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# Fredholm Type Integral Equations and Certain Polynomials

V. B. L. Chaurasia and Ashok Singh Shekhawat
*Department of Mathematics, University of Rajasthan, Jaipur-302004, India*
*e-mail* : csmaths2004@yahoo.com

Abstract. This paper deals with some useful methods of solving the one-dimensional integral equation of Fredholm type. Application of the reduction techniques with a view to inverting a class of integral equation with Lauricella function in the kernel, Riemann-Liouville fractional integral operators as well as Weyl operators have been made to reduce to this class to generalized Stieltjes transform and inversion of which yields solution of the integral equation. Use of Mellin transform technique has also been made to solve the Fredholm integral equation pertaining to certain polynomials and $H$-functions.

## 1. Introduction

In recent years several authors (see, e.g. Srivastava and Buschman [13], Prabhakar and Kashyap [10], Srivastava and Raina [17], Love [8], [9], Higgins [6], Buschman, Koul and Gupta [1], Chaurasia and Patni [2], [3] have made significant contributions to the integral equations pertaining to various functions.

A detailed and systematic discussion of the various methods of solvability of certain interesting cases of Fredholm type integral equation have been given by Srivastava and Raina [17]:

$$(1.1)\quad \int_0^\infty x^{-\mu} H_{P,Q}^{M,N}\left[\, E\left(\tfrac{t}{x}\right)^m \;\middle|\; \begin{matrix} (a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q} \end{matrix} \right] f(x)\,dx = g(t), \qquad (0<t<\infty).$$

An extension of these results pertaining to the Fredholm integral equations is the main object of the present study:

$$(1.2)\qquad \int_0^\infty w^{-\alpha} H_{A,C:[B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,\mu:(u',v');\cdots;(u^{(r)},v^{(r)})}\left[z_1\left(\frac{t}{w}\right)^q,\cdots,z_r\left(\frac{t}{w}\right)^q\right] f(w)dw$$
$$= \quad g(t), \qquad (0<t<\infty)$$

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and

$$\int_0^\infty w^{-\alpha} S_n^m \left[ A' \left( \frac{t}{w} \right)^p \right] H_{B,D}^{u,v} \left[ z \left(\tfrac{t}{w}\right)^q \;\middle|\; \begin{matrix} (b_B, \beta_B) \\ \\ (d_D, \delta_D) \end{matrix} \right] f(w)\, dw \tag{1.3}$$

$$= \quad g(t), \qquad (0 < t < \infty).$$

For the multivariable $H$-function (see [15]).

The multiple integral in (1.2) converges absolutely if

$$|\arg(z_i)| < \frac{1}{2}\pi T_i, \quad i = 1, \cdots, r, \tag{1.4}$$

where

$$T_i = - \sum_{j=1+\lambda}^{A} \theta_j^{(i)} + \sum_{j=1}^{v^{(i)}} \Phi_j^{(i)} - \sum_{j=1+v}^{B^{(i)}} \Phi_j^{(i)} - \sum_{j=1}^{C} \Psi_j^{(i)} + \sum_{j=1}^{u^{(i)}} \delta_j^{(i)} - \sum_{j=1+u^{(i)}}^{D^{(i)}} \delta_j^{(i)} \tag{1.5}$$

$$> 0, \qquad \text{for all} \;\; i \in (1, \cdots, r).$$

$S_n^m[A]$ in (1.3) denotes the general class of polynomials (see Srivastava [11])

$$S_n^m[A] \;=\; \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} E_{n,k} A^k, \qquad (n = 0, 1, 2, \cdots) \tag{1.6}$$

where $m$ is an arbitrary positive integer and the coefficients $E_{n,k}$ $(n,\ k \geq 0)$ are arbitrary constants real or complex.

Let the space of all functions $f$ defined on $R^+ = [0, \infty)$ be represented by $\int$ and satisfy

(i) $f \in b^\infty(R^+)$,

(ii) $\lim_{t \to \infty} [t^\gamma f^r(t)]$ for all non negative integer $\gamma$ and $r$,

(iii) $f(t) = 0(1)$ as $t \to \infty$.

For correspondence to the space of good functions defined on the whole real line $(-\infty, \infty)$ see (Lighthill [7]).

The Riemann-Liouville fractional integral (of order $\lambda$) is given by:

$$D^{-\lambda}[f(t)] = {}_0D_t^\lambda[f(t)] = \frac{1}{\Gamma(\lambda)} \int_0^t (t-y)^{\lambda-1} f(y)\, dy, \quad (\mathrm{Re}(\lambda) > 0 : \; f \in \int), \tag{1.7}$$

where $D^\lambda[f(t)] = \Phi(t)$ is understood to mean that $\Phi$ is a locally integrable solution of $f(t) = D^{-\lambda}[\Phi(t)]$, implies that $D^\lambda$ is the inverse of the fractional integral operator $D^{-\lambda}$ (for brevity, $D_t^{-\lambda}$ stands for ${}_0D_t^{-\lambda}$).

The Weyl fractional integral (of order $\nu$) is defined by:

$$(1.8)\quad \begin{aligned} W^{-\nu}\{f(t)\} &= {}_tD_{\infty}^{-\nu}\{f(t)\} \\ &= \frac{1}{\Gamma(\nu)}\int_t^{\infty}(\xi-t)^{\nu-1}f(\xi)\,d\xi, \quad (\mathrm{Re}(\nu)>0:\ f\in \int). \end{aligned}$$

## 2. Preliminary results

**Lemma 1.** *Assuming the following that*

(i) $\mu, \rho, u^{(i)}, v^{(i)}, A, B^{(i)}, C, D^{(i)}$ *be positive integers such that* $0 \le \mu \le A$, $0 \le u^{(i)} \le D^{(i)}$, $c \ge 0$ *and* $0 \le v^{(i)} \le B^{(i)}$, *for all* $i = 1, \cdots, r$;

(ii) $Re(\alpha) > Re(\beta)$; $Re\left[\beta + q\sum_{i=1}^{r}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ $(j = 1, \cdots, u^{(i)})$; $q > 0$;

(iii) $|arg(z_i)| < \frac{1}{2}\pi T_i$, *where* $T_i$ *is given by* (1.5);

(iv) $m_1$ *be an arbitrary positive integer and the coefficients* $A_{n_1,k_1}$ $(n_1, k_1 \ge 0)$ *be arbitrary constants, real or complex.*

*Then*

$$(2.1)\quad W^{\beta-\alpha}\left[w^{-\alpha}S_{n_1}^{m_1}\left(\frac{t}{w}\right)^{\rho} H^{0,\mu:(u',v');\cdots;(u^{(r)},v^{(r)})}_{A,C:[B',D'];\cdots;[B^{(r)},D^{(r)}]}\left[z_1\left(\frac{t}{w}\right)^{q},\cdots,z_r\left(\frac{t}{w}\right)^{q}\right]\right]$$

$$= w^{-\beta}\sum_{k_1=0}^{[n_1/m_1]}\frac{(-n_1)_{m_1k_1}}{k_1\,!}A_{n_1,k_1}\left(\frac{t}{w}\right)^{\rho k_1} H^{0,\mu+1:(u',v');\cdots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\cdots;[B^{(r)},D^{(r)}]}$$
$$\left[\begin{array}{ll} [1-\beta-\rho k_1:q,\cdots,q] & [(a):\theta';\cdots;\theta^{(r)}]:[(b'):\Psi'];\cdots;[(b^{(r)}):\Psi^{(r)}] \\ [(c):\Psi';\cdots;\Psi^{(r)}] & [1-\alpha-\rho k_1:q,\cdots,q]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}], \end{array}\right.$$
$$\left.\left|\ z_1\left(\frac{t}{w}\right)^{q},\cdots,z_r\left(\frac{t}{w}\right)^{q}\right.\right].$$

*Proof.* Let $\Delta(w)$ stands for the first member of the assertion (2.1). Then making use of (1.8) and the definition of the multivariable $H$-function [15], we get

$$(2.2) \quad \Delta(w) = \sum_{k_1=0}^{[n_1/m_1]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} A_{n_1,k_1} \frac{1}{\Gamma(\alpha-\beta)} \int_w^\infty (x-w)^{\alpha-\beta-1} \left( \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \Phi_1(s_1)\cdots\Phi_r(s_r) \times \left[\prod_{i=1}^r \Psi(s_i)\right] z_1^{s_1}\cdots z_r^{s_r} t^{q\sum_{i=1}^r s_i+\rho k_1} x^{-\alpha-\rho k_1-q\sum_{i=1}^r s_i} ds_1\cdots ds_r \right) dx.$$

The change of order of integration in (2.2) is supposed to be permissible under the absolute (and uniform) condition of convergence of the integral; we find

$$(2.3) \quad \Delta(w) = \sum_{k_1=0}^{[n_1/m_1]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} A_{n_1,k_1} \frac{1}{\Gamma(\alpha-\beta)} \frac{1}{(2\pi i)^r} \int_{L_1} \cdots \int_{L_r} \Phi_1(s_1)\cdots\Phi_r(s_r)\Psi(s_1,\cdots,s_r) \times z_1^{s_1}\cdots z_r^{s_r} t^{q\sum_{i=1}^r s_i+\rho k_1} \left( \int_w^\infty x-w^{\alpha-\beta-1} x^{-\alpha-\rho k_1-q\sum_{i=1}^r s_i} dx \right) ds_1\cdots ds_r.$$

The inner integral in (2.3) can be solved under hypothesis (ii) of Lemma 1, and we get

$$(2.4) \quad \Delta(w) = \frac{w^{-\beta}}{(2\pi i)^r} \sum_{k_1=0}^{[n_1/m_1]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} A_{n_1,k_1} \int_{L_1} \cdots \int_{L_r} \Phi_1(s_1)\cdots\Phi_r(s_r)\Psi(s_1,\cdots,s_r) \times \frac{\Gamma(\beta+q\sum_{i=1}^r s_1+\rho k_1)}{\Gamma(\alpha+q\sum_{i=1}^r s_1+\rho k_1)} z_1^{s_1}\cdots z_r^{s_r} \left(\frac{t}{w}\right)^{q\sum_{i=1}^r s_i+\rho k_1} ds_1\cdots ds_r$$

which gives the second member of (2.1) by reinterpreting the multivariable $H$-function. □

The multivariable $H$-function appearing in (2.1) exist (and are analytic) under the conditions (i) and (ii) of Lemma 1 and the Weyl fractional integral converges absolutely under the condition (ii).

**Theorem 1.** *Under the set of sufficient conditions* (*i*), (*ii*) *and* (*iii*) *of Lemma 1,*

$$(2.5) \qquad \int_0^\infty w^{-\beta} \sum_{k_1=0}^{[n_1/m_1]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} A_{n_1,k_1} \left(\frac{t}{w}\right)^{\rho k_1} H^{0,\mu+1:(u',v');\cdots;(u^{(r)},v^{(r)})}_{A+1,C+1:[B',D'];\cdots;[B^{(r)},D^{(r)}]}$$

$$\left[\begin{array}{ll} [1-\beta-\rho k_1 : q,\cdots,q] & [(a):\theta';\cdots;\theta^{(r)}]:[(b'):\Psi'];\cdots;[(b^{(r)}):\Psi^{(r)}] \\ [(c):\Psi';\cdots;\Psi^{(r)}], & [1-\alpha-\rho k_1 : q,\cdots,q]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}] \end{array}\right.$$

$$\left.\Bigg| \quad z_1\left(\frac{t}{w}\right)^q,\cdots,z_r\left(\frac{t}{w}\right)^q \right] f(w)dw$$

$$= \sum_{k_1=0}^{[n_1/m_1]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} A_{n_1,k_1} \int_0^\infty w^{-\alpha-\rho k_1} t^{\rho k_1} H^{0,\mu}_{A,C}\left[z_1\left(\frac{t}{w}\right)^q,\cdots,z_r\left(\frac{t}{w}\right)^q\right]$$

$$D^{\beta-\alpha}\{f(w)\}dw, \qquad \textit{where } f\in \int \textit{ and } t>0.$$

*Proof.* Suppose that the first member of (2.5) is represented by $\eta$ and by applying Lemma 1 and then changing the order of summation and integration (which is permissible under given conditions), we arrive at the result (2.5). □

## 3. Solution of a Lauricella form of the integral equation (1.2)

We use the reduction technique by which a given integral equation may be reduced to some simpler integral transform with the aid of the result derived in the preceding section, to obtain the solution of a certain Lauricella's hypergeometric form of the integral equation (1.2).

Using the following result [4] in Theorem 1 at first

$$(3.1) \qquad H^{0,0:(u',v');\cdots;(u^{(r)},v^{(r)})}_{0,0:[B',D'];\cdots;[b^{(r)},D^{(r)}]}$$

$$\left[\begin{array}{ll} [(a):\theta',\cdots,\theta^{(r)}]:[(b'):\Phi'];\cdots;[(b^{(r)}):\Phi^{(r)}] \;\; ; & \\ & z_1,\cdots,z_r \\ [(c):\Phi',\cdots,\Phi^{(r)}:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}] \;\; ; & \end{array}\right]$$

$$= \prod_{i=1}^{r} H^{u^{(i)},v^{(i)}}_{B^{(i)},D^{(i)}}\left[\, z_i \;\middle|\; \begin{array}{l} [(b^{(i)}):\Phi^{(i)}] \\ \\ [(d^{(i)}):\delta^{(i)}] \end{array}\right].$$

Also, we set $\Phi_j^{(i)} = 1$ ($j = 1,\cdots,B^{(i)}$) and $\delta_j^{(i)} = 1$ ($j = 1,\cdots,D^{(i)}$), modify the parameters of the $H$-function of several complex variables in such a way that use can be made of the relationship [14] on the left hand side of (2.5) and of the relationship [15]

$$(3.2)\qquad H_{1,1}^{1,1}\left[\, z \;\middle|\; \begin{matrix}(1-\alpha,1)\\ \\ (0,1)\end{matrix}\,\right] = \frac{\Gamma(\alpha)}{(1+z)^{\alpha}} = \Gamma(\alpha)\,{}_1F_0[\alpha\,;\,\_\,;\,-z]$$

on the right hand side of (2.5). Thus the solution of the Corollary 1 emerges from Theorem 1.

**Corollary 1.** *Let* $a_1,\cdots,a_r$, $\alpha$ *and* $\beta$ *be complex parameters,* $Re(\alpha) > Re(\beta) > 0$, $f \in \int$ *and* $q \in \mathbb{N}$, *then for all* $t > 0$,

$$\sum_{k_1=0}^{[n_1/m_1]} \frac{(n_1)_{m_1k_1}}{k_1\,!} A_{n_1,k_1} \int_0^\infty w^{-\beta-\rho k_1} t^{\rho k_1} \frac{\prod_{i=1}^{r}\Gamma(a_i)\Gamma(\beta+\rho k_1)}{\Gamma(\alpha+\rho k_1)}$$
$$F\left[a_1,\cdots,a_r,\xi(q;\,\beta+\rho k_1);\xi(q;\alpha+\rho k_1);\,-\left(\frac{t}{w}\right)^q,\cdots,-\left(\frac{t}{w}\right)^q\right] f(w)dw$$
$$= \sum_{k_1=0}^{[n_1/m_1]} \frac{(n_1)_{m_1k_1}}{k_1\,!} A_{n_1,k_1} \int_0^\infty \prod_{i=1}^{r}\Gamma(a_i) t^{\rho k_1} \frac{w^{q\sum_{i=1}^{r} a_i-\alpha-\rho k_1} D^{\beta-\alpha}\{f(w)\}dw}{(t^q+w^q)^{\sum_{i=1}^{r} a_i}}$$

OR

$$= \sum_{k_1=0}^{[n_1/m_1]} \frac{(n_1)_{m_1k_1}}{k_1\,!} A_{n_1,k_1} \int_0^\infty w^{-\beta-\rho k_1} t^{\rho k_1} \frac{\prod_{i=1}^{r}\Gamma(a_i)\Gamma(\beta+\rho k_1)}{\Gamma(\alpha+\rho k_1)} F_{1:0,\cdots,0}^{1:1,\cdots,1}$$
$$\left[\begin{matrix}[\beta+\rho k_1;q,\cdots,q]:[a_1:1]:\cdots:[a_r:1]\\ \\ [\alpha+\rho k_1;q,\cdots,q]:[0:0]:\cdots:[0:0]\end{matrix}\;\middle|\; -\left(\frac{t}{w}\right)^q,\cdots,-\left(\frac{t}{w}\right)^q\right] f(w)dw$$
$$= \sum_{k_1=0}^{[n_1/m_1]} \frac{(n_1)_{m_1k_1}}{k_1\,!} A_{n_1,k_1} \int_0^\infty \prod_{i=1}^{r}\Gamma(a_i) t^{\rho k_1} \frac{w^{q\sum_{i=1}^{r} a_i-\alpha-\rho k_1} D^{\beta-\alpha}\{f(w)\}dw}{(t^q+w^q)^{\sum_{i=1}^{r} a_i}}$$

where

$$(3.3)\quad \xi(q:\alpha+\rho k_1) = \left(\frac{\alpha+\rho k_1}{q}, \frac{\alpha+\rho k_1+1}{q},\cdots,\frac{\alpha+\rho k_1+q-1}{q}\right),\qquad (q\in\mathbb{N}).$$

**Theorem 2.** *Let* $a_1,\cdots,a_r$, $\alpha$ *and* $\beta$ *be complex parameters such that* $Re\left(\sum_{i=1}^{r} a_i\right) >$

$1$ $(i = 1, \cdots, r)$ *and* $Re(\alpha) > Re(\beta) > 0$ *and* $q \in 1\mathbb{N}$, $f \in \int$. *Then for all* $t > 0$ *and* $g \in \int$, *the integral equation*

$$(3.4)\quad \sum_{k_1=0}^{[n_1/m_1]} \frac{(n_1)_{m_1 k_1}}{k_1\,!} A_{n_1,k_1} \int_0^\infty w^{-\beta-\rho k_1} t^{\rho k_1} \frac{\prod_{i=1}^{r} \Gamma(a_i)\Gamma(\beta+\rho k_1)}{\Gamma(\alpha+\rho k_1)}$$
$$F\left[a_1, \cdots, a_r, \xi(q;\, \beta+\rho k_1); \xi(q; \alpha+\rho k_1); -\left(\frac{t}{w}\right)^q, \cdots, -\left(\frac{t}{w}\right)^q\right] f(w)dw$$
$$= \quad g(t)$$

has a solution given by

$$(3.5)\quad f(t) \;=\; \frac{q\Gamma(a_1+\cdots+a_r)}{\prod_{i=1}^{r}\Gamma(a_i)} D_t^{\alpha-\beta}$$
$$\left[t^{q(1-\sum_{i=1}^{r} a_i)+\alpha+\rho k_1-1} D_{t^n}^{1-\sum_{i=1}^{r}\alpha_i} \left\{(1+t) \lim_{n'\to\infty} L_{n',t^n}[g(t)]\right\}\right]$$

where

$$(3.6)\qquad L_{n',t^n}[g(t)] \;=\; \frac{(-t)^{n'-1}}{n'(n'-2)\,!} \frac{d^{2n'-1}}{dx^{2n'-1}}\{t^{n'} g(t)\} \qquad (n' = 2, 3, 4, \cdots).$$

*Proof.* By using Corollary 1 and making use of given result [12], [9] and [19], we arrive at the result given in Theorem 2. □

## 4. Use of other methods

By resorting to the application of the Mellin transforms one-dimensional Fredholm integral equation (1.3) concerning the Fox's $H$-function and a general class of polynomials $S_n^m[E]$ in the kernel can also be solved. We first prove the following result, which we required in proving Theorem 3 below.

**Lemma 2.** *Let*

(i) $u, v, B, D$ *be positive integers such that* $1 \le u \le D$, $0 \le v \le B$,

(ii) $Re(\alpha) > Re(\beta)$; $Re\left[\beta + q\sum_{i=1}^{r} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$ $(j = 1, \cdots, u^{(i)})$; $q > 0$;

(iii) $m$, $m_1$ *be an arbitrary positive integer and the coefficients* $A_{n,k}$ $(n, k \ge 0)$, $A_{n_1,k_1}$ $(n_1, k_1 \ge 0)$ *be arbitrary constants, real or complex.*

(iv) $|arg(z)| < \frac{1}{2}\pi T$,

*where*

$$T = \sum_{j=1}^{u} \delta_j - \sum_{j=1+u}^{D} \delta_j + \sum_{j=1}^{v} \beta_j - \sum_{j=1+v}^{B} \beta_j > 0.$$

*Then*

$$(4.1) \qquad W^{\beta-\alpha}\left\{ w^{-\alpha} S_{n_1}^{m_1}\left[A'\left(\frac{t}{w}\right)^{\rho}\right] S_n^m\left[E'\left(\frac{t}{w}\right)^{\rho}\right] H_{B,D}^{u,v}\left[ z\left(\frac{t}{w}\right)^{q} \,\middle|\, \begin{matrix} (b_B, \Phi_B) \\ \\ (d_D, \delta_D) \end{matrix} \right]\right\}$$

$$= w^{-\beta} \sum_{k_1=0}^{[n_1/m_1]} \sum_{k=0}^{[n/m]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} \frac{(-n)_{mk}}{k\,!} A_{n_1,k_1} A_{n,k} \left(\frac{t}{w}\right)^{\rho k_1 + \rho k} {A'}^{k_1} {E'}^{k}$$

$$\times H_{B+1,D+1}^{u,v+1}\left[ \begin{matrix} (1-\beta-\rho k-\rho k_1 : q), (b_B, \Phi_B) \\ \\ (d_D, \delta_D), (1-\alpha-\rho k-\rho k_1 : q) \end{matrix} \,\middle|\, z\left(\frac{t}{w}\right)^{q} \right].$$

*Proof.* To prove Lemma 2, firstly we use the definition of Weyl fractional integral given in (1.8), express the Fox's $H$-function in a contour integral of Mellin-Barnes type and a general class of polynomials, then we change the order of summation and integrations (which is justified under the stated conditions), evaluate the $t$-integral and reinterpreting the resulting Mellin-Barnes contour integral in terms of the $H$-function, we get the desired result. □

**Theorem 3.** *With the sufficient conditions (i), (ii), (iii) and (iv) of Lemma 2,*

$$(4.2) \quad \int_0^{\infty} w^{-\beta} \sum_{k_1=0}^{[n_1/m_1]} \sum_{k=0}^{[n/m]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} \frac{(-n)_{mk}}{k\,!} A_{n_1,k_1} A_{n,k} \left(\frac{t}{w}\right)^{\rho k_1 + \rho k} {A'}^{k_1} {E'}^{k}$$

$$\times H_{B+1,D+1}^{u,v+1}\left[ \begin{matrix} (1-\beta-\rho k-\rho k_1 : q), (b_B, \Phi_B) \\ \\ (d_D, \delta_D), (1-\alpha-\rho k-\rho k_1 : q) \end{matrix} \,\middle|\, z\left(\frac{t}{w}\right)^{q} \right] f(w)dw$$

$$= \int_0^{\infty} w^{-\alpha} S_{n_1}^{m_1}\left[A'\left(\frac{t}{w}\right)^{\rho}\right] S_n^m\left[E'\left(\frac{t}{w}\right)^{\rho}\right] H_{B,D}^{u,v}\left[ z\left(\frac{t}{w}\right)^{q} \,\middle|\, \begin{matrix} (b_B, \Phi_B) \\ \\ (d_D, \delta_D) \end{matrix} \right]$$

$$D^{\beta-\alpha}\{f(w)\}dw.$$

*Proof.* Theorem 3 can be proved with the help of Lemma 2 and the equation (1.8), on proceeding on similar lines as given in the proof of Theorem 1. □

**Theorem 4.** *Suppose that* $f \in \int$ , $D^{\alpha-\beta} f(w)$ *exists* $q > 0$, $t > 0$, $|arg(z)| < \frac{1}{2}\pi T$, $T > 0$ *(*$T$ *given in Lemma 2),* $Re(\alpha) > Re(\beta) > 0$ *and* $S_n^m[A]$ *be defined by (1.6), then the solution of the integral equation*

$$\int_0^\infty w^{-\alpha} S_{n_1}^{m_1}\left[A'\left(\frac{t}{w}\right)^\rho\right] S_n^m\left[E'\left(\frac{t}{w}\right)^\rho\right] H_{B,D}^{u,v} \left[ z\left(\frac{t}{w}\right)^q \middle| \begin{matrix} (b_B, \Phi_B) \\ \\ (d_D, \delta_D) \end{matrix} \right] f(w)dw = g(t), \qquad (0 < 1 < \infty) \tag{4.3}$$

*is as follows*

$$\begin{aligned} f(t) &= \sum_{k_1=0}^{[n_1/m_1]} \sum_{k=0}^{[n/m]} \frac{(-n_1)_{m_1 k_1}}{k_1\,!} \frac{(-n)_{mk}}{k\,!} A_{n_1,k_1} A_{n,k} {A'}^{k_1} {E'}^{k} t^{\alpha-\rho k_1 - pk} \\ &\times \lim_{\rho'\to\infty} \int_{\sigma'+i\rho'}^{\sigma'+i\rho'} \left[\theta\left(\frac{-pk-\rho k_1 - s}{q}\right)\right]^{-1} t^{-s} z^{\rho k+\rho k_1+s} \Phi(s)\, ds, \end{aligned} \tag{4.4}$$

*exists when*

$$\max\{\text{Re}[(a_\ell - 1)/\Phi]\} < \text{Re}\left(\frac{pk+s}{q}\right) < \min\left\{\text{Re}\left(\frac{d_j}{d_j}\right)\right\}, \tag{4.5}$$

$$(j = 1, \cdots, u), \quad (l = 1, \cdots, v).$$

*Proof.* Making use of Theorem 3, Lemma 2 along with the known results ([18], [5] and [6]), we have the required result. □

Letting $n_1 \to 0$ and $n \to 0$, Theorem 4 seen to correspond to a result given by Srivastava and Raina [17] under less stringent conditions. Also, for $n_1 \to 0$, Theorem 4 reduces to a Theorem recently established by Chaurasia and Patni [3].

The importance of our results lies in the manifold generality. In view of the generality of the functions and polynomials, the results encompass several special cases interest scattered hitherto in the literature.

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## References


[1] R. G. Buschman, C. L. Koul and K. C. Gupta, *Convolution integral equations involving the $H$-function of two variables*, Glasnik Mat. Ser. III, **12(32)**(1997), 61-66.

[2] V. B. L. Chaurasia and R. Patni, *A class of convolution integral equation concerning a generalized polynomial set*, Soochow J. Math., **27(1)**(2001), 37-43.

[3] V. B. L. Chaurasia and R. Patni, *Integral equation of Fredholm type with special functions*, Indian J. Pure Appl. Math., **32(5)**(2001), 747-758.

[4] C. Fox, *The $G$ and $H$-functions as symmetrical Fourier kernels*, Trans. Amer. Math. Soc., **98**(1961), 395-429.

[5] K. C. Gupta and U. C. Jain, *The $H$-function II*, Proc. Nat. Acad. Sci. India Sect., **A36**(1966), 594-609.

[6] T. P. Higgins, *A hypergeometric function transform*, J. Soc. Indust. Appl. Math., **12**(1964), 601-612.

[7] M. J. Lighthill, Introduction to Fourier analysis and Generalized Functions, Comb. Mong. Mech. Appl. Math. Cambridge University Press, Cambridge, London and New York, 1958.

[8] E. R. Love, *Some integral equations involving hypergeometric functions*, Proc. Edinburgh Math. Soc., **15(2)**(1967), 169-198.

[9] E. R. Love, In Fractional Calculus and its Applications (Ed. B. Ross) pp. 272-288, Springer-Verlag, New York, 1975.

[10] T. R. Prabhakar and N. K. Kashyap, *A new class of hypergeometric integral equations*, Indian J. Pure Appl., **11(1)**(1980), 92-97.

[11] H. M. Srivastava, *A contour integral involving Fox's $H$-function*, Indian J. Math., **14**(1972), 1-6.

[12] H. M. Srivastava, *Some remarks on a generalisation of the Stieltjes transform*, Publ. Math. Debrecen, **23**(1976), 119-122.

[13] H. M. Srivastava and R. G. Buschman, *Some convolution integral equations*, Nederl. Akad. Welensch Proc. Ser., A 77 = Indag. Math. **36**(1974), 211-216.

[14] H. M. Srivastava and C. Daoust Martha, *Certain generalized Neumann Expansions associated with the Kampé de Fériet function*, Nederl. Akad. Wetensch Proc. Ser. A 72 = Indag. Math., **31**(1969), 449-457.

[15] H. M. Srivastava, K. C. Gupta and S. P. Goyal, The $H$-functions of One and Two Variables with Applications, South Asian Publishers, New Delhi, Madras, 1982.

[16] H. M. Srivastava and R. Panda, *Some bilateral generating function for a class of generalized hypergeometric polynomials*, J. Reine Angew. Math., **283/284**(1976), 265-274.

[17] H. M. Srivastava and R. K. Raina, *On certain methods of solving a class of integral equations of Fredholm type*, J. Austral. Math. Soc., **A-52**(1992), 1-10.

[18] E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals (Second edition), Oxford University Press, London, Clarendon Press, Oxford, 1948.

[19] D. V. Widder, The Laplace Transform, Princeton University Press, Princeton New Jersey, 1946.