KYUNGPOOK Math. J. 45(2005), 73-85

On L^1-convergence of Certain Trigonometric Sums with Generalized Sequence K^α

Kulwinder Kaur
Department of Applied Sciences, GZS College of Engineering and Technology, Bathinda. Punjab - 151005. India
e-mail : mathkk@hotmail.com

Abstract. In this paper a criterion for L^1-convergence of a new modified sine sums is obtained by using Cesàro means of integral and non-integral orders.

1. Introduction

Let

$$g(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx, \tag{1.1}$$

$$g_n(x) = \frac{1}{2\sin x} \sum_{k=1}^{n} \sum_{j=k}^{n} (\triangle a_{j-1} - \triangle a_{j+1}) \sin kx. \tag{1.2}$$

Concerning the L^1-convergence of the Fourier cosine series (1.1), Kolmogorov [5] proved the following well known theorem:

Theorem A. *If $\{a_n\}$ is a quasi-convex null sequence, then for the L^1-convergence of the cosine series (1.1), it is necessary and sufficient that* $\lim_{n\to\infty} a_n \log n = 0$.

In [4] a new modified sine sums are introduced and a criterion for L^1-convergence of this modified sine sums have been obtained under a newly defined class K, by proving the following result:

Theorem B. *Let the sequence $\{a_n\}$ belongs to the class K, then $g_n(x)$ converges to $g(x)$ in L^1-norm, where K is the class of sequences defined in the following way:*

Definition ([4]). If $\{a_k\} = o(1), \quad k \to \infty$ and

$$\sum_{k=1}^{\infty} k \left|\triangle^2 a_{k-1} - \triangle^2 a_{k+1}\right| < \infty \qquad (a_0 = 0), \tag{1.3}$$

Received November 11, 2003.
2000 Mathematics Subject Classification: 42A20, 42A32.
Key words and phrases: L^1-convergence, Cesàro means, conjugate Cesàro means, Fourier cosine series.

then we say that $\{a_k\}$ belongs to the class K.

In particular, in [4] an analouge of Theorem A of Kolmogorov have been obtained as a corollary under the class K, by proving the following:

Theorem C. *If $\{a_n\}$ belongs to the class K, then the necessary and sufficient condition for for L^1-convergence of the cosine series* (1.1) *is* $\lim_{n\to\infty} a_n \log n = 0$.

We generalize the class K of sequences as follows:

Definition. If $a_k = o(1), \quad k \to \infty$ and

$$\sum_{k=1}^{\infty} k^{\alpha} \left|\triangle^{\alpha+1} a_{k-1} - \triangle^{\alpha+1} a_{k+1}\right| < \infty \quad (a_0 = 0) \quad \text{for } \alpha > 0, \tag{1.4}$$

then we say that $\{a_k\}$ belongs to the class K^{α}. For $\alpha = 1$ the class K^{α} is same as K.

The aim of this paper is to study the L^1-convergence of modified sine sums (1.2) for the class $\mathbf{K}^{\alpha}$, by using Cesàro means of integral and non-integral orders.

2. Notation and formulae

We use the following notations [7]:

Given a sequence $S_0, S_1, S_2, \cdots$, we define for every $\alpha = 0, 1, 2, \cdots$, the sequence $S_0^{\alpha}, S_1^{\alpha}, S_2^{\alpha}, \cdots$, by the conditions

$$S_n^0 = S_n,$$
$$S_n^{\alpha} = S_0^{\alpha-1} + S_1^{\alpha-1} + S_2^{\alpha-1} + \cdots + S_n^{\alpha-1} \quad (\alpha = 1, 2, \cdots, \ n = 0, 1, 2, \cdots).$$

Similarly for $\alpha = 0, 1, 2, \cdots$, we define the sequence of numbers $A_0^{\alpha}, A_1^{\alpha}, A_2^{\alpha}, \cdots$ by the conditions

$$A_n^0 = 1,$$
$$A_n^{\alpha} = A_0^{\alpha-1} + A_1^{\alpha-1} + A_2^{\alpha-1} + \cdots + A_n^{\alpha-1} \quad (\alpha = 1, 2, \cdots, \ n = 0, 1, 2, \cdots).$$

Consider $\sum a_n$ be a given infinite series. For any real number α the conjugate Cesàro sums of order α of $\sum a_n$ are defined by

$$\tilde{S}_n^{\alpha}(a_p) = \tilde{S}_n^{\alpha} = \sum_{p=0}^{n} A_{n-p}^{\alpha} a_p = \sum_{p=0}^{n} A_{n-p}^{\alpha-1} \tilde{S}_p,$$

where $\tilde{S}_n = \tilde{S}_n^0 = \tilde{D}_n$, and A_p^{α} denotes the binomial coefficients and are given by the following relations.

$\sum\limits_{p=0}^{\infty} A_p^{\alpha} x^p = (1-x)^{-\alpha-1}$ and $\tilde{S}_n^{,}\, s$ are given by

$$\sum_{p=0}^{\infty} \tilde{S}_p^{\alpha} x^p = (1-x)^{-\alpha} \sum_{p=0}^{\infty} \tilde{S}_p x^p. \tag{2.1}$$

Also

$$\begin{aligned} A_n^\alpha &= \sum_{p=0}^{n} A_p^{\alpha-1} \\ A_n^\alpha &= \binom{n+\alpha}{n} \simeq \frac{n^\alpha}{\Gamma\alpha+1} \quad (\alpha \neq -1,-2,-3,\cdots). \end{aligned}$$

The conjugate Cesàro means $\tilde{T}_n^\alpha$ of order α of $\sum a_n$ will be defined by

$$\tilde{T}_n^\alpha = \frac{\tilde{S}_n^\alpha}{A_n^\alpha}. \tag{2.2}$$

The following formulae will also be needed;

$$\tilde{S}_n^\alpha(\tilde{S}_p^r) = \tilde{S}_n^{\alpha+r+1}, \tag{2.3}$$

$$\tilde{S}_n^{\alpha+1} - \tilde{S}_{n-1}^{\alpha+1} = \tilde{S}_n^\alpha, \quad \sum_{p=0}^{n} A_{n-p}^\alpha A_p^\beta = A_n^{\alpha+\beta+1}. \tag{2.4}$$

For any positive integer α the differences of order α of the sequence $\{a_n\}$ are defined by the equations

$$\begin{aligned} \triangle^1 a_n &= a_n - a_{n+1}, \\ \triangle^\alpha a_n &= \triangle(\triangle^{\alpha-1} a_n), \quad n = 0,1,2,3,\cdots. \end{aligned}$$

For these differences we have

$$\triangle^\alpha a_n = \sum_{m=0}^{\alpha} A_m^{-\alpha-1} a_{n+m} = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m}, \tag{2.5}$$

since $A_m^{-\alpha-1} = 0$ for $m \geqslant \alpha+1$.

If the series (2.5) are convergent for some α which is not a positive integer, then we denote the differences

$$\triangle^\alpha a_n = \sum_{m=0}^{\infty} A_m^{-\alpha-1} a_{n+m}, \quad n = 0,1,2,3\cdots. \tag{2.6}$$

The broken differences $\triangle_n^\alpha a_p$ are defined by

$$\triangle_n^\alpha a_p = \sum_{m=0}^{n-p} A_m^{-\alpha-1} a_{p+m}. \tag{2.7}$$

By repeated partial summation of order α,

$$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n} \tilde{S}_p^{\alpha-1}(a_p) \triangle_n^\alpha b_p. \tag{2.8}$$

If α is positive integer then we have

$$\sum_{p=0}^{n} a_p b_p = \sum_{p=0}^{n-\alpha} \tilde{S}_p^{\alpha-1}(a_p) \triangle^{\alpha} b_p + \sum_{p=n-\alpha+1}^{n} \tilde{S}_p^{\alpha-1}(a_p) \triangle_n^{\alpha} b_p. \tag{2.9}$$

3. Lemmas

We need the following Lemmas for the proof of our result:

Lemma 3.1 ([3]). *If* $\alpha \geqslant 0,\ p \geqslant 0,$

(i) $\epsilon_n = o(n^{-p})$,

(ii) $\sum\limits_{n=0}^{\infty} A_n^{\alpha+p} \left|\triangle^{\alpha+1}\epsilon_n\right| < \infty$, *then*

(iii) $\sum\limits_{n=0}^{\infty} A_n^{\lambda+p} \left|\triangle^{\lambda+1}\epsilon_n\right| < \infty$, *for* $-1 \leq \lambda \leq \alpha$ *and*

(iv) $A_n^{\lambda+p}\triangle^{\lambda}\epsilon_n$ *is of bounded variation for* $0 \leq \lambda \leq \alpha$ *and tends to zero as* $n \to \infty$.

Lemma 3.2 ([1]). *Let* r *be the real number* $\geqslant 0$. *If the sequence* $\{\epsilon_n\}$ *satisfies the conditions:*

(i) $\epsilon_n = O(1)$ *and*

(ii) $\sum\limits_{n=1}^{\infty} n^r \left|\triangle^{r+1}\epsilon_n\right| < \infty$,

then $\triangle^{\beta}\epsilon_n = \sum\limits_{m=0}^{\infty} A_m^{r-\beta}\triangle^{r+1}\epsilon_{n+m}$, *for* $\beta > 0$.

Lemma 3.3 ([2]). *If* $0 \leqslant \delta \leq 1$ *and* $0 \leq m < n$, *then*

$$\left|\sum_{i=0}^{m} A_{n-i}^{\delta-1} S_i\right| \leq \max_{0\leq p\leq m} \left|S_p^{\delta}\right|.$$

Lemma 3.4 ([7]). *Let* $\tilde{S}_n(x)$ *and* $\tilde{T}_n^{\alpha}$ *be the* n^{th} *partial sums and Cesàro means of order* $\alpha > 0$, *respectively, of the series*

$$\sin x + \sin 2x + \sin 3x + \cdots + \sin nx + \cdots .$$

Then

(i) $\int\limits_0^{\pi} \left|\tilde{S}_n(x)\right| dx \sim \log n,$

(ii) $\int_0^\pi \left|\tilde{T}_n^\alpha\right| dx$ *remains bounded for all* n.

4. Main Result

The main result of this paper is the following Theorem:

Theorem 4.1. *Let the sequence* $\{a_n\}$ *belongs to the class* K^α, *where* $\alpha > 0$ *be a real number. Then* $g_n(x)$ *converges to* $g(x)$ *in* L^1*-norm. If we take* $\alpha = 1$, *then this Theorem reduces to Theorem B.*

Proof.

$$(4.1)\qquad \begin{aligned} g_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}\sum_{j=k}^{n}(\triangle a_{j-1}-\triangle a_{j+1})\sin kx \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle a_{k-1}-\triangle a_{k+1})\,\tilde{D}_k(x) \\ &= \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle a_{k-1}-\triangle a_{k+1})\,\tilde{S}_k(x). \end{aligned}$$

Part I. Let α be integral. Applying Abel's transformation of order α to $g_n(x)$, we have

$$\begin{aligned} g_n(x) &= \frac{1}{2\sin x}\{\sum_{k=1}^{n-\alpha}\left(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}\right)\tilde{S}_k^\alpha(x) \\ &\quad +\sum_{k=1}^{\alpha}\left(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2}\right)\tilde{S}_{n-k+1}^k(x)\}. \end{aligned}$$

Then

$$(4.2)\qquad \begin{aligned} g(x) &= \lim_{n\to\infty} g_n(x) \\ &= \frac{1}{2\sin x}\sum_{k=1}^{\infty}\left(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}\right)\tilde{S}_k^\alpha(x). \end{aligned}$$

Thus, by (4.1) and (4.2),

$$\begin{aligned} &\int_0^\pi |g(x)-g_n(x)|\,dx \\ &= \int_0^\pi \Big| \frac{1}{2\sin x}\sum_{k=n-\alpha+1}^{\infty}\left(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}\right)\tilde{S}_k^\alpha(x) \\ &\quad -\sum_{k=1}^{\alpha}\left(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2}\right)\tilde{S}_{n-k+1}^k(x)\Big|\,dx \end{aligned}$$

$$\begin{aligned}
&\leq C\int_0^{\pi}\left|\sum_{k=n-\alpha+1}^{\infty}\left(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}\right)\tilde{S}_k^{\alpha}(x)\right|dx\\
&\quad+\int_0^{\pi}\left|\sum_{k=1}^{\alpha}\left(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2}\right)\tilde{S}_{n-k+1}^{k}(x)\right|dx\\
&= C\sum_{k=n-\alpha+1}^{\infty}\int_0^{\pi}\left|\tilde{S}_k^{\alpha}(x)\right|dx\left|\left(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}\right)\right|\\
&\quad+C\sum_{k=1}^{\alpha}\int_0^{\pi}\left|\tilde{S}_{n-k-1}^{k}(x)\right|dx\left|\left(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2}\right)\right|\\
&= C\sum_{k=n-\alpha+1}^{\infty}A_k^{\alpha}\left|\left(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}\right)\right|\int_0^{\pi}\left|\tilde{T}_k^{\alpha}(x)\right|dx\\
&\quad+C\sum_{k=1}^{\alpha}A_{n-k+1}^{k}\left|\left(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2}\right)\right|\int_0^{\pi}\left|\tilde{T}_{n-k+1}^{k}(x)\right|dx\\
&= C_1\sum_{k=n-\alpha+1}^{\infty}A_k^{\alpha}\left|\left(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}\right)\right|\\
&\quad+C_1\sum_{k=1}^{\alpha}A_{n-k+1}^{k}\left|\left(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2}\right)\right|\qquad (C_1 \text{ is an absolute constant})\\
&= o(1)+o(1) = o(1),
\end{aligned}$$

by Lemma 3.1, 3.4 and hypothesis of the Theorem 4.1.

Hence

$$\lim_{n\to\infty}\int_0^{\pi}|g(x)-g_n(x)|\,dx=o(1). \tag{4.3}$$

Part II. Let α be non-integral. Let $\alpha=r+\delta$, r is the integral part of α, and δ is the fractional part, $0<\delta<1$.

Case (i). Let $r=0$.

Applying Abel's transformation of order $-\delta$, we have by (2.8)

$$\begin{aligned}
&\sum_{k=1}^{n}\tilde{S}_k^{\delta}(x)\left(\triangle^{\delta+1}a_{k-1}-\triangle^{\delta+1}a_{k+1}\right)\\
&= \sum_{k=1}^{n}\tilde{S}_k(x)\sum_{m=0}^{n-k}A_m^{\delta-1}(\triangle^{\delta+1}a_{m+k-1}-\triangle^{\delta+1}a_{m+k+1}).
\end{aligned}$$

Also by Lemma 3.2, we have

$$\begin{aligned}
&\sum_{k=1}^{n} \tilde{S}_k^{\delta}(x)\left(\triangle^{\delta+1}a_{k-1} - \triangle^{\delta+1}a_{k+1}\right)\\
=\;& \sum_{k=1}^{n} \tilde{S}_k(x)\Bigg\{ (\triangle a_{k-1} - \triangle a_{k+1})\\
&\quad - \sum_{m=n-k+1}^{\infty} A_m^{\delta-1}(\triangle^{\delta+1}a_{m+k-1} - \triangle^{\delta+1}a_{m+k+1})\Bigg\}\\
=\;& \sum_{k=1}^{n} \tilde{S}_k(x)\,(\triangle a_{k-1} - \triangle a_{k+1}) - R_n(x)
\end{aligned}$$

where

$$\begin{aligned}
R_n(x) \;=\;& \sum_{k=1}^{n} \tilde{S}_k(x)\{A_{n-k+1}^{\delta-1}(\triangle^{\delta+1}a_{n+2} - \triangle^{\delta+1}a_n)\\
&+ A_{n-k+2}^{\delta-1}(\triangle^{\delta+1}a_{n+3} - \triangle^{\delta+1}a_{n+1}) + \cdots\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\frac{1}{2\sin x}\sum_{k=1}^{n} \tilde{S}_k(x)\,(\triangle a_{k-1} - \triangle a_{k+1})\\
=\;& \frac{1}{2\sin x}\left\{\sum_{k=1}^{n} \tilde{S}_k^{\delta}(x)\left(\triangle^{\delta+1}a_{k-1} - \triangle^{\delta+1}a_{k+1}\right) + R_n(x)\right\}
\end{aligned}$$

and consequently,

$$g_n(x) \;=\; \frac{1}{2\sin x}\left\{\sum_{k=1}^{n} \tilde{S}_k^{\delta}(x)\left(\triangle^{\delta+1}a_{k-1} - \triangle^{\delta+1}a_{k+1}\right) + R_n(x)\right\}.$$

When $r = 0$, then $\alpha = \delta$ and $g(x) = \dfrac{1}{2\sin x}\left\{\sum_{k=1}^{\infty} \tilde{S}_k^{\delta}(x)\left(\triangle^{\delta+1}a_{k-1} - \triangle^{\delta+1}a_{k+1}\right)\right\}$.

Therefore,

$$\begin{aligned}
(4.4)\qquad & \int_0^{\pi} |g(x) - g_n(x)|\,dx\\
=\;& \int_0^{\pi} \left|\frac{1}{2\sin x}\left\{\sum_{k=n+1}^{\infty} \tilde{S}_k^{\delta}(x)(\triangle^{\delta+1}a_{k-1} - \triangle^{\delta+1}a_{k+1}) - R_n(x)\right\}\right| dx\\
\leqslant\;& C\left\{\sum_{k=n+1}^{\infty} \left|(\triangle^{\delta+1}a_{k-1} - \triangle^{\delta+1}a_{k+1})\right| \int_0^{\pi} \left|\tilde{S}_k^{\delta}(x)\right| dx + \int_0^{\pi} |R_n(x)|\,dx\right\}
\end{aligned}$$

$$
\begin{aligned}
&= C\left\{\sum_{k=n+1}^{\infty} A_k^{\delta}\left|(\triangle^{\delta+1}a_{k-1}-\triangle^{\delta+1}a_{k+1})\right|\int_0^{\pi}\left|\tilde{T}_k^{\delta}(x)\right|dx+\int_0^{\pi}|R_n(x)|\,dx\right\}\\
&\leqslant C_1\sum_{k=n+1}^{\infty} A_k^{\delta}\left|(\triangle^{\delta+1}a_{k-1}-\triangle^{\delta+1}a_{k+1})\right|+C\int_0^{\pi}|R_n(x)|\,dx\\
&= o(1)+C\int_0^{\pi}|R_n(x)|\,dx,\quad \text{by Lemma 3.1 and 3.4.}
\end{aligned}
$$

Now for the estimate of $\int_0^{\pi}|R_n(x)|\,dx$, we have

$$
\begin{aligned}
&\int_0^{\pi}|R_n(x)|\,dx\\
&= \int_0^{\pi}\Big|\left(\sum_{k=1}^{n}\tilde{S}_k(x)A_{n-k+1}^{\delta-1}\right)(\triangle^{\delta+1}a_{n+2}-\triangle^{\delta+1}a_n)\\
&\qquad +\left(\sum_{k=1}^{n}\tilde{S}_k(x)A_{n-k+2}^{\delta-1}\right)(\triangle^{\delta+1}a_{n+3}-\triangle^{\delta+1}a_{n+1})+\cdots\Big|\,dx\\
&\leqslant \int_0^{\pi}\left|(\triangle^{\delta+1}a_{n+2}-\triangle^{\delta+1}a_n)\right|\left|\sum_{k=1}^{n}\tilde{S}_k(x)A_{n-k+1}^{\delta-1}\right|dx\\
&\qquad +\int_0^{\pi}\left|(\triangle^{\delta+1}a_{n+3}-\triangle^{\delta+1}a_{n+1})\right|\left|\sum_{k=1}^{n}\tilde{S}_k(x)A_{n-k+2}^{\delta-1}\right|dx+\cdots\\
&\leqslant \int_0^{\pi}\left|(\triangle^{\delta+1}a_{n+2}-\triangle^{\delta+1}a_n)\right|\max_{0\leqslant p\leqslant n+1}\left|\tilde{S}_p^{\delta}(x)\right|dx\\
&\qquad +\int_0^{\pi}\left|(\triangle^{\delta+1}a_{n+3}-\triangle^{\delta+1}a_{n+1})\right|\max_{0\leqslant p\leqslant n+2}\left|\tilde{S}_p^{\delta}(x)\right|dx+\cdots,\quad \text{by Lemma 3.3.}\\
&= \left|\triangle^{\delta+1}a_{n+2}-\triangle^{\delta+1}a_n)\right|A_{n+1}^{\delta}\int_0^{\pi}\max_{0\leqslant p\leqslant n+1}\left|\tilde{T}_p^{\delta}(x)\right|dx\\
&\qquad +\left|(\triangle^{\delta+1}a_{n+3}-\triangle^{\delta+1}a_{n+1})\right|A_{n+2}^{\delta-1}\int_0^{\pi}\max_{0\leqslant p\leqslant n+2}\left|\tilde{T}_p^{\delta}(x)\right|dx+\cdots\\
&= CA_{n+1}^{\delta}\left|(\triangle^{\delta+1}a_{n+2}-\triangle^{\delta+1}a_n)\right|+CA_{n+2}^{\delta}\left|(\triangle^{\delta+1}a_{n+3}-\triangle^{\delta+1}a_{n+1})\right|+\cdots\\
&= o(1)+o(1)\ =\ o(1),\quad \text{by Lemma 3.1 and 3.4.}
\end{aligned}
$$

Thus, by (4.4),

$$\lim_{n\to\infty}\int_0^\pi |g(x)-g_n(x)|\,dx = o(1). \tag{4.5}$$

Case (ii). Let $r \geqslant 1$.

Applying Abel's transformation of order r,

$$\begin{aligned} g_n(x) &= \frac{1}{2\sin x}\sum_{k=1}^{n}(\triangle a_{k-1}-\triangle a_{k+1})\,\tilde{S}_k(x) \\ &= \frac{1}{2\sin x}\{\sum_{k=1}^{n-r}\left(\triangle^{r+1}a_{k-1}-\triangle^{r+1}a_{k+1}\right)\tilde{S}_k^r(x) \\ &\quad + \sum_{k=1}^{r}\left(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2}\right)\tilde{S}_{n-k+1}^k(x)\}. \end{aligned} \tag{4.6}$$

Again applying Abel's transformation of order -δ, we obtain

$$\begin{aligned} &\frac{1}{2\sin x}\sum_{k=1}^{n}\tilde{S}_k^\alpha(x)(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}) \\ =\ &\frac{1}{2\sin x}\sum_{k=1}^{n}\tilde{S}_k^r(x)\sum_{m=0}^{n-k}A_m^{\delta-1}(\triangle^{\alpha+1}a_{m+k-1}-\triangle^{\alpha+1}a_{m+k+1}). \end{aligned}$$

By Lemma 3.2,

$$\begin{aligned} &\frac{1}{2\sin x}\sum_{k=1}^{n}\tilde{S}_k^\alpha(x)(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}) \\ =\ &\frac{1}{2\sin x}\{\sum_{k=1}^{n}\tilde{S}_k^r(x)(\triangle^{r+1}a_{k-1}-\triangle^{r+1}a_{k+1})-R_n(x)\}, \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} R_n(x) &= \sum_{k=1}^{n}\tilde{S}_k^r(x)\{A_{n-k+1}^{\delta-1}(\triangle^{\alpha+1}a_{n+2}-\triangle^{\alpha+1}a_n) \\ &\quad + A_{n-k+2}^{\delta-1}(\triangle^{\alpha+1}a_{n+3}-\triangle^{\alpha+1}a_{n+1})+\cdots\} \\ &= \left(\sum_{k=1}^{n}\tilde{S}_k^r(x)A_{n-k+1}^{\delta-1}\right)(\triangle^{\alpha+1}a_{n+2}-\triangle^{\alpha+1}a_n) \\ &\quad + \left(\sum_{k=1}^{n}\tilde{S}_k^r(x)A_{n-k+2}^{\delta-1}\right)(\triangle^{\alpha+1}a_{n+3}-\triangle^{\alpha+1}a_{n+1})+\cdots. \end{aligned}$$

Replacing n by $n-r$ in (4.7), we have

$$\begin{aligned}(4.8)\qquad & \frac{1}{2\sin x}\sum_{k=1}^{n-r}\tilde{S}_k^{\alpha}(x)(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}) \\ = \; & \frac{1}{2\sin x}\left\{\sum_{k=1}^{n-r}\tilde{S}_k^{r}(x)(\triangle^{r+1}a_{k-1}-\triangle^{r+1}a_{k+1})-R_{n-r}(x)\right\}.\end{aligned}$$

Now by (4.6) and (4.8), we have

$$\begin{aligned}(4.9)\qquad g_n(x) \; = \; & \frac{1}{2\sin x}\sum_{k=1}^{n-r}\tilde{S}_k^{\alpha}(x)(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}) \\ & + R_{n-r}(x)+\sum_{k=1}^{r}(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2})\tilde{S}_{n-k+1}^{k}(x).\end{aligned}$$

Therefore

$$\begin{aligned}(4.10) & \int_0^{\pi}|g(x)-g_n(x)|\,dx \\ = \; & \int_0^{\pi}\Big|\,\frac{1}{2\sin x}\Big\{\sum_{k=n-r+1}^{\infty}\tilde{S}_k^{\alpha}(x)(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1}) \\ & -R_{n-r}(x)-\sum_{k=1}^{r}(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2})\tilde{S}_{n-k+1}^{k}(x)\Big\}\,\Big|\,dx \\ \leq \; & C\sum_{k=n-r+1}^{\infty}\left|(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1})\right|\int_0^{\pi}\left|\tilde{S}_k^{\alpha}(x)\right|dx \\ & +\int_0^{\pi}|R_{n-r}(x)|\,dx+\sum_{k=1}^{r}\left|(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2})\right|\int_0^{\pi}\left|\tilde{S}_{n-k+1}^{k}(x)\right|dx \\ = \; & C\sum_{k=n-r+1}^{\infty}A_k^{\alpha}\left|(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1})\right|\int_0^{\pi}\left|\tilde{T}_k^{\alpha}(x)\right|dx \\ & +\int_0^{\pi}|R_{n-r}(x)|\,dx+\sum_{k=1}^{r}A_{n-k+1}^{k}\left|(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2})\right|\int_0^{\pi}\left|\tilde{T}_{n-k+1}^{k}(x)\right|dx \\ \leq \; & C_1\sum_{k=n-r+1}^{\infty}A_k^{\alpha}\left|(\triangle^{\alpha+1}a_{k-1}-\triangle^{\alpha+1}a_{k+1})\right| \\ & +C\int_0^{\pi}|R_{n-r}(x)|\,dx+C_1\sum_{k=1}^{r}A_{n-k+1}^{k}\left|(\triangle^{k}a_{n-k}-\triangle^{k}a_{n-k+2})\right|\end{aligned}$$

$$= o(1) + o(1) + C\int_0^\pi |R_{n-r}(x)|\,dx$$
$$= o(1) + C\int_0^\pi |R_{n-r}(x)|\,dx,$$

by the hypothesis of the theorem and Lemma 3.1. But

$$\begin{aligned}
&\int_0^\pi |R_{n-r}(x)|\,dx \\
\leq\ & \int_0^\pi \left|\left(\sum_{k=1}^{n-r}\tilde{S}_k^r(x)A_{n-r-k+1}^{\delta-1}\right)(\triangle^{\alpha+1}a_{n-r+2}-\triangle^{\alpha+1}a_{n-r})\right|dx \\
&+\int_0^\pi \left|\left(\sum_{k=1}^{n-r}\tilde{S}_k^r(x)A_{n-r-k+2}^{\delta-1}\right)(\triangle^{\alpha+1}a_{n-r+3}-\triangle^{\alpha+1}a_{n-r+1})\right|dx \\
&+\int_0^\pi \left|\left(\sum_{k=1}^{n-r}\tilde{S}_k^r(x)A_{n-r-k+3}^{\delta-1}\right)(\triangle^{\alpha+1}a_{n-r+5}-\triangle^{\alpha+1}a_{n-r+2})\right|dx+\cdots \\
\leq\ & \sum_{k=1}^{n-r}A_{n-r-k+1}^{\delta-1}A_k^r\int_0^\pi\left|\tilde{T}_k^r(x)\right|dx\left|(\triangle^{\alpha+1}a_{n-r+2}-\triangle^{\alpha+1}a_{n-r})\right| \\
&+\sum_{k=1}^{n-r}A_{n-r-k+2}^{\delta-1}A_k^r\int_0^\pi\left|\tilde{T}_k^r(x)\right|dx\left|(\triangle^{\alpha+1}a_{n-r+3}-\triangle^{\alpha+1}a_{n-r+1})\right| \\
&+\sum_{k=1}^{n-r}A_{n-r-k+3}^{\delta-1}A_k^r\int_0^\pi\left|\tilde{T}_k^r(x)\right|dx\left|(\triangle^{\alpha+1}a_{n-r+5}-\triangle^{\alpha+1}a_{n-r+2})\right|+\cdots \\
\leq\ & C_1\sum_{k=1}^{n-r}A_{n-r-k+1}^{\delta-1}A_k^r\left|(\triangle^{\alpha+1}a_{n-r+2}-\triangle^{\alpha+1}a_{n-r})\right| \\
&+C_1\sum_{k=1}^{n-r}A_{n-r-k+2}^{\delta-1}A_k^r\left|(\triangle^{\alpha+1}a_{n-r+3}-\triangle^{\alpha+1}a_{n-r+1})\right| \\
&+C_1\sum_{k=1}^{n-r}A_{n-r-k+3}^{\delta-1}A_k^r\left|\triangle^{\alpha+1}(a_{n-r+5}-\triangle^{\alpha+1}a_{n-r+2})\right|+\cdots \\
\leq\ & C_1\sum_{k=1}^{n+1-r}A_{n+1-r-k}^{\delta-1}A_k^r\left|(\triangle^{\alpha+1}a_{n-r+2}-\triangle^{\alpha+1}a_{n-r})\right|
\end{aligned}$$

$$
\begin{aligned}
&\quad + C_1 \sum_{k=1}^{n+2-r} A_{n+2-r-k}^{\delta-1} A_k^r \left|(\triangle^{\alpha+1} a_{n-r+3} - \triangle^{\alpha+1} a_{n-r+1})\right| \\
&\quad + C_1 \sum_{k=1}^{n+3-r} A_{n+3-r-k}^{\delta-1} A_k^r \left|(\triangle^{\alpha+1} a_{n-r+5} - \triangle^{\alpha+1} a_{n-r+2})\right| + \cdots \\
\leq\; & C_2 A_{n+1-r}^{r+\delta} \left|(\triangle^{\alpha+1} a_{n-r+2} - \triangle^{\alpha+1} a_{n-r})\right| \\
&\quad + C_2 A_{n+2-r}^{r+\delta} \left|(\triangle^{\alpha+1} a_{n-r+3} - \triangle^{\alpha+1} a_{n-r+1})\right| \\
&\quad + C_2 A_{n+3-r}^{r+\delta} \left|(\triangle^{\alpha+1} a_{n-r+5} - \triangle^{\alpha+1} a_{n-r+2})\right| + \cdots \\
\leq\; & C_2 A_{n+1-r}^{\alpha} \left|(\triangle^{\alpha+1} a_{n-r+2} - \triangle^{\alpha+1} a_{n-r})\right| \\
&\quad + C_2 A_{n+2-r}^{\alpha} \left|(\triangle^{\alpha+1} a_{n-r+3} - \triangle^{\alpha+1} a_{n-r+1})\right| \\
&\quad + C_2 A_{n+3-r}^{\alpha} \left|(\triangle^{\alpha+1} a_{n-r+5} - \triangle^{\alpha+1} a_{n-r+2})\right| + \cdots \\
=\; & o(1) + o(1) + \cdots \\
=\; & o(1), \quad \text{by the hypothesis of the theorem.}
\end{aligned}
$$

Hence (4.10) implies

$$\lim_{n\to\infty} \int_0^\pi |g(x) - g_n(x)|\, dx = o(1). \tag{4.11}$$

Thus by (4.5) and (4.11)

$$\lim_{n\to\infty} \int_0^\pi |g(x) - g_n(x)|\, dx = o(1), \quad \text{when } \alpha \text{ is non} - \text{integral.} \tag{4.12}$$

Hence by (4.3) and (4.12)

$$\lim_{n\to\infty} \int_0^\pi |g(x) - g_n(x)|\, dx = o(1), \quad \text{for any } \alpha > 0.$$

This gives $g_n(x) \to g(x)$ in L^1-norm. This proves the theorem. □

References

[1] A. F. Andersen, *On extensions within the theory of Cesàro summability of a classical convergence theorem of Dedekind*, Pro. London Math. Soc., **8**(1958), 1-52.

[2] L. S. Bosanquet, *Note on the Bohr-Hardy theorem*, J. London Math Soc., **17**(1942), 166-173.

[3] L. S. Bosanquet, *Note on convergence and summability factors* (*III*), Proc. London Math. Soc., (1949), 482-496.

[4] K. Kaur, S. S. Bhatia and B. Ram, *On L^1-convergence of certain Trigonometric Sums*, Georgian journal of Mathematics, **11(1)**(2004), 99-104.

[5] A. N. Kolmogorov, *Sur l'ordere de grandeur des coefficients de la series de Fourier - Lebesque*, Bull. Polon. Sci. Ser. Sci. Math. Astronom. Phys., (1923), 83-86.

[6] S. A. Teljakovskii, *Some estimates for trigonometric series with quasi-convex coefficients*, Mat. Sbornik, **63(105)**(1964), 426-444.

[7] A. Zygmund, Trigonometric Series, Cambridge University Press, **1(II)**(1959).