International Journal of Reliability and Applications
Vol. 6, No. 2, pp. 117-133, 2005

Estimations of the Parameters in a Two-component System Using Dependent Masked Data

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Abstract. Estimations of the parameters included in a two-component system are derived based on masked system life test data, when the probability of masking depends upon the exact cause of system failure. Also estimations of reliability for the individual components at a specified mission time are derived. Maximum likelihood and Bayes methods are used to derive these estimators. The problem is explained on a series system consisting of two independent components each of which has a Pareto distributed lifetime. Further we present numerical studies using simulation.

Key Words : *Pareto distribution,Bayes estimator, maximum likelihood estimator, reliability.*

1. INTRODUCTION

One of the interesting problems is to estimate the parameters included in a multi-component system based on system life test data. Such estimates are extremely useful because they can be used to deduce estimations of the reliability of the individual components of the system. These estimations reflect the components' reliability after their assembly into an operational system. Therefore, these estimates can be used, under some certain conditions, to predict the reliability of a new configuration of components in a new system.

Generally, one can derive the estimations of reliability components based on system life-test data. In many reliability situations and life testing, the exact cause of system failure is often unknown. Sometimes the exact cause of system failure can be isolated to a subset of the system components, but it remains unknown. Such type of data is called masked system life-test data. Masking is usually due to limited

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resources for diagnosing the causes of system failures as well as the modular nature of the system.

In such type of data, there are two possible observable quantities for each system on the life test: (*1*) the system life time, and (*2*) the set of components that contains the component causing the system failure.

Several papers used masked system life test data to estimate the parameters of the life time distributions of the individual components in multi-component systems. Miyakawa (1984) studied the problem of a 2-component series system when the system's components have constant failure rates. He derived closed form expressions for the maximum likelihood estimators for the parameters included based on masked data. Usher and Hodgson (1988) extended Miyakawa's results to a three-component series system using the same assumption that the failure rates of the components are constant. Guess, et al.(1991) extended and clarified the derivation of the likelihood function under the assumption that masking is independent of the exact failure cause. The exact maximum likelihood estimates using masked system data are derived by Lin, et al. (1993) based on the same assumption that the system components have constant failure rates. Sarhan (2001) derived maximum likelihood and Bayes estimates of the values of reliability of system's components in the case of n component series system when the components have constant failure rates. Iterative maximum likelihood procedure is used by Usher (1996) in the case of 2-component series system when the system's components life times have Weibull distributions. He illustrated the approach with a simple numerical example. The maximum likelihood estimators for the parameters included in the cases of 2-component and 3-component series systems are derived by Sarhan (2003) under the assumption that the lifetime distributions of the components are Weibull. He derived closed-form expressions for maximum likelihood estimates in some particular cases, which generalize the results obtained by Usher and Hodgson (1988). Sarhan (2004) derived Bayes and maximum likelihood estimators of the unknown parameters in the case of series system when the component lives having linear failure rate distributions. Sarhan and El-Gohary (2003) derived Bayes and maximum likelihood estimators for the parameters included in the cases of 2-component series systems when the component's lives having Pareto distributions.

In the previous developed models which used the masked system life-test data, it is assumed that masking occurs independently of the exact cause of system failure. This means that, the probability of observing a particular masked set does not depend on which component causes the system failure. In some cases, such assumption may not hold. For example, consider a system with two components where, under certain environmental conditions, the failure of either component can result in a fire and complete destruction of the system. If the system is destroyed, then the cause of failure can not be identified. Dependence occurs when the probability of the system's destruction differs based on which component causes the system failure. Moreover this probability of destruction given that a particular cause of failure may depend on time but it seems reasonable to assume that the ratio of probabilities will not be a function of time.

In this paper we use the masked system life test data, when the probability of masking is dependent upon the exact cause of system failure, to derive estimations of the parameters indexed to the distributions of the individual components in a series. Also we derive estimations of the reliability measures of the system components. Bayes and maximum likelihood procedure will be followed to derive such estimates. The problem is illustrated for a series system consisting of two independent components each of which has a a Pareto distributed lifetime.

The main objective in this paper is to derive the maximum likelihood and Bayes estimators of the parameters included in the system life tim distribution based on masked system life test data. It is assumed here that the set of the components that contains the component causing the system failure is dependent on the exact cause of failure.

Pareto distribution has many applications in socio-economics, statistics and reliability models, see for example (Johnson et al. (1994) and Soliman (1999)). The Pareto distribution arises as a mixture of exponential distributions and in simulation, Kumar and Tiwari (1989) and Lindley and Singpurwalla (1986).

Bayes estimation for the Pareto distribution with one unknown parameter is obtained by Arnold and Press (1983). Tiwari et al. (1996) developed a fully Bayes approach of estimation of reliability measures of the Pareto distribution with two unknown parameters wherein the range depends on one of the parameters, using the Gibbs sampler and the rejection/acceptance algorithm. Upadhyay and Shastri (1997) considered the full Bayes analysis of the Pareto distribution when the observations are doubly censored.

The assumptions and notations on which the model is based on are presented in Section 2. The likelihood function of the model is given in Section 3. The maximum likelihood estimators (mles) for the unknown parameters included in the model and their properties are given also in Section 3. We derived in this section the mles for the parameters in the case of independent masking data as a special case. Bayes analysis of the model is discussed in Section 4. Finally, numerical results and conclusions are given in Section 5.

2. NOTATIONS AND MODEL ASSUMPTIONS

Throughout this paper, the following assumptions are used:

Assumption A

A.1 The system is a series system consists of m independent components.

A.2 The liver of system's components have Pareto distributions with different parameters.

A.3 A number of n independent and identical systems are put on the life test. The test is terminated when all systems failed (there is no either censoring or replacement).

A.4 There are two observable quantities for each system on the life test: (*i*) the system lifetime, and (*ii*) the set of components that contains the component causing the system failure.

A.5 The cause of system failure may be either completely unknown or isolated to a subset of system components or exactly known.

A.6 The probability of masking is dependent on the true component causing the system failure.

In addition, we use the following notation:

n : the number of observations (sample size).

T_{ij} : The random lifetime of component $j\,(j = 1, \ldots, m)$ in system $i\,(i = 1, \ldots, n)$.

T_i : the random lifetime of system $i\,(i = 1, \ldots, n)$.

K_i : random index of the component causing the failure of system $i\,(i = 1, \ldots, n)$.

S_i : the random set of the system components that contains the component causes the failure system $i\,(i = 1, \ldots, n)$.

s_i : realization of S_i $(i = 1, \ldots, n)$.

t_i : realization of T_i $(i = 1, \ldots, n)$.

f_j : the probability density function (pdf) of component j $(j = 1, \ldots, m)$ in system $i\,(i = 1, \ldots, n)$.

$\bar{F}_j$: the reliability function of component j $(j = 1, \ldots, m)$ in system $i\,(i = 1, \ldots, n)$.

h_j : the failure rate function of component $j\,(j = 1, \ldots, m)$ in system $i\,(i = 1, \ldots, n)$.

θ : vector of unknown parameters indexed to the distributions of components lives, T_{ij}.

According to assumptions A.2 and A.3, the random variables T_{1j}, T_{2j}, , ... , T_{nj} $(j = 1, \ldots, m)$ being identical with the pdf of component j $(j = 1, \ldots, m)$ given by

$$f_j(x) = \frac{\theta_j}{\tau} \left(\frac{\tau}{x}\right)^{\theta_j+1}, \; \theta_j > 0, \, x \geq \tau, \tag{2.1}$$

its reliability function is

$$\bar{F}_j(x) = \left(\frac{\tau}{x}\right)^{\theta_j}, \ \theta_j > 0, \ x \geq \tau, \tag{2.2}$$

and its hazard rate and cumulative hazard functions, respectively, are

$$h_j(x) = \frac{\theta_j}{x}, \ \theta_j > 0, \ x \geq \tau \tag{2.3}$$

and

$$H_j(x) = \int_\tau^x h_j(u)du = \theta_j \ \ln\left(\frac{x}{\tau}\right), \ \theta_j > 0, \ x \geq \tau. \tag{2.4}$$

It is assumed here that the parameter τ is known while the parameters θ_1, θ_2, ..., θ_m are unknown.

3. MAXIMUM LIKELIHOOD ESTIMATORS

In this section we derive the likelihood function of the masked system life test data. Then it will be used to derive the maximum likelihood estimators for the vector of unknown parameters $\theta = (\theta_1, \theta_2, \ldots, \theta_m)$.

Given the set of observations (t_1, s_1), (t_2, s_2), ..., (t_m, s_m), the likelihood function is, Guess et al. (1991),

$$L(data, \theta) = \prod_{i=1}^{n} \left\{ \sum_{j \in s_i} P(S_i = s_i | T_i = t_i, \ K_i = j) \, f_j(t_i) \cdot \prod_{\ell=1, \, \ell \neq j}^{m} \bar{F}_\ell(t_i) \right\}. \tag{3.1}$$

Using the relation between $f_j(t)$, $h_j(t)$, $H_j(t)$ and $\bar{F}_j(t)$ given by

$$f_j(t) = h_j(t) \, \bar{F}_j(t) \ \text{ and } \bar{F}_j(t) = \exp\{-H_j(t)\}$$

one can write the likelihood function $L(data, \theta)$ as:

$$L(data, \theta) = \exp\left\{ -\sum_{i=1}^{n} \sum_{j=1}^{m} H_j(t_i) \right\} \prod_{i=1}^{n} \left[\sum_{j \in s_i} P(S_i = s_i | T_i = t_i, \ K_i = j) \, h_j(t_i) \right] \tag{3.2}$$

The assumption A.6 means that for any $j \neq \ell \in s_i$:

$$P(S_i = s_i | T_i = t_i, \ K_i = j) \neq P(S_i = s_i | T_i = t_i, \ K_i = \ell) \tag{3.3}$$

These probabilities are called the masking probabilities, see Guttman et al. (1995). These masking probabilities are, of course, dependent on S_i. If we think of the example of two components presented in the Introduction, it seems natural to assume proportional probabilities for $j,\ \ell \in S_i$ and $j \neq \ell$. That is,

$$P(S_i = s_i|T_i = t_i,\ K_i = j) = \pi \times P(S_i = s_i|T_i = t_i,\ K_i = \ell) \tag{3.4}$$

where $\pi > 0$ is implicity a function of j, ℓ, but not of t_i. Note that if $\pi \neq 1$ there is dependent masking over S_i, while if $\pi = 1$, the special case of independent masking over S_i holds.

Generally, based on the relation (3.3), it is so difficult to study the problem for a general value of m. So, we shall study the problem when $m = 2$. In this case we need the following auxiliary notations. Let n_1 be the number of observations when the component 1 causes the failure of the system and the system lifetimes are x_1, $x_2, \ldots, x_{n_1}$. Let n_2 be the number of observations when the component 2 causes the failure of the system and the system lifetimes are $y_1,\ y_2, \ldots,\ y_{n_2}$. Namely, n_j $(j = 1, 2)$ is the number of observation when the cause of system is known and $s_i = \{j\}$ $(j = 1, 2)$. Let n_{12} be the number of observations when the cause system failure is unknown (either component 1 or component 2) and the system lifetimes are $z_1,\ z_2, \ldots,\ z_{n_{12}}$. That is, n_{12} is the number of observation when $s_i = \{1, 2\}$. Note that $n = n_1 + n_2 + n_{12}$ and $\{t_1,\ t_2, \ldots,\ t_n\} = \{x_1,\ x_2, \ldots,\ x_{n_1},\ y_1,\ y_2, \ldots,\ y_{n_2},\ z_1,\ z_2, \ldots,\ z_{n_{12}}\}$.
We need also the following assumptions. Let

$$\begin{aligned}
\lambda_j(t_i) &= P\left(S_i = \{j\}|T_i = t_i,\ K_i = j\right),\ j = 1, 2\,, \\
\lambda_3(t_i) &= P\left(S_i = \{1,\ 2\}|T_i = t_i,\ K_i = 1\right)\,, \\
\lambda_4(t_i) &= P\left(S_i = \{1,\ 2\}|T_i = t_i,\ K_i = 2\right)\,,
\end{aligned}$$

Based on the above assumptions and notations, the likelihood function (3.2) reduces to the following form

$$\begin{aligned}
L(data, \theta) &= \exp\left\{-\sum_{i=1}^{n}(H_1(t_i) + H_2(t_i))\right\} \prod_{i=1}^{n_1} \{\lambda_1(x_i)h_1(x_i)\} \\
&\quad \times \prod_{i=1}^{n_2} \{\lambda_2(y_i)h_2(y_i)\} \prod_{i=1}^{n_{12}} \{\lambda_3(z_i)h_1(z_i) + \lambda_4(z_i)h_2(z_i)\}
\end{aligned} \tag{3.5}$$

Substituting (2.3) and (2.4) into (3.5) we get

$$L(data, \theta) = \theta_1^{n_1}\, \theta_2^{n_2}\, B_1^{\theta_1+\theta_2} \prod_{i=1}^{n} t_i^{-1} \prod_{i=1}^{n_1} \lambda_1(x_i) \prod_{i=1}^{n_2} \lambda_2(y_i) \prod_{i=1}^{n_{12}} \left\{\lambda_3(z_i)\left[\theta_1 + \frac{\lambda_4(z_i)}{\lambda_3(z_i)}\,\theta_2\right]\right\} \tag{3.6}$$

where $B_1 = \tau^n \prod_{i=1}^{n} t_i^{-1}$.

Since the conditional probabilities λ_j ($j = 1, 2, 3, 4$) do not depend on the parameters, so one can rewrite the likelihood function (3.6) up to a constant as

$$L(data, \theta) = \theta_1^{n_1} \theta_2^{n_2} B_1^{\theta_1+\theta_2} (\theta_1 + \pi\theta_2)^{n_{12}} \tag{3.7}$$

where $\pi = \frac{\lambda_4}{\lambda_3}$. Following the discussion about the relation (3.4), π is not a function of t_i, and $\pi > (<)1$ if $\lambda_4 > (<)\lambda_3$ and $\pi > 0$. Without loss of generality we assume that $0 < \pi < 1$.

Theorem 3.1 The maximum likelihood estimators for the parameters θ_1 and θ_2 are

$$\hat{\theta}_1 = \frac{-b + \sqrt{b^2 + 4\pi(1-\pi)n_1 n}}{-2\,(1-\pi)\,\ln B_1}, \tag{3.8}$$

$$\hat{\theta}_2 = \frac{b + 2\,(1-\pi)\,n - \sqrt{b^2 + 4\pi(1-\pi)n_1 n}}{-2\,(1-\pi)\,\ln B_1} \tag{3.9}$$

where $b = n_2 + \pi\, n_1 - (1-\pi)n$.

Proof. Since the log-likelihood function is

$$\mathcal{L} = n_1 \ln\theta_1 + n_2 \ln\theta_2 + (\theta_1 + \theta_2)\ln B_1 + n_{12}\ln(\theta_1 + \pi\theta_2)$$

then the likelihood equations become

$$\begin{aligned} 0 &= \frac{n_1}{\theta_1} + \frac{n_{12}}{\theta_1 + \pi\theta_2} + \ln B_1\,, \\ 0 &= \frac{n_2}{\theta_2} + \frac{\pi\, n_{12}}{\theta_1 + \pi\theta_2} + \ln B_1 \end{aligned} \tag{3.10}$$

Solving the above system of two equations with respect to θ_1 and θ_2 we can get the mles for θ_1 and θ_2 as given respectively by (3.8) and (3.9), which completes the proof.

Corollary 3.1 If the masking is independent with the true cause of system failure the the mles for θ_1 and θ_2 become

$$\hat{\theta}_j = -\frac{n_j}{\ln B_1}\left(1 + \frac{n_{12}}{n_1 + n_2}\right), \; j = 1, 2. \tag{3.11}$$

Proof. Setting $\pi \to 1$ in (3.8) and (3.9), one gets (3.11) which completes the proof.

The above corollary means that the results given in Sarhan and Gohary (2003) can be derived as special case of the results presented here.

One of the weakness of the mles of θ_1 and θ_2 given in theorem (3.1) arises when the available data is completely masked. Since the statements given in the theorem will not be defined when $n_1 = n_2 = 0$.

Some statistical properties of the mles obtained are given in the following theorem.

Theorem 3.2 The mle for $\theta_1 + \theta_2$, $\hat{\theta}_1 + \hat{\theta}_2$, satisfies the following properties:

1. it is an asymptotic unbiased estimator,
2. it is a consistent estimator,
3. it is a sufficient statistic.

Proof. Since T_{ij} has Pareto distribution with parameters θ_j and τ. Then each of $T_i = \min(T_{i1}, T_{i2})$, $i = 1, 2, \ldots, n$ has also Pareto distribution with parameters $\theta_1 + \theta_2$ and τ. Therefore, by using the transformation method, the pdf of $Y_i = \frac{\tau}{T_i}$ takes the following form

$$f_{Y_i}(y) = (\theta_1 + \theta_2)\, y^{\theta_1+\theta_2-1},\ 0 < y < 1. \tag{3.12}$$

Using the transformation method together with (3.12) and the relation $Z_i = -\ln Y_i$, one can show that Z_i has exponential distribution with the following pdf

$$f_{Z_i}(z) = (\theta_1 + \theta_2)\, \exp\{-(\theta_1 + \theta_2)\, z\},\ z > 0. \tag{3.13}$$

Therefore, $-\ln B_1$ has gamma distribution with parameters n and $\theta_1 + \theta_2$ and hence $W = -\frac{1}{\ln B_1}$ has inverted gamma distribution with the following pdf

$$f_W(w) = \frac{(\theta_1 + \theta_2)^n}{\Gamma(n)} \frac{1}{w^{n+1}} \exp -(\theta_1 + \theta_2)/w,\ w > 0. \tag{3.14}$$

Then

$$E[\hat{\theta}_1 + \hat{\theta}_2] = E\left[-\frac{n}{\ln B_1}\right] = n\, E\,[W] = \frac{n(\theta_1 + \theta_2)}{n-1}. \tag{3.15}$$

and

$$\lim_{n\to\infty} E[\hat{\theta}_1 + \hat{\theta}_2] = \theta_1 + \theta_2 \tag{3.16}$$

which implies statement 1. Also, using (3.14) we have

$$\mathrm{Var}[\hat{\theta}_1 + \hat{\theta}_2] = \mathrm{Var}\left[-\frac{n}{\ln B_1}\right] = n^2 \mathrm{Var}[W] = n^2 \frac{(\theta_1 + \theta_2)^2}{(n-1)^2(n-2)}$$

Therefore,

$$\lim_{n\to\infty} \mathrm{Var}[\hat{\theta}_1 + \hat{\theta}_2] = (\theta_1 + \theta_2)^2 \lim_{n\to\infty} \frac{1}{(1-\frac{1}{n})^2\,(n-2)} = 0$$

which proves statement 2. Finally, we can write the likelihood function given by (3.6) as in the following form

$$\begin{aligned} L(data,\theta) &= \left\{\prod_{i=1}^{n} t_i^{-1} \prod_{i=1}^{n_1} \lambda_1(x_i) \prod_{i=1}^{n_2} \lambda_2(y_i) \prod_{i=1}^{n_{12}} \lambda_3(z_i)\right\} \\ &\quad \times \theta_1^{n_1}\, \theta_2^{n_2}\, (\theta_1 + \pi\,\theta_2)^{n_{12}} \,\exp\{(\theta_1+\theta_2)\,\ln B_1\}. \end{aligned}$$

But $\ln B_1 = \frac{-n}{\hat{\theta}_1+\hat{\theta}_2}$, then

$$\begin{aligned} L(data,\theta) &= \left\{\prod_{i=1}^{n} t_i^{-1} \prod_{i=1}^{n_1} \lambda_1(x_i) \prod_{i=1}^{n_2} \lambda_2(y_i) \prod_{i=1}^{n_{12}} \lambda_3(z_i)\right\} \\ &\quad \times \theta_1^{n_1}\, \theta_2^{n_2}\, (\theta_1 + \pi\,\theta_2)^{n_{12}} \cdot \exp\left\{\frac{-n(\theta_1+\theta_2)}{\hat{\theta}_1+\hat{\theta}_2}\right\} \end{aligned} \tag{3.17}$$

which can be written as

$$L(data,\theta) = H(t_1,\, t_2, \ldots,\, t_n) \cdot K(\theta_1,\, \theta_2,\, \hat{\theta}_1,\, \hat{\theta}_2). \tag{3.18}$$

That is, the likelihood function can be written as a product of two functions L_1 and L_2, where L_1 depends only on the data while the function L_2 depends on the parameters and on the data through the estimators obtained. This completes the proof of the statement 3 and then the proof of the theorem is completed.

4. BAYES ANALYSIS

As we mentioned in section 2, the mles of the parameters θ_j $(j = 1, 2)$ are not available when the observations are completely masked. Therefore, another approach takes place. As we shall see, the Bayes procedure provides estimators based on different types of masked data, also it provides a good estimator. In order to derive the Bayes estimators of the unknown parameters, we need the following set of additional assumptions:

Assumption B

B.1 The parameters θ_1 and θ_2 are independent random variables.

B.2 The prior distribution of the θ_j is uniform on the interval $A_j = [a_j,\, b_j] \subset (0, \infty)$ $(j = 1, 2)$ with the following pdf

$$g_j(\theta_j) = \begin{cases} \frac{1}{b_j - a_j}, & \text{if } \theta_j \in A_j, \\ 0, & \text{otherwise}. \end{cases} \tag{4.1}$$

B.3 The loss incurred when the parameters θ_1 and θ_2 are estimated respectively by $\hat{\theta}_1$ and $\hat{\theta}_2$ is a quadratic:

$$l\left(\theta_1,\, \theta_2,\, \hat{\theta}_1,\, \hat{\theta}_2\right) = k_1\left(\theta_1 - \hat{\theta}_1\right)^2 + k_2\left(\theta_2 - \hat{\theta}_2\right)^2,\ k_1, k_2 > 0. \tag{4.2}$$

To drive the Bayes estimators for the parameters θ_1 and θ_2, we need the following lemma, which can be proved by using integration by parts.

Lemma 4.1 Let $I_{a,b}(n, Q)$ be define by

$$I_{a,b}(n, Q) = \int_a^b u^n\, Q^u\, du,\ Q > 0,\ n = 1, 2, \ldots \tag{4.3}$$

Then

$$I_{a,b}(n, Q) = \sum_{i=1}^{n} \frac{(-1)^i\, n!}{(n-i)!\,(\ln Q)^{i+1}} \left[b^{n-i}\, Q^b - a^{n-i}\, Q^a\right] \tag{4.4}$$

The following corollary can be easily proved by using binomial expansion of $(\theta_1 + \pi\theta_2)^{n_{12}}$.

Corollary 4.1 The likelihood function (3.6) can be written as

$$L(data, \theta) = \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \theta_1^{n_1+j}\, \theta_2^{n_2+n_{12}-j} \pi^{n_{12}-j} B_1^{\theta_1+\theta_2} \tag{4.5}$$

Theorem 4.1 Based on the groups of assumptions A and B, the joint posterior pdf of $\theta = (\theta_1,\, \theta_2)$ is

$$g(\theta|data) = \frac{1}{\Psi(0,0)} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \theta_1^{n_1+j}\, \theta_2^{n_2+n_{12}-j}\, \pi^{n_{12}-j}\, B_1^{\theta_1+\theta_2}, \tag{4.6}$$

for $(\theta_1,\, \theta_2) \in A_1 \times A_2$, where

$$\Psi(0,0) = \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \pi^{n_{12}-j}\, I_{a_1,b_1}(n_1 + j, B_1)\, I_{a_2,b_2}(n_2 + n_{12} - j, B_1)\,. \tag{4.7}$$

Proof. The joint posterior pdf of $\theta = (\theta_1,\, \theta_2)$ is related to the joint prior pdf of θ, say $g(\theta)$, and the likelihood function according to the following formula, Martz and Waller (1982),

$$g(\theta|data) = \frac{g(\theta)\, L(data,\, \theta)}{f(data)}, \tag{4.8}$$

where $f(data) = \int_{-\infty}^{\infty} g(\theta)\, L(data,\, \theta)\, d\theta$. Since θ_1 and θ_2 are independent, then the joint prior pdf of (θ_1, θ_2) becomes $g(\theta) = g_1(\theta_1)\, g_2(\theta_2)$. Substituting from (3.7) and (4.1) into (4.8),

$$g(\theta)L(data) \;=\; \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \frac{\pi^{n_{12}-j}\, B_1^{\theta_1+\theta_2} \theta_1^{n_1+j} \theta_2^{n_2+n_{12}-j}}{(b_2-a_2)(b_1-a_1)} \tag{4.9}$$

and

$$\begin{aligned} f(data) &= \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \pi^{n_{12}-j} \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{B_1^{\theta_1+\theta_2} \theta_1^{n_1+j} \theta_2^{n_2+n_{12}-j}}{(b_2-a_2)(b_1-a_1)}\, d\theta_2\, d\theta_1 \\ &= \frac{\sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \pi^{n_{12}-j}\, I_{a_1,b_1}(n_1+j, B_1)\, I_{a_2,b_2}(n_2+n_{12}-j, B_1)}{(b_2-a_2)(b_1-a_1)} \\ &= \frac{\Psi(0,0)}{(b_2-a_2)(b_1-a_1)}, \end{aligned} \tag{4.10}$$

Substituting from (4.9) and (4.10) into (4.8) one gets (4.6), which completes the proof.

The following corollary gives the marginal posterior pdf's of θ_1 and θ_2.

Corollary 4.2 The marginal posterior pdf's of θ_ℓ ($\ell = 1, 2$) are

$$g_1(\theta_1|data) = \frac{1}{\Psi(0,0)} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \pi^{n_{12}-j}\, B_1^{\theta_1} \theta_1^{n_1+j}\, I_{a_2,b_2}(n_2+n_{12}-j,\, B_1)\,, \tag{4.11}$$

for $\theta_1 \in A_1$, and for $\theta_2 \in A_2$

$$g_2(\theta_2|data) = \frac{1}{\Psi(0,0)} \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \pi^{n_{12}-j}\, B_1^{\theta_2} \theta_2^{n_2+n_{12}-j}\, I_{a_1,b_1}(n_1+j,\, B_1)\,. \tag{4.12}$$

Proof. Starting with the relation between the marginal posterior pdf of θ_ℓ ($\ell = 1, 2$) and the joint posterior pdf of $(\theta_1,\, \theta_2)$ given by

$$g_1(\theta_1|data) = \int_{a_2}^{b_2} g(\theta_1, \theta_2|data)\, d\theta_2 \;\text{ and }\; g_2(\theta_2|data) = \int_{a_1}^{b_1} g(\theta_1, \theta_2|data)\, d\theta_1 \tag{4.13}$$

Then substituting from (4.6) into (4.13) and making some simple calculations, one reaches the proof of the Corollary.

Corollary 4.3 The posterior r−th moment ($r = 1, 2, \ldots$) of θ_ℓ ($\ell = 1, 2$) are:

$$\mu_{\theta_\ell}^{(r)} = \frac{\Psi(r\delta_{1\ell}, r\delta_{2\ell})}{\Psi(0,0)}\,, \tag{4.14}$$

where

$$\Psi(r\delta_{1\ell}, r\delta_{2\ell}) = \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \pi^{n_{12}-j} \prod_{i=1}^{2} I_{a_i,b_i}\left(n_i - (-1)^i j + n_{12}\,\delta_{i2} + r\delta_{i\ell}, B_1\right) \quad (4.15)$$

and $\delta_{j\ell} = 1$, if $\ell = j$ and 0 if $\ell \neq j$.

Proof. Since the $r-$th posterior moment of θ_ℓ is given by

$$\mu_{\theta_\ell}^{(r)} = \int_{a_\ell}^{b_\ell} \theta_\ell^r\, g_\ell(\theta_\ell)\, d\theta_\ell \quad (4.16)$$

Substituting from (4.11) and (4.12) into (4.16) and making some simple integrations, we get (4.14) which completes the proof.

Now we are ready to present the Bayes estimators for the unknown parameters θ_1 and θ_2 and their associated minimum posterior risks.

Theorem 4.2 Under the groups A and B of assumptions:

1. The Bayes estimators for θ_ℓ ($\ell = 1, 2$) are

$$\tilde{\theta}_\ell = \frac{\Psi(\delta_{1\ell}, \delta_{2\ell})}{\Psi(0,0)}, \quad (4.17)$$

2. The minimum posterior risks associated with $\tilde{\theta}_\ell$ ($\ell = 1, 2$) are

$$\mathrm{Var}(\theta_\ell|data) = \frac{\Psi(2\,\delta_{1\ell}, 2\,\delta_{2\ell})}{\Psi(0,0)} - \left\{\frac{\Psi(\delta_{1\ell}, \delta_{2\ell})}{\Psi(0,0)}\right\}^2, \quad (4.18)$$

Proof. Since under the squared error loss, the Bayes estimator for θ_ℓ is defined as the posterior expectation of θ_ℓ and the associated minimum posterior risk is the posterior variance, see Martz and Waller (1982). Then for $\ell = 1, 2$, we have

$$\tilde{\theta}_l = \int_{a_\ell}^{b_\ell} \theta_\ell\, g_\ell(\theta_\ell|data)\, d\theta_\ell = \mu_{\theta_\ell}^{(1)} \quad (4.19)$$

and the minimum posterior risk associated with $\tilde{\theta}_\ell$ is

$$\begin{aligned} \mathrm{Var}(\theta_\ell|data) &= \int_{a_\ell}^{b_\ell} \theta_\ell^2\, g_\ell(\theta_\ell|data)\, d\theta_\ell - \left(\int_{a_\ell}^{b_\ell} \theta_\ell\, g_\ell(\theta_\ell|data)\, d\theta_\ell\right)^2 \\ &= \mu_{\theta_\ell}^{(2)} - \left(\mu_{\theta_\ell}^{(1)}\right)^2 \end{aligned} \quad (4.20)$$

Substituting from (4.14) into (4.19) and (4.20), one reaches the proof of theorem.

Finally, the Bayes estimators for the values of component's reliability functions and their associated minimum posterior risks are given in the following theorem.

Theorem 4.3 Under the groups A and B of assumptions:

1. The Bayes estimators for $\bar{F}_\ell(t_0)$ $(\ell = 1, 2)$ are

$$\tilde{\bar{F}}_\ell(t_0) = \frac{\Delta_\ell^{(1)}}{\Psi(0,0)} \tag{4.21}$$

2. The minimum posterior risks associated with $\tilde{\bar{F}}_\ell$ $(\ell = 1, 2)$ are

$$\mathrm{Var}(\bar{F}_\ell|data) = \frac{\Delta_\ell^{(2)}}{\Psi(0,0)} - \left[\frac{\Delta_\ell^{(1)}}{\Psi(0,0)}\right]^2, \tag{4.22}$$

where for $r = 1, 2$

$$\Delta_\ell^{(r)} = \sum_{j=0}^{n_{12}} \binom{n_{12}}{j} \pi^{n_{12}-j} \prod_{i=1}^{2} I_{a_i,b_i}\left(n_i - (-1)^i j + \delta_{i2} n_{12},\ [1 + ((\tau/t_0)^r - 1)\ \delta_{i\ell}]\, B_1\right). \tag{4.23}$$

Proof. Since the Bayes estimator and its associated minimum posterior risk for the reliability function $\bar{F}_\ell(t_0)$ are the posterior expectation and posterior variance of $\bar{F}_\ell(t_0)$, respectively, see Martz and Waller (1982). Then, the Bayes estimator for $\bar{F}_\ell(t_0)$ $(\ell = 1, 2)$ are

$$\tilde{\bar{F}}_\ell(t_0) = \int_{a_\ell}^{b_\ell} \bar{F}_\ell(t_0)\, g_\ell(\theta_\ell|data) d\theta_\ell \tag{4.24}$$

and the associated minimum posterior risks are

$$\mathrm{Var}(\bar{F}_\ell|data) = \int_{a_\ell}^{b_\ell} (\bar{F}_\ell(t_0))^2\ g_\ell(\theta_\ell|data) d\theta_\ell - \left\{\int_{a_\ell}^{b_\ell} \bar{F}_\ell(t_0)\, g_\ell(\theta_\ell|data) d\theta_\ell\right\}^2 \tag{4.25}$$

Substituting from (2.2), (4.11) and (4.12) into (4.24) and (4.25) and making some simple calculations we get (4.21) and (4.22), which completes the proof.

5. SIMULATION STUDY AND CONCLUSION

We present in this section numerical results based on large simulation studies. This studies have been made to introduce two examples. In these examples, it is assumed that the system consists of two independent components. The lifetimes of components 1 and 2 have Pareto distributions with parameters $\theta_1 = 2.8$, $\theta_2 = 3.5$ and $\tau = 0.1$. For Bayes procedure it is assumed that θ_1 and θ_2 having uniform prior distributions on intervals $\mathbf{A}_1 = [0.01,\ 5.59]$ and $\mathbf{A}_2 = [0.03,\ 6.97]$, respectively.

Example 5.1 In this example it is assumed that 30 identical systems were put on the life test. Table 1 shows the data simulated in this example.

Table 1. The simulated data for example 1.

i	t_i	S_i	i	t_i	S_i	i	t_i	S_i
1	0.101	{1,2}	11	0.104	{1,2}	21	0.112	{2}
2	0.112	{2}	12	0.112	{1,2}	22	0.186	{2}
3	0.119	{1}	13	0.106	{1,2}	23	0.104	{1,2}
4	0.116	{1,2}	14	0.113	{1,2}	24	0.118	{1,2}
5	0.121	{1,2}	15	0.101	{1,2}	25	0.112	{1,2}
6	0.116	{1,2}	16	0.108	{2}	26	0.108	{1}
7	0.103	{1,2}	17	0.119	{1,2}	27	0.119	{1}
8	0.104	{1}	18	0.133	{1,2}	28	0.119	{2}
9	0.104	{1,2}	19	0.104	{1,2}	29	0.105	{2}
10	0.102	{1,2}	20	0.103	{1,2}	30	0.119	{1,2}

Using the data given in Table 1, we get that $B_1 = \prod_{i=1}^{n}\left(\frac{\tau}{t_i}\right) = 2.869 \times 10^{-2}$, $n_1 = 4$, $n_2 = 6$ and $n_{12} = 20$.

Using these information and according to the theoretical results derived sections 3 and 4 we calculated the mles and Bayes estimates of θ_ℓ $(\ell = 1, 2)$ and their associated percentage errors. We derived such estimates when $\pi = 0.1$, 0.5 and 0.9. Table 2 gives the estimates obtained and their associated percentage errors. The percentage error associated with the estimate $\hat{\theta}$ of θ is given by

$$\mathrm{PE}_{\hat{\theta}} = \frac{|\hat{\theta} - \text{exact value of } \theta|}{\text{exact value of } \theta} \times 100\%. \tag{5.1}$$

Table 2 gives the MLE and Bayes estimates of θ_1, θ_2 and their associated percentage errors.

Table 2. Estimates and associated percentage errors.

		ML		Bayes	
π	Parameter	Estimate	PE	Estimate	PE
0.1	θ_1	6.605	135.91	3.857	37.756
	θ_2	1.842	47.360	2.125	39.278
0.5	θ_1	5.632	101.14	3.265	16.604
	θ_2	2.816	19.544	2.842	18.792
0.9	θ_1	3.822	36.499	2.674	4.513
	θ_2	4.626	32.169	3.473	0.770

It seems from results presented in Table 2 that:

1. the percentage error associated with Bayes estimate of each parameter is smaller than that associated with the maximum likelihood estimate of that parameter for all cases of π.

2. the percentage error associated with both Bayes and maximum likelihood estimates become smaller when the value of π approaches to 0.8 $\left(= \frac{h_1(x)}{h_2(x)} = \frac{\theta_1}{\theta_2}\right)$.

Accordingly, we can say that, for the data given in this example, Bayes procedure provides a better estimator than the maximum likelihood procedure in the sense of having smaller percentage error. Also, the cases in which the value of π closes to its exact value give better estimates in the sense of having smaller percentage error.

To investigate the influence of the value of π on the accuracy of the estimators given in this paper we present the following example.

Example 5.2 In order study the influence of the value of π on the accuracy of the mle and Bayes estimators, a large simulation study is carried out in this example according to the following scheme:

1. specify the value of π.
2. Generate a random sample with size $n = 30$ of the system life time, (t_1, s_1), $\ldots$, (t_n, s_n). Then the values of n_1, n_2 and n_{12} are determined.
3. Calculate $\hat{\theta}_1$, $\hat{\theta}_2$ as given in theorem 3.1 and $\tilde{\theta}_1$, $\tilde{\theta}_2$ as given in theorem 4.2.
4. Calculate the squared error for each estimate, $(\theta_\ell - \hat{\theta}_\ell)^2$ and $(\theta_\ell - \tilde{\theta}_\ell)^2$, $l = 1, 2$.
5. Repeat steps (2-4) 5000 times.
6. Calculate the mean squared error associated with each estimate of θ_ℓ ($\ell = 1, 2$) according to the following relation
$$\mathrm{MSE}_{\hat{\theta}_\ell} = \frac{\sum_{i=1}^{5000}(\theta_\ell - \hat{\theta}_\ell^{(i)})^2}{5000} \text{ and } \mathrm{MSE}_{\tilde{\theta}_\ell} = \frac{\sum_{i=1}^{5000}(\theta_\ell - \tilde{\theta}_\ell^{(i)})^2}{5000}, \tag{5.2}$$
where $\hat{\theta}_\ell^{(i)}$ and $\tilde{\theta}_\ell^{(i)}$ are the mle and Bayes estimate of θ_ℓ using sample i.
7. Steps 1-6 are done for $\pi = 0.1, 0.2, \ldots, 1.0$.

Table 3 shows the values of MSE's associated with both mles and Bayes estimates of the parameters θ_ℓ ($\ell = 1, 2$) and $\theta_1 + \theta_2$ against the value of π.

Table 3. The MSE associated with $\hat{\theta}_\ell$ and $\tilde{\theta}_\ell$ ($\ell = 1, 2$).

	MLE			Bayes		
π	θ_1	θ_2	$\theta_1 + \theta_2$	θ_1	θ_2	$\theta_1 + \theta_2$
0.1	4.87143	3.40430	8.27573	2.47509	2.71376	5.18886
0.2	4.39812	3.11155	7.50967	2.28000	2.47766	4.75766
0.3	3.98430	2.74879	6.73309	2.07813	2.19742	4.27555
0.4	3.40098	2.41231	5.81329	1.91266	2.01893	3.93159
0.5	2.83013	2.13676	4.96689	1.73030	1.93244	3.66274
0.6	2.41660	1.78747	4.20406	1.57685	1.80747	3.38432
0.7	2.09663	1.59869	3.69532	1.44700	1.83104	3.27803
0.8	1.64313	1.44158	3.08471	1.37359	1.81190	3.18549
0.9	1.41918	1.39508	2.81426	1.29047	1.83406	3.12453

Based on the results in Table 3, we see that

1. The MSE becomes smaller when the value of π approaches to 0.8.
2. The MSE associated with Bayes estimate is smaller than that associated with mle for all values of π.

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