

# Parameter Identification Using Static Compliance Dominant Frequencies

## 정유연성 지배주파수를 이용한 매개변수 인식기법

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### 국문요약

본 논문에서는 정유연성 지배주파수를 이용한 개선된 매개변수 인식기법이 제안되었다. 정유연성 지배주파수를 이용할 경우 주파수응답함수에서 고유주파수 보다 다수의 정보를 추출할 수 있어 매개변수 인식의 성능을 향상시킬 수 있는 장점이 있다. 정유연성 지배주파수를 매개변수 인식에 이용하기 위하여 기존의 고유주파수 민감도에 기반한 구조계 인식기법이 확장되었다. 정유연성 지배주파수의 이용을 통한 매개변수 인식의 성능향상은 수치예제를 통해 증명하였다. 수치예제는 스프링과 질량으로 이루어진 간략 모델이 사용되었으며, 고유주파수만을 이용하여 구한 인식값과 비교한 결과 보다 정확한 매개변수 값의 인식이 가능함을 알 수 있었다.

**주요어** : 정유연성 지배주파수, 매개변수인식법, 고유주파수

### ABSTRACT

This paper presents an improved system identification methodology for structural systems by applying static compliance dominant (SCD) frequencies. The existing sensitivity-based system identification technique is extended to adopt the static compliance dominant frequencies, and the performance of the additional spectral information, i.e., SCD frequencies, is compared with that of the natural frequencies only via a numerical example of a mechanical system. The results of the numerical study indicate that the additional use of the SCD frequencies improves accuracy in system identification problems.

**Key words** : static compliance dominance frequency, system identification, natural frequency

## 1. Introduction

To date, a corpus of interdisciplinary knowledge from physics, mathematics, materials science, applied mechanics, and computer engineering have coalesced to provide reliable nondestructive evaluation for monitoring the health of critical structures. However, nondestructive evaluation would have been an empty promise if applied identification methods were not reliable enough to obtain the confidence level needed to make structural safety or useful life decisions. Even with many system identification (SI) techniques and nondestructive damage evaluation (NDE) algorithms developed so far<sup>(1)</sup>, the SI and the NDE procedures for complex structural systems still present many problems, and improving generality and reliability of the SI and the NDE are paramount and one of the more pressing problems in the civil engineering field.<sup>(2)</sup>

To date, among the spectral quantities usually uti-

lized in SI techniques are natural frequencies, modeshapes, and frequency response functions (FRF). However, while natural frequencies are identified accurately, in real application, only a few frequencies can be extracted with acceptable accuracy, and measured modeshapes and FRFs are usually accurate to within 10% at best.<sup>(3)</sup> Also, even though the direct use of FRF in SI problem can simplify modal analysis procedures, it requires intensive degree of computational works.<sup>(4),(5)</sup> Naturally, this motivated the introduction of another type of quantities in FRF to the problem of identifying structural systems, and the use of the antiresonant frequencies is getting attention.<sup>(6)</sup> The main advantages on the use of antiresonances are like natural frequencies antiresonance frequencies are located along the frequency axis and can be easily and accurately measurable. To date, The antiresonant frequencies have also been applied in the problems of model updating<sup>(7),(8)</sup> and crack detection.<sup>(9),(10)</sup> However, so far no attempt has been made to utilize other information from a FRF than natural and antiresonant frequencies.

In this paper, the use of other spectral information from an FRF, i.e., static compliance domi-

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nant (SCD) frequencies<sup>(11)</sup>, to an SI problem is investigated. The possible use of the SCD frequencies to the SI problem is explored using a sensitivity-based SI technique. The following steps are performed: (1) SCD frequencies that can be possibly utilized in developing more accurate and effective SI techniques is introduced; (2) a sensitivity-based SI technique is extended to accommodate the SCD frequencies; and (3) numerical studies are conducted to investigate the performance of SI using SCD frequencies.

## 2. SCD Frequencies

The frequencies that can be obtained from the same static compliance values in the FRF can be defined as the SCD frequencies. The FRF matrix,  $H(\omega)$ , can be written in terms of the modeshapes as<sup>(12)</sup>:

$$H_{ij}(\omega) = \sum_{k=1}^n \frac{\Phi_{ik} \Phi_{jk}}{(\lambda_k - \omega^2)} \quad (1)$$

where  $\lambda_k$  is the  $k^{\text{th}}$  eigenvalue, and  $\Phi_{ik}$  and  $\Phi_{jk}$  are the elements  $i$  and  $j$  of the  $k^{\text{th}}$  mode shape. From Eq. (1), the  $i - j^{\text{th}}$  transfer FRF can be rewritten as:

$$H_{ij}(\omega) = \sum_{k=1}^n \frac{\Phi_{ik} \Phi_{jk}}{(\omega_k^2 - \omega^2)} \quad (2)$$

where  $\omega_k$  is the  $k^{\text{th}}$  natural frequency. The static compliance can be obtained from Eq. (2) with  $\omega = 0$  as follows:

$$H_{ij}(0) = \sum_{k=1}^n \frac{\Phi_{ik} \Phi_{jk}}{\omega_k^2} \quad (3)$$

For different values of the parameter  $\omega$ , let a function be defined as:

$$\bar{H}_{ij}(\omega) = \sum_{k=1}^n \frac{\Phi_{ik} \Phi_{jk}}{(\omega_k^2 - \omega^2)} - \sum_{k=1}^n \frac{\Phi_{ik} \Phi_{jk}}{\omega_k^2} \quad (4)$$

Then, roots of Eq. (4) can be found by setting the whole expression equals to zero.

$$\bar{H}_{ij}(\omega) = 0 \quad (5)$$

The positive roots of Eq. (5) are the frequencies of the  $i - j^{\text{th}}$  transfer FRF that can yield the same static compliance of the structural system. Those frequen-

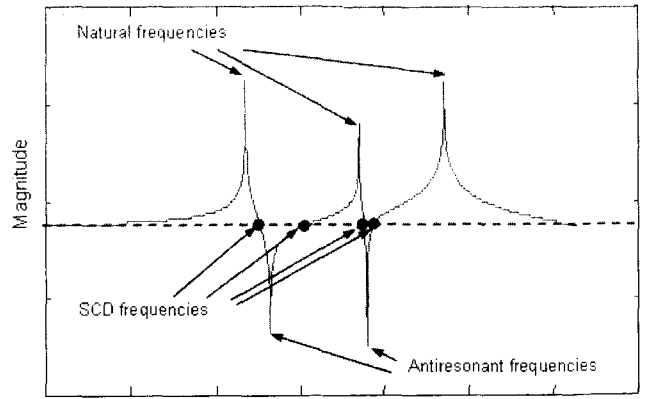


Fig 1 Graphical Representation of SCD Frequencies

cies are defined as SCD frequencies. The graphical representations of the SCD frequencies are depicted in Fig. 1 along with the natural and antiresonant frequencies. In the figure, the corresponding frequencies to the dots represent the SCD frequencies, while the peaks and the dips correspond to the natural and the antiresonant frequencies, respectively. The SCD frequencies can be extracted from experimental FRFs as well as analytical FRFs. To obtain the SCD frequencies from experiments, they can be achieved by a picking technique that picks the same values from the magnitude plot of a given frequency response function at  $\omega = 0$ .

## 3. System Identification Theory

In this section, a sensitivity-based SI method proposed by Stubbs and Osegueda<sup>(13)</sup> is presented and extended to accommodate the SCD frequencies.

Let the mathematical model of a system be defined by  $n$  scalar parameters  $x_i$  in which  $i = 1, 2, \dots, n$ . The parameter,  $x_i$ , can be collected into the vector  $x$ . Using Taylor series expansion

$$f(x + dx) = \sum_{k=0}^n \frac{1}{k!} f^{(k)}(x) dx^k + E_n(dx) \quad (6)$$

where

$$E_n(dx) = \frac{1}{(n+1)!} f^{(n+1)}(\xi) dx^{n+1} \quad (7)$$

in which the point  $\xi$  is between  $x$  and  $x + dx$ .

Considering the first term in Eq. (6) and assuming that  $f(x + dx)$  represents the true solutions for a real system and that  $f(x)$  describes the approximate sol-

utions for a mathematical model, the error,  $e$ , can be expressed as:

$$e = f(x + dx) - f(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i \quad (8)$$

If Eq. (8) is applied to  $m$  functions, the  $j^{\text{th}}$  expression for the error is given by:

$$e_j = \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i \quad (j = 1, 2, \dots, m) \quad (9)$$

Normalizing Eq. (9) by dividing by  $f_j$  yields<sup>(14)</sup>

$$\frac{e_j}{f_j} = \sum_{i=1}^n \frac{\partial f_j}{\partial x_i} \frac{x_i}{f_j} \frac{dx_i}{x_i} \quad (10)$$

or

$$z_j = \sum_{i=1}^n f_{ji} \alpha_i \quad (11)$$

where  $z_j = \frac{e_j}{f_j}$ ;  $\alpha_i = \frac{dx_i}{x_i}$ ; and  $f_{ji} = \frac{\partial f_j}{\partial x_i} \frac{x_i}{f_j}$ , represent the  $j^{\text{th}}$  fractional change of the solution vectors between real and mathematical system, the  $i^{\text{th}}$  fractional changes of the scalar parameters, and the  $j^{\text{th}}$  sensitivities with respect to the  $i^{\text{th}}$  scalar parameters, respectively. Eq. (11) can be rewritten in matrix form as:

$$\mathbf{Z} = \mathbf{F}\boldsymbol{\alpha} \quad (12)$$

The vector  $\mathbf{Z}$  is obtained from results for a real and a mathematical system. The gradient of  $f$ , the scalar parameters  $x$ , and the solution of the function  $f$  determine the sensitivity matrix  $\mathbf{F}$ . The vector  $\boldsymbol{\alpha}$  is the only unknown to estimate. The vector  $\mathbf{Z}$  includes  $Z_n$  and  $Z_s$  that represent the fractional change in the square of the natural frequencies and the SCD frequencies, respectively

$$\mathbf{Z} = \{Z_n \quad Z_s\}^T \quad (13)$$

or

$$\mathbf{Z} = \{z_{n1}, z_{n2}, \dots, z_{ni}, \dots, z_{nm} : z_{s1}, z_{s2}, \dots, z_{sj}, \dots, z_{sp}\}^T \quad (14)$$

( $i = 1, 2, \dots, m; j = 1, 2, \dots, p$ )

where  $m$  and  $p$  represent the number of natural frequencies and SCD frequencies available, respectively,

and  $z_{ni}$  and  $z_{sj}$  are the fractional change in the  $i^{\text{th}}$  natural frequency and the fractional change in the  $j^{\text{th}}$  SCD frequency given by:

$$z_{ni} = \frac{\bar{\omega}_{ni}^2 - \omega_{ni}^2}{\omega_{ni}^2}, \quad z_{sj} = \frac{\bar{\omega}_{sj}^2 - \omega_{sj}^2}{\omega_{sj}^2} \quad (15)$$

when  $\omega_{ni}$  is the measured natural frequency;  $\bar{\omega}_{ni}$  is the corresponding value for the initial structure;  $\omega_{sj}$  is the measured SCD frequency; and  $\bar{\omega}_{sj}$  is the corresponding value for the initial structure.

From Eq. (12), it can be deduced that the fractional change in structural stiffness,  $\alpha_j$ , may be expressed by:

$$\boldsymbol{\alpha} = \mathbf{F}^{-1}\mathbf{Z} = \mathbf{F}^{-1} \begin{Bmatrix} Z_n \\ Z_s \end{Bmatrix} \quad (16)$$

where  $\mathbf{Z}$  is the  $(m + p) \times 1$  column matrix,  $\mathbf{F}$  is the  $(m + p) \times q$  matrix, and is the  $(q \times 1)$  column matrix when there are  $q$  unknown structural parameters to be identified.

The relationship between the fractional stiffness change and the stiffness parameters for the  $j^{\text{th}}$  element is defined in terms of a damage severity index<sup>(15)</sup>:

$$\alpha_j = \frac{\Delta k_j}{k_j} \quad (17)$$

where  $\alpha_j$  is the fractional change in stiffness of the  $j^{\text{th}}$  element  $\Delta k_j = k_j^* - k_j$ ;  $k_j^*$  is the unknown stiffness parameter of the  $j^{\text{th}}$  element of the existing structure and  $k_j$  is the known stiffness parameter of the  $j^{\text{th}}$  element of the baseline (i.e. undamaged) structure. Eq. (17) can be rewritten as:

$$k_j^* = k_j(1 + \alpha_j) \quad (18)$$

Suppose that the corresponding sets of  $m$  resonant frequencies and  $p$  SCD frequencies of the initial FE model and the existing structure are known. Before implementing Eq. (16) to get the fractional stiffness changes of  $q$  elements between the initial and the existing structures, the sensitivity matrix,  $\mathbf{F}$ , should first be developed. The sensitivity matrix represents the relation between fractional changes in stiffness and fractional changes in the frequencies of two structures.

The following procedure can be used to determine the sensitivity matrix: first,  $m$  resonant frequencies are numerically generated for the initial FE model of a system; second,  $p$  SCD frequencies are computed from the FRF of the FE model using Eq. (5); third, a known severity of damage,  $\alpha_j$ , at element  $j$  of the FE model is introduced and the corresponding  $m$  resonant frequencies for the damaged model are numerically generated; fourth, based on a known damage,  $\alpha_j$ , at element  $j$  of the FE model,  $p$  SCD frequencies are computed from the FRF such as the second step; fifth, the fractional changes between the  $(m + p)$  initial frequencies and the  $(m + p)$  damaged frequencies are computed using Eq. (15); sixth, each component of the  $j^{\text{th}}$  column of the matrix  $F$  is computed dividing the fractional changes in each frequency by the simulated severity at the element  $j$  and finally, the  $(m + p) \times q$  matrix  $F$  is generated repeating the procedure for all  $q$  elements.

With the  $F$  matrix obtained, the following 8-step algorithm is proposed to identify a target structure:

1. Extract FRFs and natural frequencies from the target structure (i.e. an existing structure).
2. Compute the SCD frequencies of the target structure.
3. Select an initial FE model of the structure utilizing all possible knowledge about the design and construction of the structure.
4. Compute the natural and the SCD frequencies of the initial FE model.
5. Compute the sensitivity matrix,  $F$ , for the FE model.
6. Compute the fractional changes in frequencies between the FE model and the target structure.
7. Fine-tune the FE model by first solving Eq. (16) to estimate stiffness changes and next solving Eq. (18) to update stiffness parameters of the FE model.
8. Repeat steps 1-7 until  $Z \approx 0$  or  $\alpha \approx 0$  when the structural parameters of the FE model are identical to the existing target structure.

### 3. Verification of Methodology Using Numerical Simulation

Using simulated data from a simple mechanical system, the performance of SI using SCD frequencies

is investigated. The physical model selected here is shown in Fig. 2. The three-degrees-of-freedom mechanical system consists of three unequal masses,  $m_i$  ( $i = 1, 2, 3$ ) that are interconnected and anchored to their supports by six linear spring elements,  $k_i$  ( $i = 1, 2, \dots, 6$ ). The stiffnesses, masses, and the material properties of the model are listed in Table 1. In this study, the masses of the model are assumed to be known and thus the six stiffness parameters of the system are the only elements to be identified.

For following initial values of the structural properties are assumed as a starting point:

- (1) Case 1 - the initial assumed stiffnesses are 10% less than the true stiffnesses
- (2) Case 2 - the initial assumed stiffnesses are 20% less than the true stiffnesses
- (3) Case 3 - the initial assumed stiffnesses are 50% less than the true stiffnesses

These three cases are selected to demonstrate the rate of convergence and the accuracy of the prediction as a function of the distance of the initial guesses from the true values. The spectral information of the unperturbed system and the three initial systems are summarized in Table 2.

Table 1 Material Properties of the Mechanical System

Stiffness	Element					
	$k_1$	$k_2$	$k_3$	$k_4$	$k_5$	$k_6$
	2	1	1	2	2	1
Mass	$m_1$		$m_2$		$m_3$	
	0.8		2.0		1.2	

Table 2 Natural and SCD Frequencies

Mode	Frequency (rad/sec)				
	Target	Case 1	Case 2	Case 3	
Natural Frequency	1	1.173	1.113	1.049	0.830
	2	1.848	1.753	1.653	1.306
	3	2.354	2.234	2.106	1.665
SCD Frequency	1-2 ( $H_{11}$ )	1.253	1.189	1.121	0.886
	2-3 ( $H_{11}$ )	1.878	1.782	1.680	1.328
	1-2 ( $H_{22}$ )	1.494	1.417	1.336	1.056
	2-3 ( $H_{22}$ )	2.041	1.937	1.826	1.443
	1-2 ( $H_{33}$ )	1.276	1.210	1.141	0.903
	2-3 ( $H_{33}$ )	2.182	2.070	1.952	1.544

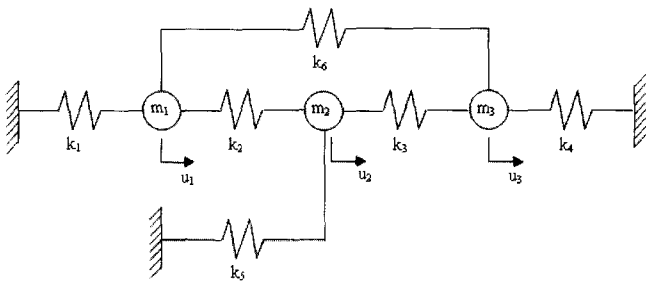


Fig 2 Configuration of the Mechanical System

The SI procedure defined in the previous section was performed to identify the structural stiffness parameters of the system. First, the identification was performed with 3 natural frequencies. The results can be seen in Tables 3 through 5. The results show that the maximum errors in the estimated structural parameters for Case 1, Case 2, and Case 3 are 1.81%, 3.56%, and 8.93%, respectively. Fig. 3 shows the magnitude and phase plots of the FRFs for the system with the identified stiffness and the true stiffness for Case 3. In the figures, it is shown that the FRFs of the system do not match up exactly even though the natural frequencies fit.

To investigate the performance of SI including additional spectral information, numerical simulation sets were conducted using the following combination

of natural frequencies and SCD frequencies: (1) 3 natural frequencies and 2 SCD frequencies (5 pieces of information) and (2) 3 natural frequencies and 6 SCD frequencies (9 pieces of information). The 5 pieces of information consist of 3 natural frequencies and 2 SCD frequencies obtained from a point frequency response function ( $H_{11}$ ), and the 9 pieces of information consist of 3 natural frequencies and 6 SCD frequencies obtained from a point frequency response function ( $H_{11}$ ,  $H_{22}$ , and  $H_{33}$ ).

The system identification results using the additional spectral information are summarized in Tables 3 to 5 for Cases 1 to 3, respectively. The accuracies of the different parameter identification results are compared via the maximum percentage errors of the identified stiffness of elements as shown in the tables. From Table 3, while the maximum error in the identified stiffness parameters obtained by using 3 natural frequencies is 1.81%, the maximum errors by including SCD frequencies are 0.93% using 5 pieces of information, and 0% using 9 pieces of information. From Table 4, while the maximum error in the identified stiffness parameters obtained by using 3 natural frequencies is 3.56%, the maximum errors by including SCD frequencies are 1.18% using 5

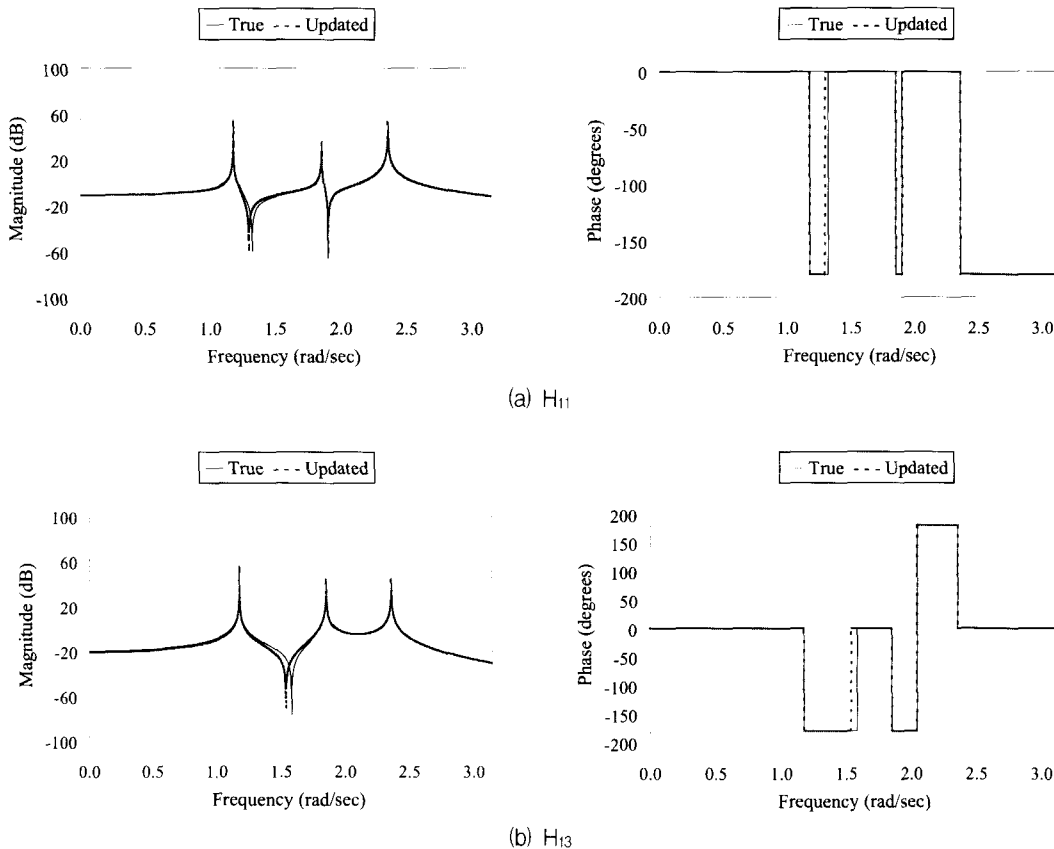


Fig 3 FRF Amplitude and Phase Plots for Case 3 - NF only

pieces of information, and 0% using 9 pieces of information. From Table 5, while the maximum error in the identified stiffness parameters obtained by using 3 natural frequencies is 8.93%, the maximum errors by including SCD frequencies are 4.93% using 5 pieces of information, and 0% using 9 pieces of information.

Table 3 Comparison of Identified Parameters (Case 1)

Parameters	True	NF only	NF + 2 SCDF	NF + 6 SCDF
$k_1$	2.0	2.034	1.998	2.000
$k_2$	1.0	0.986	1.006	1.000
$k_3$	1.0	0.992	0.991	1.000
$k_4$	2.0	2.036	2.019	2.000
$k_5$	2.0	1.976	1.992	2.000
$k_6$	1.0	0.987	0.997	1.000
Max. error		1.81%	0.93%	0.00%

Table 4 Comparison of Identified Parameters (Case 2)

Parameters	True	NF only	NF + 2 SCDF	NF + 6 SCDF
$k_1$	2.0	2.069	1.989	2.000
$k_2$	1.0	0.972	1.007	1.000
$k_3$	1.0	0.983	0.989	1.000
$k_4$	2.0	2.071	2.024	2.000
$k_5$	2.0	1.955	1.990	2.000
$k_6$	1.0	0.975	0.996	1.000
Max. error		3.56%	1.18%	0.00%

Table 5 Comparison of Identified Parameters (Case 3)

Parameters	True	NF only	NF + 2 SCDF	NF + 6 SCDF
$k_1$	2.0	2.179	1.989	2.000
$k_2$	1.0	0.928	1.028	1.000
$k_3$	1.0	0.951	0.953	1.000
$k_4$	2.0	2.178	2.099	2.000
$k_5$	2.0	1.900	1.962	2.000
$k_6$	1.0	0.937	0.982	1.000
Max. error		8.93%	4.93%	0.00%

From the tables, it can be observed that better results could be obtained by including SCD frequencies. This shows that the SCD frequencies can be utilized in SI problems like natural frequencies. Notably, with additional 6 SCD frequencies, the error in the prediction is zero, regardless of the starting point. This shows significant advantage of utilizing additional spectral information in SI problems. In most real applications, only a few natural frequencies can be extracted from dynamic testing, and this limitation has been one of the major disadvantages of frequency-based SI methods. However, as shown in Table 2, utilizing additional spectral information can expand data space and accordingly can enhance the capability as well as the accuracy of SI. Fig. 4 shows the magnitude and phase plots of the FRFs for the system with the identified stiffness and the true stiffness using 9 pieces of information. To compare with the results using only natural frequencies in Fig. 3, the results of the same case, Case 3, are depicted. It is shown that the FRFs of the system do match up in Fig 4.

#### 4. Conclusions

In this paper, application of SCD frequencies to an SI problem was presented. The motivation of this study was to investigate the feasibility of utilizing the additional spectral information to SI problems and to improve the performance of SI by supplying more information to a sensitivity-based SI technique. For the purpose of this study, the sensitivity-based method was extended to accommodate the SCD frequency, and the performance of SI including the SCD frequencies is compared with that of natural frequencies only via a numerical example of a mechanical system.

From the numerical study, the following observations can be drawn:

1. with the addition of SCD frequencies, the accuracy and consistency of SI can be improved;
2. with the additional 6 SCD frequencies, the error in the prediction is zero, regardless of the starting point.
3. the SCD frequencies can be utilized in SI problems like natural frequencies; and
4. utilizing additional spectral information such as SCD frequencies can expand data space and accordingly can enhance the capability of SI.

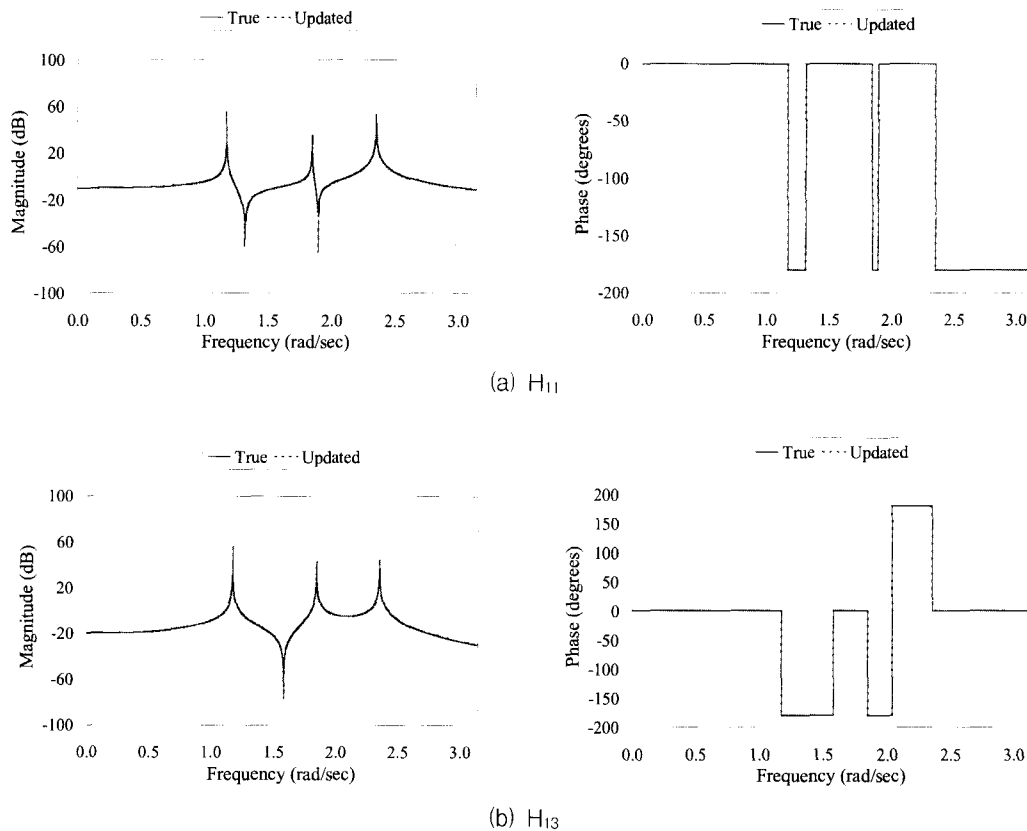


Fig 4 FRF Amplitude and Phase Plots for Case 3 - NF + SCDF

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