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A History of Dynamical Systems and the H-Shadowing

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In this paper, we introduce the history of dynamical systems and the notion of the H-shadowing property. And we study some relationships between the H-shadowing property and other dynamical properties such as expansivity and topological stability.

Key words: dynamical system, shadowing property, H-shadowing property, topological stability, expansivity, topological conjugacy

0. A history of dynamical systems

Dynamics has evolved into three parts: applied, mathematical, and experimental. Applied dynamics is the oldest. Originally regarded as a branch of natural philosophy, or physics, it goes back to Galileo(1564-1642) at least. It deals with the concept of change, rate of change, rate of rate of change, and so on, as they occur in natural phenomena. We take these concepts for granted, but they emerged into our consciousness only in the fourteenth century.

Mathematical dynamics began with Newton(1642-1727) and has become a large and active branch of pure mathematics. This includes the theory of ordinary differential equations, now a classical subject. But since Poincare(1854-1912), the newer methods of topology and geometry have dominated the field. It is concerned with the clarification of the nature of the behavior of integral curves in the entire domain of their existence without integration of the equations.

Experimental dynamics is an increasingly important branch of the subject.

Founded by Galileo, it showed little activity until Rayleigh(1842-1919), Duffing (1861-1944), and Van der Pol(1889-1959). Experimental techniques have been revolutionized with each new development of technology. Analog and digital computers are now accelerating the advance of the research frontier, making experimental work more significant than ever.

Chaotic limit sets had been known to theoretical dynamics since Poincare. The first chaotic attractor in a dynamical system was discovered in 1932 by Birkhoff.

Helmholtz, Rayleigh, Duffing, Van der Pol, and Hayashi relied on experimental simulations to discover the main properties of periodic motions in nonlinear oscillations. Since 1950, digital simulations have become increasingly important, especially since the experimental discovery of chaotic attractors in 1962 by Lorenz. It took many years before these objects emerged into the theoretical literature, in the works of Charpentier, Levinson, Cartwright, Littlewood, Smale, and others.

Many research workers in the fields of mathematical, physical, mechanical, biological and social sciences are eager to learn some of the background of this subject.

The theory of dynamical systems may be said to have begun as a special topic in the theory of differential equations. The shadowing property starting with the work of Anosov(1970) and Bowen(1975), often appears in several branches of the modern theory of dynamical systems, and usually plays an important role in the investigation of dynamical systems. The theory of shadowing has developed intensively in recent years and become a significant part of the qualitative theory of dynamical systems containing at lot of interesting and deep results. Especially, the topologically stable, structurally stable dynamical systems and orbit structure of a dynamical system are investigated by the mathematicians.

When we study a dynamical system from the numerical point of view, usually, we consider the geometric pattern of the set of orbits of a system under investigation. In this case, the main object of interest is just the geometric shape of the orbit as a set while the orbit behaviour of the system under special time is irrelevant. To develop this approach, the notion of the various shadowing has been introduced by Pilyugin, Rodionova and Sakai(2003).

The shadowing property, which is also well known as the pseudo orbit tracing property, is one of the interesting concepts in the qualitative theory of dynamical systems. The notion of shadowing property of a dynamical system is used to justify the validity of computer simulations of the system, asserting that there is a true orbit of the system close to the computed orbit. Many people have studied the relations between the shadowing property and the classical notions in the qualitative theory of dynamical systems.

1. Dynamical system

Let X be a compact metric space with a metric d, and let Z stands for the set of integers and ℝ be the set of real numbers.

Throughout the paper, we denote H(X) by the set of all homeomorphisms on X with the C^0-metric: for any $f, g \in$ H(X),

$$d_0(f, g) = \sup\{ d(f(x), g(x)) : x \in X\}.$$

A dynamical system on X is the triple(X, ℝ, F), where F is a continuous map from the product space X×ℝ into the space X satisfying the following axioms:

(a) $F(x,0) = x$ (identity axiom)

(b) $F(F(x,s),t) = F(x, s+t)$ (group axiom)

for every $x \in$ X and $s, t \in$ ℝ. We say that the triple (X, Z, F) which satisfy the above properties (a) and (b) is a discrete dynamical system.

Let (X, ℤ, F) be a discrete dynamical system. Then it is easy to show that the map f: X →X defined by

$$f(x) = F(x, 1) \text{ for all } x \in X$$

is homeomorphism on X. Conversely, let f: X →X be a homeomorphism and define a map

$$F(x, n) = f^n(x) = f \circ \cdots \circ f(x)$$

for all n∈ Z and $x \in$X.

Let $f \in$H(X) and $x \in$X. The set

$$O(f, x) = \{ f^n(x) : n \in Z \}$$

is said to be the *orbit* of f through $x \in X$.

Let $\delta > 0$ be an arbitrary number. A δ*-pseudo orbit* of f is a sequence of points $\xi = \{x_n \in X : n \in \mathbf{Z}\}$ such that

$$d(f(x_n), x_{n+1}) < \delta$$

for all $n \in \mathbf{Z}$.

The notion of a pseudo orbit plays an important role in the general qualitative theory of dynamical systems. Usually, a δ-pseudo orbit is a natural model of computer output in a process of numerical investigation of the dynamical system f in X. In this case, the value δ measures one step errors of the method and round-off errors. It is also used to define some types of invariant sets such as the chain recurrent set or chain prolongation sets(see [6], [7], [10]).

A pseudo orbit $\xi = \{x_n \in X : n \in \mathbf{Z}\}$ is δ*-shadowed* by a point $x \in X$ if the inequality

$$d(f^n(x), x_n) < \delta$$

for all $n \in \mathbf{Z}$, holds.

Thus the existence of a shadowing point for a pseudo orbit ξ means that ξ is close to a real orbit of f.

A homeomorphism $f \in H(X)$ is said to have the *shadowing property* (or the pseudo orbit tracing property) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit $\{x_n\}_{n \in Z}$ in X is ε-shadowed by some point $x \in X$:

$$d(f^n(x), x_n) \leq \varepsilon$$

for all $n \in \mathbf{Z}$.

The theory of shadowing was developed intensively in recent years and became a significant part of the qualitative theory of dynamical systems containing a lot of interesting and deep results(see [8]).

A homeomorphism $f \in H(X)$ is said to be *expansive* if there is a constant $e > 0$ such that if

$$d(f^n(x), f^n(y)) \leq e$$

for all $n \in \mathbf{Z}$, then $x=y$. Such a number e is called an *expansive constant* of f.

A homeomorphism $f \in H(X)$ is said to be *topologically stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$, $g \in H(X)$, then there is a continuous surjection h : X → X with $f \circ h = h \circ g$ and $d_0(h, I_X) < \varepsilon$, where I_X : X → X stands for the identity homeomorphism. The map h is called a *semiconjugacy* from f to g.

It is well known that if M is a compact smooth manifold and $f \in H(M)$ is topologically stable then it has the shadowing property. Moreover, it was proved that if $f \in H(M)$ is an expansive homeomorphism which has the shadowing property then it is topologically stable.

However, the above results do not hold in general if M is not compact smooth manifold.

2. H-shadowing for discrete dynamical systems

Recently Diamond et al ([3]) obtained a necessary and sufficient condition under which a homeomorphism on a compact smooth manifold has the shadowing property.

Let M be a compact smooth manifold. A homeomorphism f on M has the shadowing property if and only if for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ for any $g \in H(X)$, then every g-orbit is ε-shadowed by a f-orbit[3].

The above theorem can be used to motivate the notion of another shadowing property for discrete dynamical systems on metric spaces as follows.

Definition 2.1. A dynamical system $f \in H(X)$ is said to have the H-*shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ for every $g \in H(X)$, then any g-orbit is ε-shadowed by a f-orbit : for every $x \in X$, there exists $x_0 \in X$ such that

$$d(f^n(x_0), g^n(x)) \leq \varepsilon$$

for all $n \in \mathbf{Z}$.

Theorem 2.2. If a dynamical system $f \in H(X)$ has the shadowing property then it has the H-shadowing property.

Proof. Let $f \in H(X)$ has the shadowing property. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-pseudo orbit $\xi = \{x_n : n \in \mathbf{Z}\}$ in X is ε-shadowed by a f-orbit. Let $g \in H(X)$ be $d_0(f, g) < \delta$. Then we have

$$d(f(g^n(x)), g^{n+1}(x)) = d(f(g^n(x)), g(g^n(x))) < \delta$$

for all $n \in \mathbf{Z}$ and $x \in X$, and so $\xi = \{g^n(x) = x_n : n \in \mathbf{Z}\}$ is a δ-pseudo orbit of f in X. By assumptions, there exists $x_0 \in X$ such that $d(f^n(x_0), x_n) = d(f^n(x_0), g^n(x)) \leq \varepsilon$ for all $n \in \mathbf{Z}$. This means that f has the H-shadowing property. □

However the converse is not true in general as we can see in the following example.

Example 2.3. Consider the circle $S^1 = \left\{(\zeta, \eta) : (\zeta - \frac{1}{2})^2 + \eta^2 = \frac{1}{4}\right\}$, coordinatized by $\theta \in [0, 1)$, and define the homeomorphism f_1 on S^1 as follows:

$$f_1(\theta) = \theta \quad \text{if} \quad \theta = 0 \text{ or } \theta = \frac{1}{2};$$

$$f_1(\theta) > \theta \quad \text{if} \quad \theta \in (0, \frac{1}{2});$$

$$f_1(\theta) < \theta \quad \text{if} \quad \theta \in (\frac{1}{2}, 1).$$

Let $L = \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$, and consider the homeomorphism f_2 on L given by $f_2(x, 0) = (x^2, 0)$. Define a homeomorphism f on $X = S^1 \cup L$ by

$$f(x) = \begin{cases} f_1(x) & \text{if} \quad x \in S^1, \\ f_2(x) & \text{if} \quad x \in L. \end{cases}$$

It is easily checked that f has the H-shadowing property, but it does not have the shadowing property.

We say that $f,\ g \in \mathrm{H(X)}$ are *topologically conjugate* if there exists $h \in \mathrm{H(X)}$ such that $hg = fh$. The $h \in \mathrm{H(X)}$ is called a *topologically conjugacy* between f and g. In the following theorem, we see that H-shadowing property is invariant under a topological conjugacy.

Theorem 2.4. Suppose $f_1 \in \mathrm{H(X)}$ is topologically conjugate to $f_2 \in \mathrm{H(X)}$. Then f_1 has the H-shadowing property if and only if f_2 has the H-shadowing property.

Proof. Let $f_1 \in \mathrm{H(X)}$ has the H-shadowing property, and suppose that $f_1, f_2 \in \mathrm{H(X)}$ are topologically conjugate. Let $h \in \mathrm{H(X)}$ be a topological conjugacy between f_1 and f_2. Let $\varepsilon > 0$ be arbitrary, and choose $0 < \varepsilon_1 < \varepsilon$ such that if $d(a, b) < \varepsilon_1$, then $d(h^{-1}(a),\ h^{-1}(b)) < \varepsilon$ for $a, b \in \mathrm{X}$.

We shall complete the proof by showing that f_2 has the H-shadowing property. Since f_1 has the H-shadowing property, given $\varepsilon_1 > 0$ there exists $\delta > 0$ such that if $d_0(f_1, g) < \delta$ for $g \in \mathrm{H(X)}$ then for every $x \in X$, there exists $x_1 \in X$ such that

$$d(f_1^n(x_1),\ g^n(x)) < \varepsilon_1 \text{ for all } n \in \mathbb{Z}.$$

For the $\delta > 0$, choose $0 < \delta_1 < \delta$ such that if $d(a, b) < \delta_1$, for every $a, b \in X$, then $d(h(a), h(b)) < \delta$. Let $g_1 \in \mathrm{H(X)}$ be such that $d_0(f_2, g_1) < \delta_1$, and let $g = h g_1 h^{-1}$. Then we have

$$d(h(f_2(x)), h(g_1(x))) = d(f_1(h(x)), g(h(x))) < \delta$$

for any $x \in X$, and so $d_0(f_1, g) < \delta$.

By the assumption, we have that for any $x \in X$, there exists $h(y) \in X$ such that

$$d(f_1(h(y)), g(h(x))) = d(h(f_2(y)), h(g_1(x)) < \varepsilon_1.$$

Hence we have

$$\begin{aligned} d(f_1^n(h(y)), g^n(h(x))) &= d(f_1^{n-1}(h(f_2(y))), g^{n-1}(h(g_{1(x)))}) \\ &= d(h(f_2^n(y)), h(g_1^n(x))) \\ &< \varepsilon_1 \end{aligned}$$

for all $n \in \mathbf{Z}$, and so we get

$$d(f_2^n(x_2), g_1^n(x)) < \varepsilon.$$

Consequently we have shown that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d_0(f_2, g_1) < \delta$ for $g_1 \in \mathrm{H(X)}$ then for every $x \in X$, there exists $x_2 \in X$ such that

$$d(\, f_2^n(x_2),\ g_1^n(x)\,) < \varepsilon \text{ for all } n \in \mathbf{Z}.$$

This means that f_2 has the H-shadowing property, and so completes the proof. □

Theorem 2.5. If $f \in \mathrm{H(X)}$ is topologically stable then it has the H-shadowing property.

Proof. Let $\varepsilon > 0$ be arbitrary. By the assumption, we can choose a constant $\delta > 0$ such that if $d_0(f, g) < \delta$ and $g \in \mathrm{H(X)}$, then there is a continuous surjection $h : \mathrm{X} \to \mathrm{X}$ satisfying $f \circ h = h \circ g$ and $d_0(h, I_X) < \varepsilon$.

Then we have $h \circ g^{-1} = f^{-1} \circ h$, and so we get

$$h(g^n(x)) = h\,g(g^{n-1}(x)) = f\,h(g^{n-1}(x)) = \cdots = f^n(h(x)),$$

for any $x \in \mathrm{X}$ and $n \in \mathbf{Z}$. Put $y = h(x)$. Then we obtain

$$d(g^n(x),\ f^n(y)) < \varepsilon$$

for all $n \in \mathbf{Z}$. This implies that any g-orbit is ε-shadowed by a f-orbit and so completes the proof. □

Definition 2.6. A homeomorphism $f \in \mathrm{H(X)}$ is said to have the H-*shadowing uniqueness property* if there exists a constant $\varepsilon > 0$ such that if $d_0(f, g) < \delta$ for any $\delta > 0$ and $g \in \mathrm{H(X)}$, then every g-orbit is ε-shadowed by not more than one f-orbit : for every $x \in \mathrm{X}$, there exists $x_0 \in \mathrm{X}$ such that

$$d(f^n(x_0), g^n(x)) \le \varepsilon$$

for all $n \in \mathbf{Z}$.

Theorem 2.7. A homeomorphism $f \in \mathrm{H(X)}$ has the H-shadowing property. Then f has the H-shadowing uniqueness property if and only if f is expansive.

Proof. Let $f \in \mathrm{H(X)}$ has the H-shadowing uniqueness property. Then we can choose a constant $\varepsilon > 0$ such that if $d_0(f, g) < \delta$ for any $\delta > 0$ and $g \in H(X)$, then every g-orbit is ε-shadowed not more than one f-orbit. Then we claim that ε is an expansive costant for f. To show this, we suppose that

$$d(f^n(x), f^n(y)) < \varepsilon$$

for all $n \in \mathbb{Z}$. Since $d_0(f, f) < \delta$ for any $\delta > 0$, every f-orbit is ε-shadowed not more than one f-orbit. This means that $x = y$.

Conversely, let $f \in H(X)$ be expansive and let $e > 0$ be an expansive costant for f, then for any $\delta > 0$, choose $g \in H(X)$ with $d_0(f, g) < \delta$. Suppose there is a g-orbit, $\{g^n(p)\}$, which is $\frac{e}{2}$-shadowed by two f-orbits $\{f^n(x) : n \in \mathbb{Z}\}$ and $\{f^n(y) : n \in \mathbb{Z}\}$.

Then we have

$$d(g^n(p), f^n(x)) < \frac{e}{2} \quad \text{and} \quad d(g^n(p), f^n(y)) < \frac{e}{2},$$

for all $n \in \mathbb{Z}$. Hence we get

$$d(f^n(x), f^n(y)) < e$$

for all $n \in \mathbb{Z}$. Since f is expansive we obtain $x = y$. This means that every g-orbit is $\frac{e}{2}$-shadowed by not more than one f-orbit. □

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역학계의 역사와 H-추적 성질

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본 논문에서는 역학계의 역사와 H-추적 성질(H-shadowing property)의 개념을 소개한다. 또한 콤팩트 거리 공간에서 H-추적 성질과, 확정성, 위상적 안정성 등 다른 역학적 성질 사이의 관계를 연구하였다.

주제어 : 역학계, 추적성, H-추적성, 위상적 안정성, 확장성, 위상적 공액성

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