output MIMO system like Fig. 9.

In the same way, the dynamic O. F. compensator $H(s)^{SFG} \in (s)^{p\times m}$ between output vector $y(s) = [y_1(s), y_2(s), \dots, y_p(s)]^t$ and input vector $r(s) = [r_1(s), r_2(s), \dots, r_m(s)]^t$ of a m-input, p-output MIMO system can be decomposed into "augmented static O. F. compensator $K^{SFG} \in {}^{(p+1)\times(m+1)}$" and "its associated dynamic (transfer function) elements $G_{m+1,p+1}(s)$, $\{G_{i,p+1}(s), G_{m+1,j}(s)$ for all $i = 1, \dots, m$ and $j = 1, \dots, p\}$" of an augmented $(m+1)$-input, $(p+1)$-output system; see Fig. 10. But in this case, the dynamic compensator $H(s)^{SFG}$ is not minimum order. Consider *maximally degenerated* dynamic element case where only one dynamic element locates in a crossed position of $G_{m+1,p+1}(s)$ like Fig. 11,

Hence if the order of $G_{m+1,p+1}(s)$ is "1", then the dynamic compensator $H(s)^{SFG}$ is *to be minimum order* by "1" by definition of system order (the order of LCM

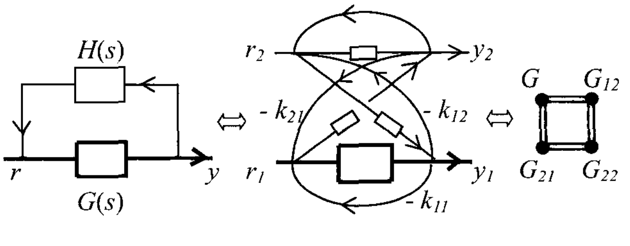

Fig. 9. Augmentation of SISO dynamic O. F. system into 2-input, 2-output static O. F. system.

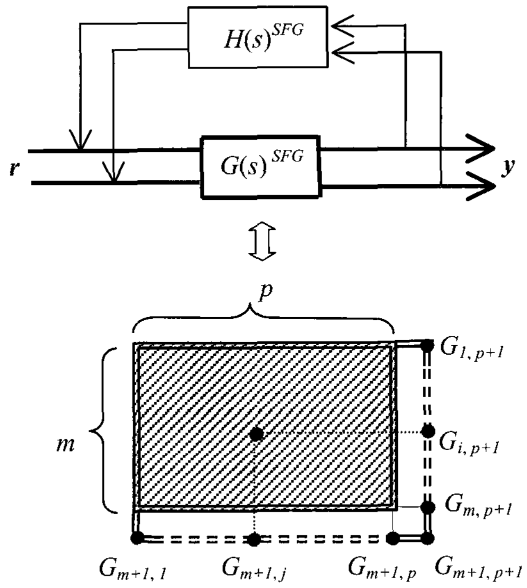

Fig. 10. Augmentation of m-input, p-output dynamic O. F. system into $(m+1)$-input, $(p+1)$-output static O. F. system.

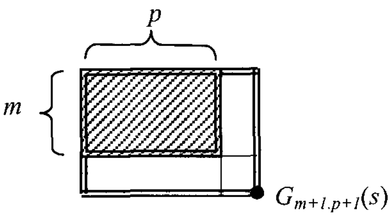

Fig. 11. Lattice diagram with a dynamic element in a crossed position.

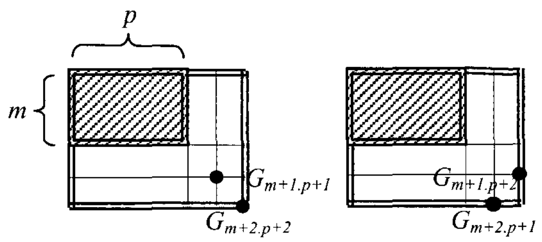

Fig. 12. Lattice diagram with 2 dynamic elements in two crossed minimal positions.

(least common multiplying denominator) of all minors in $G(s)^{SFG} \in (s)^{(m+1)\times(p+1)}$).

If the order of $G_{m+1,p+1}(s)$ is "2", one can make further input-output augmentation in the two crossed minimal positions, $\{G_{m+1,p+1}(s), G_{m+2,p+2}(s)\}$ or $\{G_{m+1,p+2}(s), G_{m+2,p+1}(s)\}$ like Fig. 12,

In the same way, if the order of $G_{m+1,p+1}(s)$ is "d" in Fig. 11, one can make input-output augmentations in the "d! number of d-crossed minimal positions" like Fig. 13.

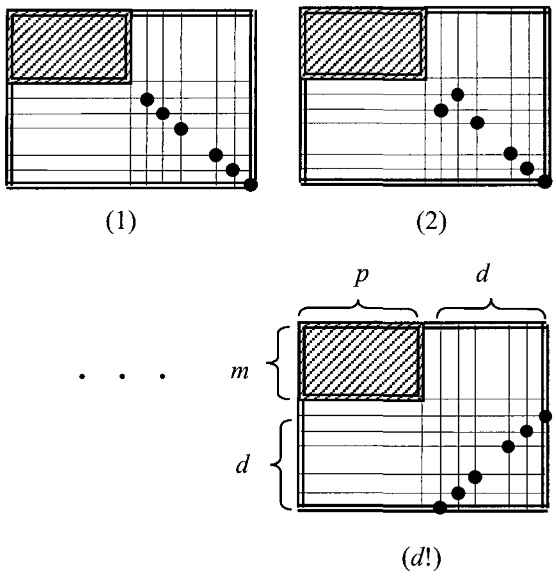

Fig. 13. Lattice diagrams with 1st order dynamic elements in $d!$ number of minimal positions.

As shown in Fig. 11 - Fig. 13, the augmented (input and output) numbers are *maximized*, when all the dynamic elements are *minimized* by 1st order in a d-crossed minimal position. We shall call this decomposition mode by "maximal gain-loop decomposition mode" of dynamic O. F. compensator. Hence we can easily prove that the minimum order dynamic O. F. compensator $H(s)^{SFG.min} \in \mathbb{R}(s)^{p\times m}$ for pole-assignment is earned in the maximal gain-loop decomposition mode of dynamic O. F. compensator.

Lemma 1 (decomposition mode of minimum order dynamic O. F. compensator): The pole-assignment conditions by minimum order dynamic O. F. compensator in linear system *are equivalently obtained* by the pole-assignment conditions by static O. F. compensator in augmented linear system which are augmented into the mode of maximal gain-loop decomposition of the dynamic. O. F. compensator.

Proof: In maximal gain-loop decomposition mode of dynamic O. F. compensator, the numbers of input and output of system are "maximally augmented" since the dynamic elements locate in a minimal position (of a well-posed $d \times d$ sublattice), and the orders of dynamic elements are minimized by 1st orders. Hence we need to check following 2 necessary conditions which naturally cover sufficient conditions.

i) Dimensional necessary condition: In this decomposition mode, the number (#) of O. F. gain *variables* is maximized and the number (#) of O. F. gain *equations* is minimized so that the necessary condition of (complete) static O. F. pole-assignment on $\mathbb{R}$ and $\mathbb{C}$ (obtained by dimension argument in nonlinear O. F. equations),

$$\#\text{variables} \geq \#\text{equations} \tag{8}$$

is well satisfied within minimum number of dynamic elements, i.e., *minimum dynamic order*.

ii) Invariant necessary condition: By SFG theory, the invariant necessary condition of (complete) static O. F. pole-assignment of augmented systems in maximal gain-loop decomposition mode of dynamic elements, "full-rank of Plücker submatrices of L^{aug}" on $\mathbb{R}$ and $\mathbb{C}$ (obtained by dimension argument in linear vector O. F. equation $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$), *is always preserved as* the invariant necessary condition of (complete) minimum order dynamic O. F. pole-assignment; for the meaning of the symbols, L^{aug}, $\boldsymbol{k}^{aug}$ and $\boldsymbol{a}^{aug}$, refer to Definition 4 and Section 5.2 in next section. i) and ii) complete the proof. □

4.2. Structural quantitative relationship

From Lemma 1, one can also obtain immediately Theorem 1, which shows the "structural quantitative relationship" between static O. F. and minimum order dynamic O. F. for pole-assignment.

Theorem 1 (structural quantitative relationship): The minimum d-th order dynamic O. F. compensator for pole-assignment in original m-input, p-output, n-th order linear systems *is quantitatively decomposed into* static O. F. compensator and its associated d number of 1st-order dynamic elements in augmented $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order linear systems.

Proof: From Lemma 1, the minimum d-th order dynamic O. F. compensator for pole-assignment is decomposed into the mode of maximal gain-loop decomposition.

Thus, the minimum d-th order dynamic O. F. compensator is quantitatively decomposed into static O. F. compensator and its associated d number of 1st-order dynamic elements in augmented $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order linear systems, by definition of maximal gain-loop decomposition. □

Remark 3: As mentioned in Section 1, this Theorem 1 is significant, because any outcomes (like generic or complete necessary and/or sufficient condition, dynamic invariants, canonical forms, etc.) regarding to minimum order dynamic O. F. pole-assignment can *be induced directly* from the pre-known outcomes (of generic or complete necessary and/or sufficient condition, static invariants, canonical forms, etc.) regarding to static O. F. pole-assignment, or *vice versa*.

Remark 4: From the Theorem 1, a *general gain formula* for computation of minimum order dynamic O. F. compensator $H(s)^{SFG.min} \in \mathbb{R}(s)^{p\times m}$ for pole-assignment is derived (or induced) from the static O. F. compensator $K^{SFG.aug} \in \mathbb{R}^{(p+d)\times(m+d)}$ and associated d number of arbitrary 1st order dynamic elements. Considering a diagonally descending minimal position in Fig. 13(1), the general gain formula for computation of each element $H_{ji}(s)^{SFG..min}$ of $H(s)^{SFG..min}$ is obtained by

$$-H_{ji}(s)^{SFG\,min} = k_{ji} + \Big(k_{j,m+1}\frac{e_1(s)}{1-e_1(s)W_1^{ij}(s)}k_{p+1,i} + \dots$$
$$+ k_{j,i+d}\frac{e_d(s)}{1-e_d(s)W_d^{ij}(s)}k_{p+d,i} \Big)$$

where $W_1^{ij}(s)$, ... , $W_d^{ij}(s)$ are provided by

$$W_1^{ij}(s) = k_{p+1,\,m+1} + \sum_{\lambda=2}^{d} k_{p+1,m+\lambda}\cdot\frac{e_\lambda(s)}{1-e_\lambda(s)k_{p+\lambda,\,m+\lambda}}\cdot k_{p+\lambda,m+1}$$

$$\cdots\cdots\cdots\cdots$$

$$W_d^{ij}(s) = k_{p+d,\,m+d} + \sum_{\lambda=1}^{d-1} k_{p+d,m+\lambda}\cdot\frac{e_\lambda(s)}{1-e_\lambda(s)k_{p+\lambda,\,m+\lambda}}\cdot k_{p+\lambda,m+d}$$

for all j = 1, ... , p and i = 1, ... , m (this derivation is not hard if the SFG theory is precisely employed).

4.3. Induction of (new) N and S condition

From Theorem 1, we can induce a (new) N and S condition of *generic* minimum order dynamic O. F. pole-assignment on $\mathbb{C}$ from the well-known N and S condition of *generic* static O. F. pole-assignment on $\mathbb{C}$, $mp \ge n$ in strictly proper systems.

Theorem 2 (induction of new N and S condition): In m-input, p-output, n-th order strictly proper linear systems, $(m+d)(p+d) \ge n+d$ or $(m+d)(p+d) \ge n+d+1$ is induced as new N and S condition of *generic* minimum d-th order dynamic O. F. pole-assignment on complex field $\mathbb{C}$.

Proof: From Theorem 1 (structural quantitative relationship), new N and S condition of *generic* minimum d-th order dynamic O. F. pole-assignment in strictly proper systems on $\mathbb{C}$ *can be directly induced* from the N and S condition of *generic* static O. F. pole-assignment $mp \ge n$ on $\mathbb{C}$ in strictly proper systems.

i) If all the d number of 1st order dynamic elements are to be "strictly proper", new N and S condition of *generic* minimum d-th order dynamic O. F. pole-assignment in strictly proper systems on $\mathbb{C}$ *is directly induced* by $(m+d)(p+d) \ge n+d$.

ii)If all or some of the d number of 1st order dynamic elements are to be "proper", the N and S condition is directly induced by $(m+d)(p+d) \ge n+d+1$ [12,Theorem 5.3(c)]. i) and ii) complete the proof. □

As mentioned in Section 1, the *induced* N and S condition of generic minimum order dynamic O. F. pole-assignment on $\mathbb{C}$, $(m+d)(p+d) \ge n+d$ or $(m+d)(p+d) \ge n+d+1$ are dissimilar with *original* N and S condition of generic minimum order dynamic O. F. pole-assignment on $\mathbb{C}$, $mp + d(m+p) \ge n+d$ by Rosenthal [4, Theorem 5.11]. We shall call this dissimilarity by "mismatching problem" *between two (incomplete) generic outcomes* on static O. F. pole-assignment condition, $mp \ge n$ and minimum order dynamic O. F. pole-assignment condition, $mp + d(m+p) \ge n+d$ on $\mathbb{C}$.

4.4. Induction of dynamic Grassmann invariant

From Theorem 1, we can induce and define dynamic Grassmann invariant from the (static) Grassmann invariant in augmented static O. F. linear systems as following.

Definition 4 (dynamic Grassmann invariant, L^{aug}): The Grassmann invariant, so-called Plücker matrix L^{aug} in *augmented* $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order static O. F. linear systems (which is augmented by d number of 1st-order dynamic elements) is defined by "(minimum order) dynamic Grassmann invariant" in m-input, p-output, n-th order dynamic O. F. linear systems.

5. DECISION OF 1ST ORDER DYNAMIC ELEMENTS

The decision of "properness or strictly properness" of the (arbitrary) 1st order dynamic elements is made through the rank analysis of dynamic Grassmann invariant L^{aug} whose construction algorithm is provided in Section 5.2.

5.1. Complete parameterization

The Grassmann invariant (so-called Plücker matrix) L is theoretically derived from the "polynomial Grassmann-representative" of column-spanned polynomial vector space of MFD (matrix fraction description) of closed-loop transfer function matrix by Giannakopoulos and Karcanias [11] (refer to Appendix. B). Whence it is proved that the Grassman invariant is to be a complete invariant of the column-spanned polynomial vector space [11, 12(Theorem 4.1)]. And it is also shown that from the linear vector equation $L\boldsymbol{x} = \boldsymbol{a}$, the full-rank condition of the (first column and first row curtailed) Plücker submatrix $L^{sub} \in \mathbb{R}^{n\times\sigma}$,

$$\text{rank}\, L^{sub} = n \tag{9}$$

is to be a necessary condition of *complete* O. F. pole-assignment as well as a well-known necessary condition $mp \ge n$ in strictly proper systems [12, Corollary 5.1.1] (where $\sigma = \binom{m+p}{m}-1$). And in the sameway the full-rank of the (first column curtailed) Plücker submatrix $L^{sub'} \in \mathbb{R}^{(n+1)\times\sigma}$, i.e.,

$$\text{rank}\, L^{sub'} = n+1 \tag{10}$$

is to be a necessary condition of *complete* O. F. pole-assignment as well as a well-known necessary condition $mp \geq n+1$ in proper systems[12, Corollary 5.3.1]. Thus, we can check possible none pole-assignability on $\mathbb{C}$ by rank test of Plücker submatrix L^{sub} in the generic static O. F. pole-assignment condition $mp \geq n$ in strictly proper systems. In the same way, from Theorem 1, we can also check none pole-assignability on $\mathbb{C}$ by rank test of Plücker submatrix $L^{aug.sub}$ in the generic minimum order dynamic O. F. pole-assignment condition $mp + d(m+p) \geq n+d$ and $(m+d)(p+d) \geq n+d$ in "augmented static O. F. strictly proper" systems, and by rank test of Plücker submatrix $L^{aog.sub'}$ in $(m+d)(p+d) \geq n+d+1$ in "augmented static O. F. proper" systems.

5.2. Construction of dynamic Grassmann invariant

The internal structure of dynamic Grassmann invariant L^{aug} (as Grassmann invariant in augmented system) is presented in the linear vector product formula $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$ in Fig. 14 [12], where $\boldsymbol{k}^{aug} = [1\ k_{11}\ k_{21} \ldots k_{p+d,m+d}\ k_{i1} \ldots k_{ir}]^t$ indicates O. F. gain vector whose elements consists of Plücker coordinates in projective space $\mathbb{R}^{\sigma^*}$; $q^* = (p+d)\times(m+d)$, $r = \sigma^* - q^*$, and $\boldsymbol{a}^{aug} = [1\ a_1 \ldots a_{n+d}]^t$ indicates arbitrary real coefficient vector of closed-loop characteristic polynomial $D_{C\text{-}L}{}^{aug}(s)$ of augmented system (where $\sigma^* = \binom{m+d+p+d}{m+d} - 1$). And $D_{C\text{-}L}{}^{aug}(s)$ and $D_{O\text{-}L}{}^{aug}(s)$ indicate real coefficient vector columns of open-loop and closed-loop characteristic polynomials of augmented systems, respectively. The $-N_{11}, \ldots, -N_{mp}$ and $-n_1, \ldots, -n_d$ indicate the "-1" multiplied real coefficient vector columns of numerators of augmented transfer function matrix and dynamic elements $e_1(s), \ldots, e_d(s)$, respectively (whose all denominators are normalized by LCD, i.e., open-loop characteristic polynomial $D_{O\text{-}L}{}^{aug}$)[16]. And $I_{i1}, \ldots, I_{ir}$ indicate the real coefficient vector columns of the nonlinear interacting factors of augmented transfer function matrix (whose denominator is also normalized by $D_{O\text{-}L}{}^{aug}$). Under SFG analysis, the nonlinear interacting factors of augmented static O. F. linear (MIMO) systems are constructed by

$$\begin{aligned}
I_{i1}:\ & 1^{st}\,(-1)^2\,\{2\times 2 \text{ minors of } G^{SFG,aug}(s)\} \\
& \cdot \{\text{correspondent } 2\times 2 \text{ minors of } K^{SFG,aug}\} \\
& \cdots\cdots\cdots \\
I_{ir}:\ & r^{th}\,(-1)^z\,\{z\times z \text{ minors of } G^{SFG,aug}(s)\} \\
& \cdot \{\text{correspondent } z\times z \text{ minors of } K^{SFG,aug}\}
\end{aligned} \tag{11}$$

(where $r = \sigma^* - q^*$, and $2 \leq z \leq N$, $N = 2, \ldots, \min\{m, p\}$).

5.3. Decision of 1st order dynamic elements

Prior to application of Theorem 1 (structural quantitative relationship) to static O. F. pole-assignment condition for direct induction of minimum order dynamic O. F. pole-assignment condition, we need to investigate whether there is certain intrinsic rank reduction effect, depending upon the "properness and strictly properness" of the arbitrary 1st order dynamic elements. We shall check possible intrinsic rank reduction effect by counting the rank of Plücker submatrices $L^{aug.sub} \in \mathbb{R}^{(n+d)\times\sigma^*}$ and $L^{aug.sub'} \in \mathbb{R}^{(n+d+1)\times\sigma^*}$ in the maximum number of (possible linearly independent) columns *under assumption that* row vectors are linearly independent, because the degeneracy of augmented transfer function matrix of augmented static O. F. system is heavily related with the number of nonzero columns but is hardly related with the number of nonzero rows.

Lemma 2 (column rank reduction by *strictly proper* dynamic elements in $d \geq 2$): In maximal

$$\begin{array}{c}
\begin{array}{cccccccccc} D_{O\text{-}L}{}^{aug} & -N_{11} & -N_{21} & \ldots & -N_{pm} & -n_1 \ldots -n_d & I_{i1} & I_{i2} & \ldots & I_{ir} \end{array} \\
\begin{array}{c} s^{n+d} \\ s^{n+d-1} \\ \cdot \\ \cdot \\ \cdot \\ s \\ 1 \end{array}
\left[\begin{array}{c|c|c} \begin{array}{c} 1 \\ b_1 \\ \cdot \\ \cdot \\ \cdot \\ b_{n+d-1} \\ b_{n+d} \end{array} & \begin{array}{c} \textit{linear} \\ \textit{terms' coefficients} \end{array} & \begin{array}{c} \textit{nonlinear (interacting)} \\ \textit{terms' coefficients} \end{array} \end{array}\right]
\begin{bmatrix} 1 \\ k_{11} \\ k_{21} \\ \cdot \\ \cdot \\ k_{p+d,\,m+d} \\ k_{i1} \\ \cdot \\ k_{ir} \end{bmatrix}
= \begin{bmatrix} 1 \\ a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n+d-1} \\ a_{n+d} \end{bmatrix}
\end{array}$$

with the right-hand column labelled $D_{C\text{-}L}{}^{aug}$.

Fig. 14. Internal structure of $L^{aug}\boldsymbol{k}^{aug} = \boldsymbol{a}^{aug}$.

gain-loop decomposition of dynamic O. F. compensator, if the arbitrary 1st order dynamic elements of augmented static O. F. systems are *strictly proper* and $d \geq 2$, then the column rank of Plücker submatrix *is intrinsically reduced* by

$$1 + [(mp)^{\wedge} + \Sigma N{\times}N \text{ minor}] \cdot \lfloor d/2 \rfloor$$

by strictly proper dynamic elements (where $(mp)^{\wedge}$($\leq mp$) indicates actual number of nonzero transfer functions, and $\Sigma N{\times}N$ minor indicates sum of all kinds of $N{\times}N$ minors for $N = 2, \ldots, z$ ($\leq \min\{m, p\}$), and $\lfloor x \rfloor$ indicates the nearest integer lower than or equal to x).

Proof: See the Appendix A.

Lemma 3 (no column rank reduction in *proper* dynamic elements in $d \geq 2$): In maximal gain-loop decomposition of dynamic O. F. compensator, if the arbitrary 1st order dynamic elements of augmented static O. F. systems are *proper* and $d \geq 2$, then there is *no* intrinsic column rank reduction of Plücker submatrix by the proper dynamic element.

Proof: See the Appendix A.

In similar manner of Lemma 2 and Lemma 3, we can prove that in $d = 1$ case, there is no intrinsic column rank reduction effect in strictly proper systems (unlike $d = 2$ case).

Lemma 4 (no column rank reduction by *strictly proper* dynamic element in $d = 1$): In maximal gain-loop decomposition of dynamic O. F. compensator for pole-assignment, if the arbitrary 1st order dynamic elements of augmented static O. F. systems are *strictly proper* and $d = 1$, then there is *no* intrinsic column rank reduction of Plücker submatrix by the strictly proper dynamic element.

Proof: See the Appendix A.

(For the relation of column or row rank with matrix rank, see the Appendix C.)

6. ANALYSIS OF MISMATCHING PROBLEM UNDER INVARIANT RANK CONDITION

We shall check the rank of real matrix $L^{aug.sub} \in {}^{(n+d)\times\sigma^*}$ by calculation of maximal number of columns — the numbers (#) of linear terms and nonlinear terms, under assumption that row vectors are linearly independent, in a lattice diagram of augmented $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order linear static O. F. systems. For convenience, we shall consider a diagonally descending minimal position as shown in Fig. 13(1).

6.1. SISO system case (where $m = 1, p = 1$)

From Fig. 14 and Fig. 15, the number of nonzero columns in dynamic Grassmann invariant (which should be greater than and equal to the system order $n{+}d$ or $n{+}d{+}1$ for the full-rank of $L^{aog.sub}$ or $L^{aog.sub'}$) is calculated by

$$\begin{aligned} &\#\text{linear terms} + \#2{\times}2 \text{ minors} + \#3{\times}3 \text{ minors} + \ldots + \#(1{+}d){\times}(1{+}d) \text{ minors} \\ &= (1{+}d) + (1{+}d)\, d/2! + (1{+}d)d(d{-}1)/3! + \ldots + (1{+}d)!/(1{+}d)! \\ &\geq n{+}d \text{ or } n{+}d{+}1 \end{aligned} \tag{12}$$

6.2. SIMO system case (where $m = 1, p > 1$)

From Fig. 14 and Fig. 16, the number of nonzero columns in dynamic Grassmann invariant (which should be greater than and equal to the system order $n{+}d$ or $n{+}d{+}1$ for the full-rank of $L^{aog.sub}$ or $L^{aog.sub'}$) is calculated by

$$\begin{aligned} &\#\text{linear terms} + \#2{\times}2 \text{ minors} + \#3{\times}3 \text{ minors} + \ldots + \#(1{+}d){\times}(1{+}d) \text{ minors} \\ &= (p{+}d) + \{ p\,[d{+}d(d{-}1)/2! + d(d{-}1)(d{-}2)/3! + \ldots + d!/d!] + d(d{-}1)/2! + \ldots + d!/d!\} \\ &\geq n{+}d \text{ or } n{+}d{+}1 \end{aligned} \tag{13}$$

6.3. MISO system case (where $m > 1, p = 1$)

In the same way with SIMO case above, the number of nonzero columns is calculated by

$$\begin{aligned} &(m{+}d) + \{m[d{+}d(d{-}1)/2! + d(d{-}1)(d{-}2)/3! + \ldots + d!/d!] + d(d{-}1)/2! + \ldots + d!/d!\} \\ &\geq n{+}d \text{ or } n{+}d{+}1. \end{aligned} \tag{14}$$

6.4. MIMO system case (where $m \geq 2, p \geq 2$)

In m-input, p-output MIMO systems case, the number by nonzero columns is obtained by

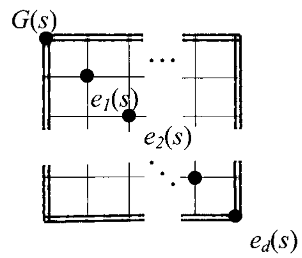

Fig. 15. Lattice diagram of augmented static O. F. systems in SISO system case.

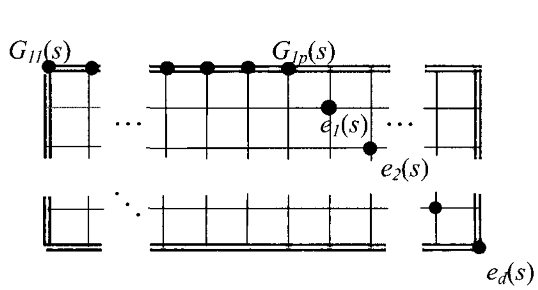

Fig. 16. Lattice diagram of augmented static O. F. systems in SIMO system case.

$$((mp)^{\wedge}+d) \;+\; \{(mp)^{\wedge}\,[d+d(d\text{-}1)/2! + d(d\text{-}1)(d\text{-}2)/3! + \ldots + d!/d!] \;+\; d(d\text{-}1)/2! + \ldots + d!/d!\} \;+\; \{(mp)^{\wedge}\}^{\Sigma} \;\geq\; n{+}d \;\text{ or }\; n{+}d{+}1 \qquad (15)$$

(where $(mp)^{\wedge}$ implies $(mp)^{\wedge} = mp$ in nondegenerate systems having no zero-transfer function(s), and implies $(mp)^{\wedge} < mp$ in degenerate systems having some zero transfer function(s), and $\{(mp)^{\wedge}\}^{\Sigma} = \{\Sigma\ N{\times}N \text{ minor} \mid N = 2, \ldots, z\ (\leq \min\{m, p\})\}$ indicates sums of all kinds of nonlinear interacting terms in m-input, p-output MIMO systems).

From (12) – (15), we can obtain total configuration on the mismatching problem in Table 1, where the degenerate case of MIMO systems is excluded where the conservatism or empty set case is not uniquely determined. From Table 1 the mismatched data are revealed as follows.

In CASE I ($\min\{m, p\} = 1$ systems with $d = 1$): The induced one $(m+d)(p+d) \geq n+d$ provides *no intersections* for pole-assignment in the marginal area of $mp + d(m+p) \sim (m+d)(p+d)$. Meanwhile the original one $mp + d(m+p) \geq n+d$ provides complex or real intersections for pole-assignment.

In CASE II ($\min\{m, p\} = 1$ systems with $d \geq 2$): In the original one $mp + d(m+p) \geq n+d$ *is conservative* by amount of d^2-1 in general. Meanwhile the induced one $(m+d)(p+d) \geq n+d+1$ provide complex or real intersections for pole-assignment in non-conservative way (except $m = p = 1, d = 2$).

In CASE III (MIMO system with $d \geq 1$): The original one $mp + d(m+p) \geq n+d$ *is conservative* by amount of d^2 (if $d = 1$) and d^2-1 (if $d \geq 2$). Meanwhile the induced one $(m+d)(p+d) \geq n+d$ and $(m+d)(p+d) \geq n+d+1$ provide complex or real intersections for pole-assignment in non-conservative way.

7. CONCLUSIONS

Major outcomes of this paper are summarized as follows.

1) It is proved using a lattice diagram analysis that minimum d-th order dynamic O. F. compensator for pole-assignment in m-input, p-output, n-th order linear systems *is decomposed into* static O. F. compensator and associated d number of arbitrary 1st-order, strictly proper or proper dynamic elements in $(m+d)$-input, $(p+d)$-output, $(n+d)$-th order linear systems.

2) From 1), it is clearly revealed that two well-known necessary and sufficient conditions $mp \geq n$ as *generic* static O. F. pole-assignment and $mp + d(m+p) \geq n+d$ as *generic* minimum d-order dynamic O. F. pole-assignment on complex field $\mathbb{C}$ do *not match up* each other in strictly proper linear systems. Total configuration of the mismatched data is presented in Table 1.

Further studies. From the mismatched data in 2), following questions are naturally occurred as future issues:

3) Is the well-known strong sufficient condition of *generic* static O. F. pole-assignment on real field $\mathbb{R}$ $mp > n$ to be the sufficient condition of *complete* static O. F. pole-assignment in strictly proper systems, combining with the full-rank condition of the Plücker submatrix of Grassmann invariant L ? — vice versa, is the $(m+d)(p+d) > n+d$ or $(m+d)(p+d) > n+d+1$ to be the sufficient condition of *complete* minimum d-th order dynamic O. F. pole-assignment on $\mathbb{R}$ in strictly proper systems, combining with full-rank condition of

Table 1. Total configuration of mismatched data in generic pole-assignabilities.

			original N and S condition on $\mathbb{C}$ $mp + d(m+p) \geq n+d$	*induced N and S condition on* $\mathbb{C}$ $(m+d)(p+d) \geq n+d+1$; $d \geq 2$ $(m+d)(p+d) \geq n+d$; $d = 1$
SISO systems ($m = 1, p = 1$)	$d = 1$		non-conservative	empty set in $n = 4$
	$d = 2$		conserv. by $d^2 - 3$	empty set in $n = 6$
	$d = 3$		conserv. by $d^2 - 2$	none (empty set)
	$d \geq 4$		conserv. by $d^2 - 1$	none
SIMO systems ($m = 1, p \geq 2$)	$d = 1$	$p \geq 2$	non-conservative	empty set in $n = 2p+1$
	$d = 2$	$p = 2$	conserv. by $d^2 - 1$	none
		$p \geq 3$	conserv. by $d^2 - 1$	none
	$d \geq 3$	$p \geq 2$	conserv. by $d^2 - 1$	none
MISO systems ($m \geq 2, p = 1$)	$d = 1$	$m \geq 2$	non-conservative	empty set in $n = 2m+1$
	$d = 2$	$m = 2$	conserv. by $d^2 - 1$	none
		$m \geq 3$	conserv. by $d^2 - 1$	none
	$d \geq 3$	$m \geq 2$	conserv. by $d^2 - 1$	none
MIMO systems (*nondegenerate*)	$d = 1$		conserv. by d^2	none
	$d = 2$		conserv. by $d^2 - 1$	none
	$d \geq 3$		conserv. by $d^2 - 1$	none

the Plücker submatrix of dynamic Grassmann invariant L^{aug} ?

4) Is the Grassmann invariant L to *be a complete invariant* (or a canonical form) under static O. F. group action for system poles ? — vice versa, is the dynamic Grassmann invariant L^{aug} to be *a complete dynamic invariant* (or a canonical form) under minimum order dynamic O. F. group action for system poles[13,14] ?

APPENDIX A

Proof of Lemma 2: Consider arbitrary 1st order dynamic elements located in a diagonally descending d-crossed minimal position like Fig. A.1 (where $I^{d\times d}(e_1, \dots, e_d)$: the interacting-factor column formulated by all dynamic elements $e_1(s), \dots, e_d(s)$, $\{I^{d\times d}(G_{ij}, e_1, \dots, e_{h-1}, e_{h+1}, \dots, e_d)\}$: the set of interacting-factor columns formulated by the elements $G_{ij}(s), e_1(s), \dots, e_{h-1}(s), e_{h+1}(s), \dots, e_d(s)$ for $h = 2, \dots, d-1$, and $i = 1, \dots, m$ and $j = 1, \dots, p$, $\{I^{(d+1)\times(d+1)}(G_{ij}, e_1, \dots, e_d)\}$: the set of interacting-factor columns formulated by the elements $G_{ij}(s)$, $e_1(s), \dots, e_d(s)$ for $i = 1, \dots, m$ and $j = 1, \dots, p$).

Hence let the strictly proper dynamic elements be $e_1(s) = 1/(s+\alpha_1)$ $(= n_1(s)/D_{O\text{-}L}{}^{aug}(s)), \dots, e_d(s) = 1/(s+\alpha_d)$ $(= n_d(s)/D_{O\text{-}L}{}^{aug}(s))$ and transfer function elements be $G_{11}(s) = n_{11}(s)/d_{11}(s), \dots, G_{mp}(s) = n_{mp}(s)/d_{mp}(s)$ (where $n_{ij}(s)/d_{ij}(s)$ are irreducible without common factors for all i and j, and $(s+\alpha_1), \dots, (s+\alpha_d)$ are relatively prime over $n_{11}(s), \dots, n_{mp}(s)$; $\alpha_i \neq \alpha_j$).

Then *possible* intrinsic column rank reductions "by dynamic elements" need to be checked in following 3 parts from the construction algorithm in (11).

1) *Linear dependency between* $-n_1, \dots, -n_d$ *and* $I^{d\times d}(e_1, \dots, e_d)$.

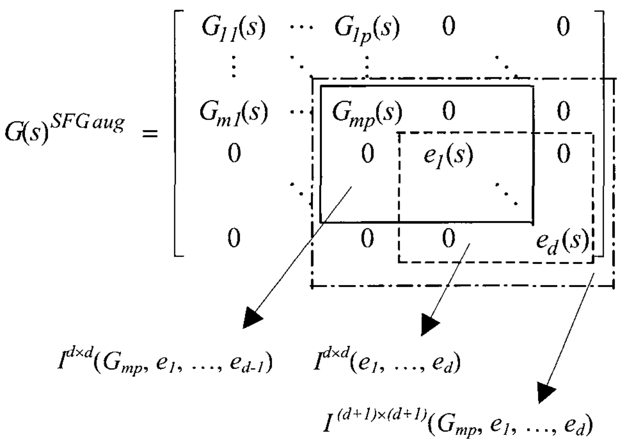

Fig. A.1. Augmented transfer function matrix and 3 kinds of interacting factors.

(i) $d = 2$ case: From Fig. A.1 and (11), following linear and nonlinear interacting terms are obtained.

- n_1 : $-d_{ij}(s)(s+\alpha_2)D_{(ij)}(s)$ $(= -(s+\alpha_2)\, D_{O\text{-}L}{}^{aug}(s)$, i.e., $d_{ij}(s)D_{(ij)}(s) := D_{O\text{-}L}{}^{aug}(s)$ for all i and j),
- n_2 : $-d_{ij}(s)(s+\alpha_1)D_{(ij)}(s)$ $(= -(s+\alpha_1)\, D_{O\text{-}L}{}^{aug}(s))$,

$I^{2\times 2}(e_1, e_2)$: $d_{ij}(s)D_{(ij)}(s)$ $(= D_{O\text{-}L}{}^{aug}(s))$.

Hence column rank *is reduced by* "1", because the $-n_1 - (-n_2)$ and $I^{2\times 2}$ are linearly dependent.

(ii) $d \geq 3$ case: The linear dependency between $-n_1, \dots, -n_d$ and $I^{d\times d}(e_1, \dots, e_d)$ is not found.

2) *Linear dependency between* $\{I^{d\times d}(G_{ij}, e_1, \dots, e_{h-1}, e_{h+1}, \dots, e_d)\}$ and $\{I^{(d+1)\times(d+1)}(G_{ij}, e_1, \dots, e_d)\}$.

(i) $d = 2$ case: From (11), following nonlinear interacting terms are obtained,

$I^{2\times 2}(G_{ij}, e_2)$: $n_{ij}(s)(s+\alpha_1)D_{(ij)}(s)$,
$I^{2\times 2}(G_{ij}, e_1)$: $n_{ij}(s)(s+\alpha_2)D_{(ij)}(s)$,
$I^{3\times 3}(G_{ij}, e_1, e_2)$: $-n_{ij}(s)D_{(ij)}(s)$

Hence column rank *is reduced by* "1", because $I^{2\times 2}(G_{ij}, e_2) - I^{2\times 2}(G_{ij}, e_1)$ and $I^{3\times 3}(G_{ij}, e_1, e_2)$ are linearly dependent for all i and j.

(ii) $d \geq 3$ case: The linear dependency between $\{I^{d\times d}(G_{ij}, e_1, \dots, e_{h-1}, e_{h+1}, \dots, e_d)\}$ and $\{I^{(d+1)\times(d+1)}(G_{ij}, e_1, \dots, e_d)\}$ is well found, and the column rank *is reduced by* $\lfloor d/2 \rfloor$ over each $G_{ij}(s)$ because any two interacting-factor columns $I^{2\times 2}(G_{ij}, e_a) - I^{2\times 2}(G_{ij}, e_b)$ and $I^{3\times 3}(G_{ij}, e_a, e_b)$ are linearly dependent for all i and j, and for all $a, b = 1, \dots, d$; $a \neq b$ (where $\lfloor x \rfloor$ indicates the nearest integer lower than or equal to x).

3) *Linear dependency between* $\{I^{(d+N-1)\times(d+N-1)}(N{\times}N$ *minor*, $e_1, \dots, e_{h-1}, e_{h+1}, \dots, e_d)\}$ and $\{I^{(d+N)\times(d+N)}$ $(N{\times}N$ *minor*, $e_1, \dots, e_d)\}$.

(i) $d = 2$ case: In the same way of 2), column rank *is reduced by* "1", because $I^{(1+N)\times(1+N)}$ $(N{\times}N$ *minor*, $e_2)$ and $I^{(1+N)\times(1+N)}(N{\times}N$ *minor*, $e_1)$ and $I^{(2+N)\times(2+N)}(N{\times}N$ *minor*, $e_1, e_2)$ are linearly dependent for all $N = 2, \dots, z$ $(\leq \min\{m, p\})$ in $G(s)^{SFG}$.

(ii) $d \geq 3$ case: In the same way of 2), column rank *is reduced by* $\lfloor d/2 \rfloor$ over each $N{\times}N$ *minor* in $G(s)^{SFG}$, because any two interacting-factor columns $I^{(1+N)\times(1+N)}(N{\times}N$ *minor*, $e_a)$ and $I^{(1+N)\times(1+N)}(N{\times}N$ *minor*, $e_b)$ and $I^{(2+N)\times(2+N)}(N{\times}N$ *minor*, $e_a, e_b)$ are linearly dependent for all $N = 2, \dots, z$ $(\leq \min\{m, p\})$, and for all $a, b = 1, \dots, d$; $a \neq b$.

From 1) - 3), the column rank of Plücker submatrix is reduced by $1 + [(mp)^{\wedge} + \Sigma\, N{\times}N \text{ minor}] \cdot \lfloor d/2 \rfloor$ (where $(mp)^{\wedge}$ $(\leq mp)$ indicates actual number of nonzero transfer functions and $N = 2, \dots, z$ $(\leq \min\{m, p\})$.

Proof of Lemma 3: (We shall prove in the same manner as the Lemma 2, setting $e_1(s) = (s+\beta_1)/(s+\alpha_1)$ $(= n_1(s)/D_{O\text{-}L}^{aug}(s))$, … , $e_d(s) = (s+\beta_d)/(s+\alpha_d)$ $(= n_d(s)/D_{O\text{-}L}^{aug}(s))$, where $(s+\alpha_1)$, … , $(s+\alpha_d)$ are relatively prime over $n_{11}(s)$, … , $n_{mp}(s)$, and $(s+\beta_1)$, … , $(s+\beta_d)$ are relatively prime over $d_{11}(s)$, … , $d_{mp}(s)$; $\beta_i \neq \beta_j$).

1) *Linear dependency between* $-n_1$, … , $-n_d$ *and* $I_i^{d\times d}(e_1, \ldots, e_d)$.

(*i*) $d = 2$ case: From Fig. A.1 and (11), following linear and nonlinear interacting terms are obtained,

$-n_1$: $-d_{ij}(s)(s+\alpha_2)(s+\beta_1)D_{(ij)}(s)$ $(= -(s+\alpha_2)(s+\beta_1)D_{O\text{-}L}^{aug}(s))$,

$-n_2$: $-d_{ij}(s)(s+\alpha_1)(s+\beta_2)D_{(ij)}(s)$ $(= -(s+\alpha_1)(s+\beta_2)D_{O\text{-}L}^{aug}(s))$,

$I^{2\times 2}(e_1, e_2)$: $d_{ij}(s)D_{(ij)}(s)$ $(=D_{O\text{-}L}^{aug}(s))$.

Hence column rank is *not* reduced, because the $-n_1 - (-n_2)$ and $I^{2\times 2}(e_1, e_2)$ are linearly independent.

(*ii*) $d \geq 3$ case: The linear dependency by linear column operations between $\{I^{d\times d}(G_{ij}, e_1, \ldots, e_{h-1}, e_{h+1}, \ldots, e_d)\}$ and $\{I^{(d+1)\times(d+1)}(G_{ij}, e_1, \ldots, e_d)\}$ is *not* found by relative primeness of $(s+\alpha_1)$, … , $(s+\alpha_d)$ and $(s+\beta_1)$, … , $(s+\beta_d)$.

2) *Linear dependency between* $\{I^{d\times d}(G_{ij}, e_1, \ldots, e_{h-1}, e_{h+1}, \ldots, e_d)\}$ and $\{I^{(d+1)\times(d+1)}(G_{ij}, e_1, \ldots, e_d)\}$.

(*i*) $d = 2$ case: From Fig. A.1 and (11), following terms are obtained,

$I^{2\times 2}(G_{ij}, e_2)$: $n_{ij}(s)(s+\alpha_1)(s+\beta_2)D_{(ij)}(s)$,

$I^{2\times 2}(G_{ij}, e_1)$: $n_{ij}(s)(s+\alpha_2)(s+\beta_1)D_{(ij)}(s)$,

$I^{3\times 3}(G_{ij}, e_1, e_2)$: $-n_{ij}(s)D_{(ij)}(s)$.

Hence column rank is *not* reduced, because $I^{2\times 2}(G_{ij}, e_2) - I^{2\times 2}(G_{ij}, e_1)$ and $I^{3\times 3}(G_{ij}, e_1, e_2)$ are linearly independent for all i and j under any linear column operations.

(*ii*) $d \geq 3$ case: The linear dependency between $\{I^{d\times d}(G_{ij}, e_1, \ldots, e_{h-1}, e_{h+1}, \ldots, e_d)\}$ and $\{I^{(d+1)\times(d+1)}(G_{ij}, e_1, \ldots, e_d)\}$ is *not* found, because $I^{d\times d}(G_{ij}, e_1, \ldots, e_{h-1}, e_{h+1}, \ldots, e_d) - I^{d\times d}(G_{ij}, e_1, \ldots, e_{t-1}, e_{t+1}, \ldots, e_d)$ and $\{I^{(d+1)\times(d+1)}(G_{ij}, e_1, \ldots, e_d)\}$ are linearly independent for all i, j and for all $d = 3, \ldots, d$ where $h \neq t$ under any linear column operations.

3) *Linear dependency between* $\{I^{(d+N-1)\times(d+N-1)}(N\times N$ *minor*, $e_1, \ldots, e_{h-1}, e_{h+1}, \ldots, e_d)\}$ and $\{I^{(d+N)\times(d+N)}$ $(N\times N$ *minor*, $e_1, \ldots, e_d)\}$. In the same way with the proof of 2), the linear dependency is *not* found for all i, j and for all $d = 3, \ldots, d$ under any linear column operations.

From 1) – 3), the column rank of Plücker submatrix is *not* reduced by proper dynamic elements. ◻

Proof of Lemma 4: If $d = 1$, then from Fig. A.1, the $I^{1\times 1}(e_1)$ indicates the numerator column formulated by a dynamic element $e_1(s)$, and $\{I^{2\times 2}(G_{ij}, e_1)\}$ indicates the interacting-factor column formulated by $e_1(s)$ and a transfer function $G_{ij}(s)$.

And let the strictly proper dynamic elements be $e_1(s) = \beta_1/(s+\alpha_1)$ $(= n_1(s)/D_{O\text{-}L}^{aug}(s))$, and the transfer function elements be $G_{ij}(s) = n_{ij}(s)/d_{ij}(s)$ (where $n_{ij}(s)/d_{ij}(s)$ are irreducible without common factors for all i and j, and $(s+\alpha_1)$ are relatively prime over all $n_{ij}(s)$ and $d_{ij}(s)$). Then the *possible* intrinsic column rank reductions *by the strictly proper dynamic elements* "do not exist" by relative primeness of $(s+\alpha_1)$ over all $n_{ij}(s)$ and $d_{ij}(s)$. ◻

APPENDIX B

If W is any nonzero m-dimensional subspace of V, then any nonzero decomposable element (in the exterior product of m vectors in W), $x_1 \wedge \ldots \wedge x_m$, $x_i \in W$, $i = 1, \ldots, m$ is called *a Grassmann-representative* for W.

APPENDIX C

Let M be a $a \times b$ matrix in field F, then *column rank* of M is defined by the maximum number of linearly independent columns of M, and the *row rank* of M is defined by the maximum number of linearly independent rows of M. Actually, these two ranks of a $a \times b$ matrix are always equal, and rank of a $a \times b$ matrix is defined by that number of the two ranks.

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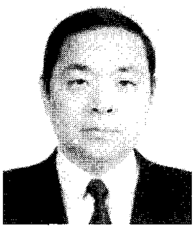

Su-Woon Kim received the B.S. degree in Electrical Engineering (with submajor in Industrial Education) in 1974 and M.S. degree in Electrical Engineering in 1979, respectively from Seoul National University. He served as an instructor from 1980 to 1983 in Ulsan University, and received Ph.D. degree in Control Science and Dynamic Systems from University of Minnesota, Minneapolis in 1996, and then engaged in post-doctoral research at the same school. Since 2003, he has been with Department of Electrical Engineering at Cheju National University. His research interests include mathematical system theory and linear system theory (especially, related problems with pole-assignment and stabilization).