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Initial structure of intuitionistic fuzzy proximity

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Abstract

In this paper, we show that the category of intuitionistic fuzzy proximity spaces has an initial structure, and consequently a subspace structure and a product space structure of them.

Key words : intuitionistic fuzzy proximity, initial intuitionistic fuzzy proximity

1. Introduction

As a generalization of fuzzy sets, the concept of intuitionistic fuzzy sets was introduced by Atanassov [1]. Recently, Çoker and his colleagues [2, 3, 4] introduced the concept of intuitionistic fuzzy topology which is a generalization of fuzzy topology.

Katsaras [5, 6] introduced the concept of fuzzy proximity, and studied the relationship between fuzzy topology and fuzzy proximity. W. Liu [9] introduced the concept of L-fuzzy proximity for a lattice L, and Y. Liu and M. Luo [10] studied the relation between L-fuzzy proximity and L-fuzzy uniformity. Also, Khare [7] studied the relationship between classical and fuzzy proximities.

In [8], the authors introduced the concept of the intuitionistic fuzzy proximity and investigate the relationship among intuitionistic fuzzy proximity and other structures.

In this paper, we get the initial structure of intuitionistic fuzzy proximity, and consequently the subspace and the product space of them.

2. Preliminaries

In this section we recall some of the definitions and theorems related to fuzzy proximity and intuitionistic fuzzy topology.

Let X be a nonempty set and I the unit interval $[0,1]$. *An intuitionistic fuzzy set* A is an ordered pair

$$A=(\mu_A,\gamma_A)$$

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where the functions $\mu_A:X\to I$ and $\gamma_A:X\to I$ denote the degree of membership and the degree of nonmembership, respectively, and $\mu_A+\gamma_A\le\tilde{1}$. Let $I(X)$ denote the set of all intuitionistic fuzzy sets in X.

Obviously every fuzzy set μ_A in X is an intuitionistic fuzzy set of the form $(\mu_A,\tilde{1}-\mu_A)$.

Definition 2.1. ([1]) Let $A=(\mu_A,\gamma_A)$ and $B=(\mu_B,\gamma_B)$ be intuitionistic fuzzy sets in X. Then

(1) $A\subseteq B$ iff $\mu_A\le\mu_B$ and $\gamma_A\ge\gamma_B$.
(2) $A=B$ iff $A\subseteq B$ and $B\subseteq A$.
(3) $A^c=(\gamma_A,\mu_A)$.
(4) $A\cap B=(\mu_A\wedge\mu_B,\gamma_A\vee\gamma_B)$.
(5) $A\cup B=(\mu_A\vee\mu_B,\gamma_A\wedge\gamma_B)$.
(6) $0_\sim=(\tilde{0},\tilde{1})$ and $1_\sim=(\tilde{1},\tilde{0})$.

Let f be a map from a set X to a set Y. Let $A=(\mu_A,\gamma_A)$ be an intuitionistic fuzzy set in X and $B=(\mu_B,\gamma_B)$ an intuitionistic fuzzy set in Y. Then $f^{-1}(B)$ is an intuitionistic fuzzy set in X defined by

$$f^{-1}(B)=(f^{-1}(\mu_B),f^{-1}(\gamma_B))$$

and $f(A)$ is an intuitionistic fuzzy set in Y defined by

$$f(A)=(f(\mu_A),\tilde{1}-f(\tilde{1}-\gamma_A)).$$

Definition 2.2. ([3]) An *intuitionistic fuzzy topology* on X is a family τ of intuitionistic fuzzy sets in X which satisfies the following properties:

(1) $0_\sim,1_\sim\in\tau$.
(2) If $A_1,A_2\in\tau$, then $A_1\cap A_2\in\tau$.
(3) If $A_i\in\tau$ for each i, then $\cup A_i\in\tau$.

The pair (X,τ) is called an *intuitionistic fuzzy topological space.* Any member of τ is called an *intuitionistic fuzzy open set* in X and the complement an *intuitionistic fuzzy closed set.*

Definition 2.3. ([3]) Let (X,τ) and (Y,ω) be intuitionistic fuzzy topological spaces. Then a map $f:X\to Y$ is said to be *continuous* if $f^{-1}(B)$ is an intuitionistic fuzzy open set in X for each intuitionistic fuzzy open set B in Y, or equivalently, $f^{-1}(B)$ is an intuitionistic fuzzy closed set in X for each intuitionistic fuzzy closed set B in Y.

Definition 2.4. ([8]) An *intuitionistic fuzzy proximity* on X is a relation δ on $I(X)$ satisfying the following properties:

(1) $A\delta B$ implies $B\delta A$.

(2) $(A\cup B)\delta C$ if and only if $A\delta C$ or $B\delta C$.

(3) $A\delta B$ implies $A\neq 0_\sim$ and $B\neq 0_\sim$.

(4) $A\not\delta B$ implies that there exists an $E\in I(X)$ such that $A\not\delta E$ and $E^c\not\delta B$.

(5) $A\cap B\neq 0_\sim$ implies $A\delta B$.

The pair (X,δ) is called an *intuitionistic fuzzy proximity space.*

Definition 2.5. ([8]) Let (X,δ_1) and (Y,δ_2) be two intuitionistic fuzzy proximity spaces and $f:X\to Y$ a map. Then f is called a *continuous* map if $A\delta_1 B$ implies $f(A)\delta_2 f(B)$.

Clearly, f is continuous if and only if $C\not\delta_2 D$ implies $f^{-1}(C)\not\delta_1 f^{-1}(D)$ for each $C,D\in I(Y)$.

3. The initial intuitionistic fuzzy proximity

Let **FProx** be the category of all fuzzy proximity spaces and proximity maps and **IFProx** the category of all intuitionistic fuzzy proximity spaces and continuous maps. In [8] we already found a categorical relationship between **FProx** and **IFProx.** In this section, we are going to show that the category of intuitionistic fuzzy proximity spaces has an initial structure.

Let $\{(X_\alpha,\delta_\alpha)\}_{\alpha\in\Gamma}$ be a family of intuitionistic fuzzy proximity spaces and X a set, and let $f_\alpha:X\to X_\alpha$ be a map for each $\alpha\in\Gamma$. For $A,B\in I(X)$, we define $A\delta B$ if and only if the following condition is satisfied: If $A=A_1\cup\cdots\cup A_n$ and $B=B_1\cup\cdots\cup B_m$ $(A_i,B_j\in I(X))$, then there exist i,j such that for each $\alpha\in\Gamma$ we have $f_\alpha(A_i)\delta_\alpha f_\alpha(B_j)$.

Remark 3.1. $A\not\delta B$ if and only if for any $A'\subseteq A$ and $B'\subseteq B$, there exists α such that $f_\alpha(A')\not\delta f_\alpha(B')$.

Theorem 3.2. *The relation δ on X defined above is the initial intuitionistic fuzzy proximity on X with respect to the family $\{f_\alpha\}_{\alpha\in\Gamma}$. That is, for any intuitionistic fuzzy proximity space (Y,δ_1) and a function $f:(Y,\delta_1)\to(X,\delta)$, f is continuous if and only if the composition $f_\alpha\circ f$ is continuous for all α.*

Proof. First, we will show that δ is an intuitionistic fuzzy proximity on X.

(1) Clearly, $A\delta B$ implies $B\delta A$.

(2) Note that if $A\delta B$ and $A\subseteq E$ then $E\delta B$. So $A\delta C$ or $B\delta C$ implies $(A\cup B)\delta C$. Suppose that $A\not\delta C$ and $B\not\delta C$. Since $A\not\delta C$, there are $A_1,\cdots,A_n$, $C_1,\cdots,C_k$ such that $A=A_1\cup\cdots\cup A_n$, $C=C_1\cup\cdots\cup C_k$ and for each pair (i,j), $1\le i\le k$, $1\le j\le n$, there exists α such that

$$f_\alpha(C_i)\not\delta_\alpha f_\alpha(A_j).$$

Since $B\not\delta C$, there are $B_1,\cdots,B_m$, $D_1,\cdots,D_l$ such that $B=B_1\cup\cdots\cup B_m$, $D=D_1\cup\cdots\cup D_l$ and for each pair (i,j), $1\le i\le l$, $1\le j\le m$, there exists β such that

$$f_\beta(D_i)\not\delta_\beta f_\beta(B_j).$$

It is clear that $C=\cup_{i,j}(C_i\cap D_j)$ and

$$A\cup B=A_1\cup\cdots\cup A_n\cup B_1\cup\cdots\cup B_m.$$

Let $1\le i\le k$, $1\le j\le l$. If $1\le r\le n$, then for this $\alpha\in\Gamma$, we have $f_\alpha(C_i)\not\delta_\alpha f_\alpha(A_r)$, and hence $f_\alpha(C_i\cap D_j)\not\delta_\alpha f_\alpha(A_r)$. Similarly, if $n+1\le r\le n+m$

then for this $\beta\in\Gamma$, we have $f_\beta(C_i\cap D_j)\not\delta_\beta f_\beta(B_{r-n})$. Therefore $C\not\delta(A\cup B)$.

(3) Let $A\delta B$. Then for each $\alpha\in\Gamma$, $f_\alpha(A)\delta_\alpha f_\alpha(B)$ and hence $f_\alpha(A)\neq 0_\sim$ and $f_\alpha(B)\neq 0_\sim$. Thus $A\neq 0_\sim$ and $B\neq 0_\sim$.

(4) Consider the set V of all pairs (A,B) such that $A\not\delta B$ and for every $E\in I(X)$ we have either $A\delta E$ or $E^c\delta B$. We claim that V is empty. Suppose $(A,B)\in V$. We first show that $f_\alpha(A)\delta_\alpha f_\alpha(B)$ for all $\alpha\in\Gamma$. Take any $\alpha\in\Gamma$ and $K\in I(X_\alpha)$. Put $L=f_\alpha^{-1}(K)\in I(X)$. Then $A\delta L$ or $L^c\delta B$. If $A\delta L$, then $f_\alpha(A)\delta_\alpha f_\alpha(L)$. Since $f_\alpha(L)=f_\alpha(f_\alpha^{-1}(K))\subseteq K$, $f_\alpha(A)\delta_\alpha K$. Similarly, $L^c\delta B$ implies $K^c\delta_\alpha f_\alpha(B)$. Since δ_α is a fuzzy proximity on X_α, it follows that $f_\alpha(A)\delta_\alpha f_\alpha(B)$. Now since $A\not\delta B$, there exist positive integers k,m and $A_1,\cdots,A_k$, $B_1,\cdots,B_m$ such that $A=A_1\cup\cdots\cup A_k$, $B=B_1\cup\cdots\cup B_m$ and for each pair (i,j), $1\le i\le k$, $1\le j\le m$, there is an $\alpha\in\Gamma$ with

$f_\alpha(A_i)\not\delta_\alpha f_\alpha(B_j)$. If $k+m=2$ then $k=m=1$ and hence $f_\alpha(A)\not\delta_\alpha f_\alpha(B)$. This is a contradiction. So $k+m>2$. Let $n=k+m$. We call such a number n the integer corresponding to (A,B). Of course n is not uniquely determined by (A,B) and we have $n>2$ Let K be the set of all integers corresponding to members of V and let n be the smallest member in K. Choose a pair $(A,B)\in V$ with the corresponding integer n. We also choose $A_1,\cdots,A_k$, $B_1,\cdots,B_m$ in $I(X)$ such that $n=k+m>2$, $A=A_1\cup\cdots\cup A_k$, $B=B_1\cup\cdots\cup B_m$ and for each pair (i,j) there exists an $\alpha\in\Gamma$ with $f_\alpha(A_i)\not\delta_\alpha f_\alpha(B_j)$. Assume that $k>1$ Let $C=A_1\cup\cdots\cup A_{k-1}$. One of the following should be true:

(A) For every $E\in I(X)$, either $C\delta E$ or $B\delta E^c$.

(B) For every $E\in I(X)$, either $A_k\delta E$ or $B\delta E^c$.

To prove this, suppose that neither (A) nor (B) holds. Then there exist $E_1,E_2\in I(X)$ such that $C\not\delta E_1$, $B\not\delta E_1^c$, $A_k\not\delta E_2$ and $B\not\delta E_2^c$. Let $E=E_1\cap E_2$. Since $C\not\delta E_1$ and $A_k\not\delta E_2$, $C\not\delta E$ and $A_k\not\delta E$ and hence $A\not\delta E$. Also $B\not\delta E_1^c$ and $B\not\delta E_2^c$ implies $B\not\delta(E_1^c\cup E_2^c)=E^c$. Hence $A\not\delta E$ and $B\not\delta E^c$ which contradicts our assumption that (A,B) is in V.

Assume (A) holds. Since $C\subseteq A$ and $A\not\delta B$, we have $C\not\delta B$ and hence $(C,B)\in V$. Thus $(k-1)+m=n-1$ belongs to K which contradicts our choice of n. Assume (B) holds. Since $A_k\subseteq A$ and $A\not\delta B$, we have $A_k\not\delta B$ and hence $(A_k,B)\in V$. Thus $1+m<k+m=n$. This is a contradiction.

(5) Suppose $A\cap B\neq 0_\sim$. Let $A=A_1\cup\cdots\cup A_n$ and $B=B_1\cup\cdots\cup B_m$. Then there are i,j such that $A_i\cap B_j\neq 0_\sim$. So for any $\alpha\in\Gamma$,

$$f_\alpha(A_i)\cap f_\alpha(B_j)\supseteq f_\alpha(A_i\cap B_j)\neq 0_\sim$$

and hence $f_\alpha(A_i)\delta_\alpha f_\alpha(B_j)$. Thus $A\delta B$.

In all, δ is an intuitionistic fuzzy proximity on X. It is clear that $f_\alpha:(X,\delta)\to(X_\alpha,\delta_\alpha)$ is continuous for each $\alpha\in\Gamma$.

Finally, let (Y,δ_1) be an intuitionistic fuzzy proximity space and $f:(Y,\delta_1)\to(X,\delta)$ a map. It is clear that if f is continuous then $f_\alpha\circ f$ is also continuous. Conversely, suppose that $f_\alpha\circ f$ is continuous for each $\alpha\in\Gamma$. Let $A,B\in I(Y)$ and $A\delta_1 B$. Take any $f(A)=A_1\cup\cdots\cup A_n$ and $f(B)=B_1\cup\cdots\cup B_m$. Then

$$\begin{aligned}A\subseteq f^{-1}f(A) &= f^{-1}(A_1\cup\cdots\cup A_n)\\ &= f^{-1}(A_1)\cup\cdots\cup f^{-1}(A_n)\end{aligned}$$

and

$$\begin{aligned}B\subseteq f^{-1}f(B) &= f^{-1}(B_1\cup\cdots\cup B_m)\\ &= f^{-1}(B_1)\cup\cdots\cup f^{-1}(B_m).\end{aligned}$$

Put $A_0=f^{-1}(A_1)\cup\cdots\cup f^{-1}(A_n)$ and $B_0=f^{-1}(B_1)\cup\cdots\cup f^{-1}(B_m)$. Then $A\subseteq A_0$ and $B\subseteq B_0$. Since $A\delta_1 B$, we have $A_0\delta_1 B_0$. Hence there exist i,j such that $f^{-1}(A_i)\delta_1 f^{-1}(B_j)$. Since $f_\alpha\circ f$ is continuous,

$$f_\alpha(ff^{-1}(A_i))\delta_\alpha f_\alpha(ff^{-1}(B_j))$$

and hence $f_\alpha(A_i)\delta_\alpha f_\alpha(B_j)$ for each $\alpha\in\Gamma$. This proves that $f(A)\delta f(B)$. Therefore f is continuous.

Definition 3.3. Let (X,δ) be an intuitionistic fuzzy proximity space and $Y\subseteq X$. Let δ_Y be the initial intuitionistic fuzzy proximity on Y with respect to the inclusion map $i:Y\to X$. Then (Y,δ_Y) is called a *subspace* of (X,δ).

Directly from the property of an initial structure, we have the following result.

Theorem 3.4. *Let (Y,δ_Y) be a subspace of (X,δ) and $i:Y\to X$ the inclusion map. Let $f:Z\to Y$ be a map between intuitionistic fuzzy proximity spaces. Then the map f is continuous if and only if the map $i\circ f$ is continuous.*

In fact, the structure of subspace is described as follows:

Theorem 3.5. *Let (Y,δ_Y) be a subspace of (X,δ) and $i:Y\to X$ the inclusion map. For $A,B\in I(Y)$, the following statements are equivalent:*

(1) *$A\delta_Y B$.*

(2) *$A_X\delta B_X$, where $A_X:X\to I\times I$ is defined by*

$$A_X(x)=\begin{cases}A(x)=(\mu_A(x),\gamma_A(x)) & \text{if} \quad x\in Y,\\ (0,1) & \text{if} \quad x\notin Y.\end{cases}$$

(3) *For all C and D in $I(X)$ whose restrictions to Y are A and B, respectively, we have $C\delta D$.*

Proof. (1) → (2) Let $A\delta_Y B$. Since $i:(Y,\delta_Y)\to(X,\delta)$ is continuous, we have $i(A)\delta i(B)$. Clearly $i(A)=A_X$.

(2) → (3) Let $C,D\in I(X)$ with $C|_Y=A$ and $D|_Y=B$. Since $A_X\delta B_X$, we have $(C|_Y)_X\delta(D|_Y)_X$. Note that

$$(C|_Y)_X=i(C|_Y)=i(C\circ i)=i(i^{-1}(C))\subseteq C.$$

Since $C\supseteq(C|_Y)_X$ and $D\supseteq(D|_Y)_X$, we have $C\delta D$.

(3) → (1) Take any $A=A_1\cup\cdots\cup A_n$ and $B=B_1\cup\cdots\cup B_m$. Since $(A_X)|_Y=A$ and $(B_X)|_Y=B$, we have $A_X\delta B_X$. Note that

$$A_X=(A_1)_X\cup\cdots\cup(A_n)_X$$

and

$$B_X=(B_1)_X\cup\cdots\cup(B_m)_X .$$

So there are j,k such that $(A_j)_X\delta(B_k)_X$ and hence $i(A_j)\delta i(B_k)$. Thus $A\delta_Y B$.

Definition 3.6. Let $\{(X_\alpha,\delta_\alpha)\}_{\alpha\in\Gamma}$ be a family of intuitionistic fuzzy proximity spaces and $X=\prod_\alpha X_\alpha$. Let δ be the initial intuitionistic fuzzy proximity on X with respect to the family of canonical projections $\{\pi_\alpha:X\to X_\alpha\}_{\alpha\in\Gamma}$. Then (X,δ) is called the *product intuitionistic fuzzy proximity space.*

Directly from the property of an initial structure, we have the following result.

Theorem 3.7. *Let $f:Y\to\prod_\alpha X_\alpha$ be a map from an intuitionistic fuzzy proximity space to the product intuitionistic fuzzy proximity space. Then the map f is continuous if and only if the map $\pi_\alpha\circ f$ is continuous for all α.*

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