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On a Normal Contact Metric Manifold

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ABSTRACT. We find the expression of the curvature tensor field for a manifold with is endowed with an almost contact structure satisfying the condition (1.7). By using this condition we obtain some properties of the Ricci tensor and scalar curvature (cf. Theorem 3.2 and Proposition 3.2).

1. Preliminaries

Let M be a real $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (f, ξ, η, g) satisfying

$$(1.1) \qquad (a)\ f^2 = -I + \eta \otimes \xi, \qquad (b)\ \eta(\xi) = 1, \qquad (c)\ f(\xi) = 0,$$
$$(d)\ \eta(X) = g(X, \xi), \qquad (e)\ g(fX, Y) + g(X, fY) = 0$$

for any vector fields X, Y tangent to M (for example, see [1]), where I is the identity of the tangent bundle TM on M. Throughout the paper, all manifolds and maps are differentiable of class C^∞. We denote by $F(M)$ the algebra of the differentiable functions on M and by $\Gamma(E)$ the $F(M)$-module of the sections of a vector bundle E over M.

The curvature tensor field, denoted by K, with respect to the Levi-Civita connection ∇, on M is given by

$$(1.2) \qquad K(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad \forall X, Y, Z \in \Gamma(T\tilde{M}).$$

If we consider an orthonormal basis on M $\{e_i, fe_i = e_{n+i}, \xi\}$, $(i = 1, \cdots, n)$ then the Ricci tensor S on M is defined by

$$(1.3) \qquad S(X,Y) = \sum_{i=1}^{2n+1} g(K(X, e_i)e_i, Y), \quad \forall\ X, Y \in \Gamma(TM),$$

and the scalar curvature r of M is

$$(1.4) \qquad r = \sum_{i=1}^{2n+1} S(e_i, e_i).$$

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The Ricci operator Q on M is given by

$$S(X,Y) = g(QX,Y), \quad \forall X,Y \in \Gamma(TM). \tag{1.5}$$

The Nijenhuis tensor field with respect to the tensor field f, denoted by N_f, is

$$N_f(X,Y) = [fX,fY] + f^2[X,Y] - f[fX,Y] - f[X,fY], \quad \forall X,Y \in \Gamma(TM).$$

The almost contact metric manifold $M(f,\xi,\eta,g)$ is called normal if

$$N_f(X,Y) + 2d\eta(X,Y)\xi = 0, \quad \forall X,Y \in \Gamma(TM).$$

According to [3], the almost contact metric manifold M is normal iff

$$(\nabla_X f)Y = f((\nabla_{fX} f)Y) - g(Y,\nabla_{fX}\xi)\xi, \quad \forall X,Y \in \Gamma(TM). \tag{1.6}$$

The purpose of this paper is to study those almost contact manifolds which satisfying the condition

$$(\nabla_X f)Y = g(\nabla_{fX}\xi,Y)\xi - \eta(Y)\nabla_{fX}\xi, \quad \forall X,Y \in \Gamma(TM). \tag{1.7}$$

S. S. Eum studied in [4], the Kaehlerian hypersurfaces isometrically immersed in an almost contact Riemannian manifold which satisfies (1.7). By straightforward calculation, using (1.6) and (1.7) we infer

Theorem 1.1. *Let M be an almost contact metric manifold such that* (1.7) *holds. Then the next assertions are true*

(1.8) (*a*) *M-is normal*; (*b*) $\nabla_{fX}\xi = f\nabla_X\xi$; (*c*) $(\nabla_\xi f)X = 0$;
(*d*) $(L_\xi\eta)X = 0; \forall X \in \Gamma(TM)$,

where L denote the Lie derivative.

By using Theorem 1.1 we deduce

Corollary 1.1. *Let M be an almost contact metric manifold such that* (1.7) *holds. Then the integral curves of the structure tensor field ξ are geodesics, that is $\nabla_\xi\xi = 0$.*

Remarks.

(1) If (1.7) is true, then the structure tensor field ξ is not necessary to be a Killing vector field.

(2) It is easy to see that, in particular, if:

 (a) $\nabla_X\xi = -fX$, then M is a Sasakian manifold,

 (b) $\nabla_X\xi = X - \eta(X)\xi$, then M becomes a Kenmotsu manifold,

 (c) $\nabla_X\xi = 0$, then M is a cosymplectic manifold,

(d) $\nabla_X \xi = -\alpha fX - \beta f^2 X, \alpha, \beta \in F(M)$, then M become a trans-Sasakian manifold,

(e) $\nabla_X \xi = -FX$, where F is a tensor field of type (1,1) with $rank F < 2n$, and ξ is a Killing vector, then M is a quasi-Sasakian manifold of $rank < 2n+1$.

Next, suppose that there exists a tensor field F of type (1,1) such that

$$\nabla_X \xi \;=\; -FX, \quad \forall X \in \Gamma(TM). \tag{1.9}$$

By using (1.1c), (1.2), (1.7), (1.8b) and (1.9), we obtain

Proposition 1.1. *Let M be an almost contact metric manifold satisfying* (1.7). *Then we have*

$$\begin{aligned}(1.10)\quad & (a)\ F(\xi) = 0; \qquad (b)\ g(FX, \xi) = 0; \qquad (c)\ fFX = FfX;\\ & (d)\ (L_\xi g)(X, Y) = -g(X, FY) - g(FX, Y);\\ & (e)\ 2d\eta(X, Y) = g(X, FY) - g(FX, Y);\\ & (f)\ K(X, Y)\xi = (\nabla_X F)Y - (\nabla_Y F)X,\ \forall X, Y \in \Gamma(TM).\end{aligned}$$

For the sake of simplicity we denote

$$\begin{aligned}(1.11)\quad U(X, Y)Z \;=\; & g(X, Z)Y - g(Y, Z)X\\ & + g(X, fZ)fY - g(Y, fZ)fX, \quad \forall X, Y, Z \in \Gamma(TM).\end{aligned}$$

By using (1.1a), (1.10a), (1.10c) and (1.11) we obtain the following

Proposition 1.2. *Let M be an almost contact metric manifold with* (1.7). *Then we have*

$$\begin{aligned}(1.12)\quad & (a)\ U(X, Y)Z = -U(Y, X)Z;\\ & (b)\ U(X, Y)\xi = \eta(X)Y - \eta(Y)X;\\ & (c)\ U(fFX, FY)Z = fU(FX, FY)Z;\\ & (d)\ g(U(X, Y)Z, W) = \ g(U(Z, W)X, Y)\\ & (e)\ U(FX, FY)fZ = -fU(FX, FY)Z;\\ & (f)\ g(U(FX, FY)fZ, fW) = -g(U(FX, FY)Z, W),\ \forall X, Y, Z, W \in \Gamma(TM).\end{aligned}$$

2. Curvature tensor

The purpose of this section is to prove some properties of the curvature tensor K of an almost contact metric manifold satisfying (1.7). First we prove

Theorem 2.1. *Let M be an almost contact metric manifold satisfying* (1.7). *Then the curvature tensor field K of M satisfies*

$$
\begin{aligned}
(2.1)\quad &(a)\quad K(X,Y)fZ = fK(X,Y)Z - \eta(Z)K(X,Y)\xi \\
&\qquad - g(K(X,Y)\xi, fZ) + U(FX,FY)fZ, \\
&(b)\quad g(K(X,Y)fZ,W) + g(K(X,Y)Z,fW) \\
&\qquad = g(K(X,Y)\xi, fU(Z,W)\xi) + g(U(FX,FY)fZ,W), \\
&(c)\quad g(K(X,Y)fZ,fW) = g(K(X,Y)Z,W) \\
&\qquad + g(K(X,Y)\xi, U(Z,W)\xi) - g(U(FX,FY)Z,W), \\
&(d)\quad g(K(fX,fY)fZ,fW) = g(KX,Y)Z,W) - g(U(X,Y)\xi, K(Z,W)\xi) \\
&\qquad - g(K(fX,fY)\xi, U(Z,W)\xi) + g(U(FX,FY)Z,W) \\
&\qquad - g(U(X,Y)FZ,FW), \qquad \forall X,Y,Z,W \in \Gamma(TM).
\end{aligned}
$$

Proof. The assertion (2.1a) follows from (1.2), (1.7), (1.12a)-(1.12e) and the Ricci identity

$$\nabla_X\nabla_Y f - \nabla_Y\nabla_X f - \nabla_{[X,Y]} f = K(X,Y)f - fK(X,Y), \quad \forall X,Y \in \Gamma(TM).$$

The assertions (2.1b) and (2.1c) follow from the assertion (2.1a), using (1.12b) and (1.2). Finally, the assertion (2.1d) is obtained from (2.1c) by straightforward calculation. □

Now from Theorem 2.1 we deduce the following

Proposition 2.1. *Let M be an almost contact metric manifold satisfying* (1.7). *Then the curvature tensor field K of M satisfying*

$$
\begin{aligned}
(2.2)\quad &(a)\quad g(K(X,fX)X,fY) \;=\; g(K(X,fX)Y,fX) \\
&\qquad + g(K(X,fX)\xi, fU(X,Y)\xi), \\
&(b)\quad g(K(X,fY)X,fY) \;=\; g(K(fX,Y)fX,Y) \\
&\qquad + g(K(X,fY)\xi - K(fX,Y)\xi, fU(X,Y)\xi), \\
&(c)\quad g(K(X,Y)fX,fY) \;=\; g(K(X,Y)X,Y) \\
&\qquad - g(U(FX,FY)X,Y) - g(K(X,Y)\xi, U(X,Y)\xi), \\
&(d)\quad g(K(X,fY)X,fY) \;=\; g(K(X,fY)Y,fX) + g(U(FX,FY)X,Y) \\
&\qquad - g(fK(X,fY)\xi, U(X,Y)\xi), \qquad \forall X,Y \in \Gamma(TM).
\end{aligned}
$$

Next suppose that the almost contact metric manifold $M(f,\xi,\eta,g)$ is of constant f-holomorphic sectional curvature c, called space form and denoted by $M(c)$. In this case we want to find the Riemannian curvature of $M(c)$.

Theorem 2.2. *Let M be an almost contact metric manifold satisfying* (1.7). *The necessary and sufficient condition for M to have constant f-holomorphic sectional* *curvature c is*

$$(2.3)\quad \begin{aligned} 8g(K(X,Y)Z,W) \;=\;& 10g(U(X,Y)FZ,FW) - 6g(U(FX,FY)Z,W) \\ &+ g(U(X,Z)FY,FW) + g(U(FX,FZ)Y,W) \\ &+ g(U(FX,FW)Z,Y) + g(U(X,W)FZ,FY) \\ &- 2c(2g(X,fY)g(Z,fW) + g(fX,fZ)g(fY,fW) \\ &- g(fX,fW)g(fY,fZ) + g(X,fW)g(fY,Z) \\ &- g(X,fZ)g(fY,W)) + 8g(U(X,Y)\xi, K(Z,W)\xi) \\ &+ 8g(K(fX,fY)\xi, U(Z,W)\xi), \\ &\qquad\qquad\qquad \forall X,Y,Z,W \in \Gamma(TM). \end{aligned}$$

Proof. First we denote by D the contact distribution on M, that is $D = \{X \in \Gamma(TM); \eta(X) = 0\}$ and we see that $fX \in \Gamma(D), \quad \forall X \in \Gamma(TM)$. Next if (2.3) is true, then, by direct calculation, we infer that

$$(2.4)\qquad g(K(X,fX)fX,X) \;=\; -\,cg^2(X,X) \quad \forall X \in \Gamma(D),$$

that is M have the constant f-sectional curvature c. Conversely, suppose that M have the constant f-sectional curvature c, that is (2.4) hold. If instead of X we put $X+Y,\ X,Y \in \Gamma(D)$, by direct calculation we infer that,

$$(2.5)\qquad \begin{aligned} & g(K(X,fX)X,fY) + g(K(X,fX)Y,fX) + g(K(X,fX)Y,fY) \\ & + g(K(X,fY)Y,fX) + g(K(X,fY)X,fY) + g(K(X,fY)Y,fY) \\ & + g(K(Y,fX)Y,fY) \;=\; -\,c(2g^2(X,Y) + 2g(X,X)g(Y,Y) \\ & + 2g(X,Y)g(Y,Y) + g(X,X)g(Y,Y)). \end{aligned}$$

By using the Bianchi identity for $g(K(X,fX)Y,fY)$ and using (2.2a)-(2.2c), from (2.5) we deduce that

$$(2.6)\qquad \begin{aligned} & 2g(K(X,fX)X,fY) + 2g(K(Y,fY)Y,fX) + 3g(K(X,fY)Y,fX) \\ & + g(K(X,Y)X,Y) \;=\; -\,c(2g^2(X,Y) + 2g(X,X)g(X,Y) \\ & + 2g(X,Y)g(Y,Y) + g(X,X)g(Y,Y)). \end{aligned}$$

Now we put $-Y$ instead of Y in (2.6) and summing it to (2.6) we get that

$$(2.7)\qquad \begin{aligned} & 3g(K(X,fY)Y,fX) + g(K(X,Y)X,Y) \\ = \;& -c(2g^2(X,Y) + g(X,X)g(Y,Y)). \end{aligned}$$

If instead of Y we put fY in (2.7), using (2.2c) and (2.2d) we derive that

$$(2.8)\qquad \begin{aligned} & 3g(K(X,Y)X,Y) + g(K(X,fY)Y,fX) - 2g(U(FX,FY)X,Y) \\ = \;& -c(2g^2(X,fY) + g(X,X)g(Y,Y)). \end{aligned}$$

Therefore (2.7) and (2.8) implies

$$(2.9)\quad 4g(K(X,Y)X,Y) = 3g(U(FX,FY)X,Y) + c(g^2(X,Y) - g(X,X)g(Y,Y) - 3g^2(X,fY)).$$

Next, by straightforward calculation, from (2.9) we deduce that

$$8g(K(X,Y)Z,Y) = 3g(U(FX,FY)Z,Y) + 3g(U(X,Y)FZ,FY) + c(2g(X,Y)g(Y,Z) - 2g(X,Z)g(Y,Y) + 6g(X,fY)g(Y,fZ)), \quad \forall X,Y,Z \in \Gamma(D).$$

Now from the above relation, if we consider $Y+W,\ Y,W \in \Gamma(D)$ instead of Y, we infer that

$$\begin{aligned}(2.10)\quad & 8(g(K(X,Y)Z,W) + g(K(X,W)Z,Y)) \\ & = 3(g(U(FX,FW)Z,Y) + g(U(FX,FY)Z,W) + g(U(X,Y)FZ,FW) \\ & + g(U(X,W)FZ,FY)) + c(2g(X,W)g(Y,Z) + g(X,Y)g(Z,W) \\ & - 4g(X,Z)g(Y,W) + 6g(X,fY)g(fZ,W) + 6g(X,fW)g(Y,fZ)).\end{aligned}$$

Interchanging Z with W and substracting it from (2.10), using the Bianchi identities, we deduce that

$$\begin{aligned}(2.11)\quad & 8g(K(X,Y)Z,W) = 2g(U(FX,FY)Z,W) + 2g(U(X,Y)FZ,FW) \\ & + g(U(FX,FW)Z,Y) + g(U(FX,FZ)Y,W) + g(U(X,W)FZ,FY) \\ & - g(U(X,Z)FW,FY) + 2c(g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\ & - g(fX,W)g(Y,fZ) + g(X,fZ)g(fY,W) + 2g(X,fY)g(fZ,W)), \\ & \qquad \forall X,Y,Z,W \in \Gamma(D).\end{aligned}$$

Finally if we consider $X-\eta(X)\xi, Y-\eta(Y)\xi, Z-\eta(Z)\xi, W-\eta(W)\xi$ instead of X,Y,Z,W, from (2.11) we obtain our assertion (2.3). □

Remark 2.1. It is easy to see that from (2.3) we obtain well known formulae for the Sasakian space form ($F=f$) or for the Kenmotsu space form ($F = I - \eta\otimes\xi$).

Next we consider the trans-Sasakian space form $M(c)$. In this case $FX = \alpha fX - \beta X,\ X\in\Gamma(D),\ \alpha,\beta\in F(M)$, and using (1.12) we deduce

$$\begin{aligned}(2.12)\quad & g(U(FX,FY)Z,W) = (\alpha^2-\beta^2)(g(X,fZ)g(Y,fW) \\ & - g(fX,W)g(fY,Z) - g(X,Z)g(Y,W) + g(Y,Z)g(X,W)) \\ & - 2\alpha\beta(g(X,Z)g(fY,W) - g(fY,Z)g(X,W) - g(X,fZ)g(Y,W) \\ & + g(Y,Z)g(X,fW)), \quad \forall X,Y,Z,W \in \Gamma(D).\end{aligned}$$

From (2.11) and (2.12) we obtain

Theorem 2.3 *A trans-Sasakian manifold M is a space form $M(c)$ if and only if*

$$g(K(X,Y)Z,W) = \frac{c+3(\alpha^2-\beta^2)}{4}(g(X,W)g(Y,Z) \\ - g(X,Z)g(Y,W)) + \frac{c-\alpha^2+\beta^2}{4}(g(X,fW)g(Y,fZ) \\ - g(X,fZ)g(Y,fW) - 2g(X,fY)g(Z,fW)), \quad X,Y,Z,W \in \Gamma(D).$$

Now from the above theorem we deduce the following

Corollary 2.1. *If $M(c)$ is a trans-Sasakian space form with $\alpha^2 = \beta^2$, then it is a η-Einstein manifold, that is $S(X,Y) = \frac{c}{2}(n+1)g(X,Y), \quad \forall X,Y \in \Gamma(D)$.*

3. Properties of the Ricci tensor

In this section we deal with the study of the some properties of the Ricci tensor on an almost contact metric manifold with (1.7). More precisely first we prove the following

Theorem 3.1. *Let M be an almost contact metric manifold so that* (1.8) *is true. Then the Ricci tensor S verifies the next assertions*

$$(3.1)\quad (a)\ S(X,Y) = \frac{1}{2}\sum_{i=1}^{2n+1} g(fK(X,fY)e_i,e_i) + \eta(K(\xi,X)Y) \\ + 2g^2(F^2X,Y) - g(FX,Y)\sum_{i=1}^{2n+1} g(Fe_i,e_i) - g(FX,fY)\sum_{i=1}^{2n+1} g(Fe_i,fe_i),$$
$$(b)\ S(X,fY) + S(fX,Y) = \eta(X)S(fY,\xi) + \eta(Y)S(fX,\xi),$$

where $\{e_i, fe_i = e_{n+i}, \xi\}$, $(i = 1,\cdots,n)$ is an orthonormal field of frame on M.

Proof. Let $X,Y \in \Gamma(TM)$ and using (2.1b), (2.1c) and the Bianchi identity we obtain

$$\begin{aligned} g(K(fe_i,X)Y,fe_i) &= -g(K(fe_i,X)fY,e_i) + g(K(e_i,X)\xi,\eta(Y)fe_i \\ &\quad - \eta(e_i)fY) + g(U(Fe_i,FX)fY,e_i) \\ &= g(K(X,fY)fe_i,e_i) - g(K(e_i,X)fY,fe_i) \\ &\quad - \eta(Y)g(fK(e_i,X)\xi,e_i) + g(U(Fe_i,FX)fY,e_i) \\ &= g(fK(X,fY)e_i,e_i) - g(K(e_i,X)Y,e_i) \\ &\quad + g(K(X,e_i)\xi,U(Y,e_i)\xi) - g(U(Fe_i,FX)Y,e_i) \\ &\quad - \eta(Y)g(fK(e_i,X)\xi,e_i) + g(U(Fe_i,FX)fY,e_i). \end{aligned}$$

The assertion (3.1a) follows from the above relation, taking into account (1.3) and (1.11). The assertion (3.1b) is a consequence of the assertion (3.1a) and (2.1c). □

Now from (3.1b) and (1.5) we infer

Proposition 3.1. *Let M be an almost contact metric manifold satisfying* (1.7). *Then the Ricci operator satisfies*

$$QfX \;=\; fQX - \eta(X)fQ\xi + g(fX, Q\xi)\xi, \quad \forall X \in \Gamma(TM). \tag{3.2}$$

Next we study the almost contact metric manifold satisfying (1.7) and the Ricci tensor η-parallel. More precisely, we give the following

Definition 3.1. Let M be an almost contact metric manifold so that (1.7) holds. Then we say that the Ricci tensor S is η- parallel if

$$(\nabla_X S)(fY, fZ) \;=\; 0, \quad \forall X, Y, Z \in \Gamma(TM). \tag{3.3}$$

Next we have the following

Theorem 3.2. *Let M be an almost contact metric manifold satisfying* (1.7). *Then the next three assertions are equivalent*
(a) *The Ricci tensor S is η-parallel,*

(b)

$$\begin{aligned}(\nabla_X S)(Y, Z) \;=\;& \eta(Y)(\nabla_X S)(Z, \xi) + \eta(Z)(\nabla_X S)(Y, \xi)\\ & - g(FX, Z)S(Y, \xi) + (\eta(Y)g(FX, Z)\\ & + \eta(Z)g(FX, Y))S(\xi, \xi) - \eta(Y)\eta(Z)(\nabla_X S)(\xi, \xi),\end{aligned} \tag{3.4}$$

(c) $$\|\nabla Q\|^2 = \sum_{i=1}^{2n+1}(2g(fQ\xi, fQ\xi)g(Fe_i, Fe_i)$$

$$+ 2g((\nabla_{e_i}Q)\xi, (\nabla_{e_i}Q)\xi) + 2g^2(Fe_i, Q\xi) - ((\nabla_{e_i}S)(\xi, \xi))^2).$$

Proof. The equivalence of (3.4a) and (3.4b) follows from (3.1b) and (3.3). Next let $\{e_i, e_{n+i} = fe_i, e_{2n+1} = \xi\}, i = (1, \cdots, n)$ be an orthonormal field of frames on M. By using (1.7) and (3.2), for $1 \le j \le 2n$, we deduce that

$$\begin{aligned}(\nabla_{e_i}Q)fe_j \;=\;& \nabla_{e_i}Qfe_j - Q(\nabla_{e_i}f)e_j - Qf\nabla_{e_i}e_j\\ =\;& \nabla_{e_i}(fQe_j + g(fe_j, Q\xi)\xi) - g(\nabla_{fe_i}\xi, e_j)Q\xi\\ & - fQ\nabla_{e_i}e_j + \eta(\nabla_{e_i}e_j)fQ\xi - g(f\nabla_{e_i}e_j, Q\xi)\xi\\ =\;& f((\nabla_{e_i}Q)e_j) - \eta(Qe_j)\nabla_{fe_i}\xi\\ & + \eta(Q\xi)g(\nabla_{fe_i}\xi, e_j) + g(fe_j, (\nabla_{e_i}Q)\xi)\xi - g(e_j, fQ\xi)\nabla_{e_i}\xi\\ & - g(e_j, fQ\xi)\nabla_{e_i}\xi - g(f\nabla_{e_i}\xi, e_j)Q\xi - g(\nabla_{e_i}\xi, e_j)fQ\xi\\ =\;& f((\nabla_{e_i}Q)e_j) + (\eta(Q\xi)g(e_j, f\nabla_{e_i}\xi) - g(e_j, f((\nabla_{e_i}Q)\xi))\xi\\ & - g(e_j, Q\xi)f\nabla_{e_i}\xi - g(e_i, fQ\xi)\nabla_{e_i}\xi - g(Fe_i, fe_j)Q\xi\\ & + g(Fe_i, e_j)fQ\xi.\end{aligned}$$

Next, by straightforward calculation, using the above relation we infer that

$$
\begin{aligned}
(3.5)\quad \|\nabla Q\|^2 &= \sum_{i,j=1}^{2n+1} g((\nabla_{e_i}Q)e_j,(\nabla_{e_i}Q)e_j)\\
&= \sum_{i=1}^{2n+1}\sum_{j=1}^{2n} g((\nabla_{e_i}Q)fe_j,(\nabla_{e_i}Q)fe_j) + \sum_{i=1}^{2n+1} g((\nabla_{e_i}Q)\xi,(\nabla_{e_i}Q)\xi)\\
&= \sum_{i=1}^{2n+1}(4g(fQ\xi,fQ\xi)g(Fe_i,Fe_i) + g(f(\nabla_{e_i}Q)\xi, f(\nabla_{e_i}Q)\xi)\\
&\quad + g((\nabla_{e_i}Q)\xi,(\nabla_{e_i}Q)\xi) + 4g^2(\nabla_{e_i}\xi, fQ\xi)\\
&\quad + 4g^2(\nabla_{e_i}\xi, Q\xi) - 4g((\nabla_{e_i}Q)\nabla_{e_i}\xi, Q\xi)\\
&\quad + 4\eta(Q\xi)g((\nabla_{e_i}Q)\xi,\nabla_{e_i}\xi) + 4g(fQ\xi,(\nabla_{e_i}Q)\nabla_{fe_i}\xi)\\
&\quad + \sum_{j=1}^{2n} g(f(\nabla_{e_i}Q)e_j, f(\nabla_{e_i}Q)e_j)).
\end{aligned}
$$

Using (1.8b) and (3.2) we get

$$
\begin{aligned}
(3.6)\qquad & g(fQ\xi,(\nabla_{e_i}Q)\nabla_{fe_i}\xi)\\
= \; & g(fQ\xi,(\nabla_{e_i}f)(Q\nabla_{e_i}\xi) + f((\nabla_{e_i}Q)\nabla_{e_i}\xi)\\
& - g(\nabla_{e_i}\xi, fQ\xi)\nabla_{e_i}\xi) - g(Fe_i,Fe_i)g(fQ\xi,fQ\xi)\\
= \; & -g^2(Q\xi,Fe_i) - g^2(Fe_i,fQ\xi) - g(Fe_i,(\nabla_{e_i}Q)Q\xi)\\
& + \eta(Q\xi)g(Fe_i,(\nabla_{e_i}Q)\xi) - g(Fe_i,Fe_i)g(fQ\xi,fQ\xi).
\end{aligned}
$$

The relations (3.5) and (3.6) imply

$$
\begin{aligned}
(3.7)\qquad \|\nabla Q\|^2 &= \sum_{i=1}^{2n+1}(2g((\nabla_{e_i}Q)\xi,(\nabla_{e_i}Q)\xi)\\
&\quad - \eta^2((\nabla_{e_i}Q)\xi) + \sum_{j=1}^{2n} g(f(\nabla_{e_i}Q)e_j, f(\nabla_{e_i}Q)e_j)).
\end{aligned}
$$

Proposition 3.2. *Let M be an almost contact metric manifold satisfying* (1.7). *Then the Ricci tensor S is η-parallel if and only if*

$$
\begin{aligned}
(3.8)\qquad & \sum_{j=1}^{2n} g(f(\nabla_{e_i}Q)e_j, f(\nabla_{e_i}Q)e_j)\\
= \; & 2g(fQ\xi,fQ\xi)g(Fe_i,Fe_i) + 2g^2(Fe_i,Q\xi).
\end{aligned}
$$

Proof. Suppose S be η-parallel and let $\{e_i, e_{n+i} = fe_i, e_{2n+1} = \xi\}$, $(i = 1, \cdots, n)$, be an orthonormal field of frames on M. By using (3.4b), for $1 \leq j \leq n$, we deduce that

$$(3.9) \qquad (\nabla_{e_i} Q)e_j = (\nabla_{e_i} S)(e_j, \xi)\xi + g(Fe_i, e_j)S(\xi, \xi)\xi - g(Fe_i, e_j)Q\xi - g(e_j, Q\xi)Fe_i.$$

Now, by straightforward calculation, using (3.9) we infer that

$$\begin{aligned} & g(f(\nabla_{e_i} Q)e_j, f(\nabla_{e_i} Q)e_j) \\ = & \; g(fQ\xi, fQ\xi)g^2(Fe_i, e_j) + g^2(e_j, Q\xi)g(Fe_i, Fe_i) \\ & + 2g(Fe_i, e_j)g(e_j, Q\xi)g(Fe_i, Q\xi), \end{aligned}$$

and the assertion (3.8) is proved. Conversely, suppose that (3.8) is true. First we see that if $Y = \xi$, the relation (3.4b) is verified for any X, Z $\in \Gamma(TM)$. Next from (3.9) we get

$$\begin{aligned} (\nabla_{e_i} S)(e_j, Z) = & \; \eta(Z)(\nabla_{e_i} S)(e_j, \xi) - g(Fe_i, e_j)S(Z, \xi) \\ & - g(Fe_i, Z)S(e_j, \xi) + g(Fe_i, e_j)\eta(Z)S(\xi, \xi), \end{aligned}$$

and therefore

$$\begin{aligned} (3.10)\,(\nabla_X S)(Y, Z) = & \; \eta(Z)(\nabla_X S)(Y, \xi) \\ & - g(FX, Y)S(Z, \xi) - g(FX, Z)S(Y, \xi) \\ & + g(FX, Y)\eta(Z)S(\xi, \xi), \quad \forall X, Z \in \Gamma(TM),\ Y \in \Gamma(D). \end{aligned}$$

□

Now if instead of Y we put $Y - \eta(Y)\xi \in \Gamma(D)$ in (3.10) by direct calculation it follows (3.4b). Finally the equivalence of (3.4b), (3.4c) follows from (3.7) and (3.8). From Theorem 3.1 we obtain

Corollary 3.2 *Let M be a Sasakian (or a Kenmotsu) manifold. Then the Ricci tensor S is η-parallel iff*

$$\|\nabla Q\|^2 = 2\|Q\|^2 + 16n^3 + 8n^2 + 4nr\epsilon,$$

($\epsilon = 1$ for the Sasakian manifold and $\epsilon = -1$ for the Kenmotsu manifold).

Next, from (1.5), (1.6) and (3.4), by direct calculation we deduce the following

Proposition 3.3. *Let M be an almost contact metric manifold satisfying* (1.7). *If the Ricci tensor S is η-parallel then*

(a) $\nabla_X r = (\nabla_X S)(\xi, \xi)$,

(b) $\nabla_X \|Q\|^2 = 2(\nabla_X S)(\xi, 2Q\xi - S(\xi, \xi)\xi), \quad \forall X \in \Gamma(TM)$.

From Lemma 3.2 we get

Corollary 3.3. *Let M be a Sasakian manifold* (*or a Kenmotsu manifold*). *If the Ricci tensor S is η-parallel then*

(a) *the scalar curvature r of M is constant*

(b) *the square of the length of the Ricci operator Q of M is constant.*

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