Honam Mathematical J. **27** (2005), No. 3, pp. 369–388

FURTHER RESULTS OF INTUITIONISTIC FUZZY COSETS

Kul Hur, Hee Won Kang and Dae Sig Kim

Abstract. First, we prove a number of results about intuitionistic fuzzy groups involving the notions of intuitionistic fuzzy cosets and intuitionistic fuzzy normal subgroups which are analogs of important results from group theory. Also, we introduce analogs of some group-theoretic concepts such as characteristic subgroup, normalizer and Abelian groups. Secondly, we prove that if A is an intuitionistic fuzzy subgroup of a group G such that the index of A is the smallest prime dividing the order of G, then A is an intuitionistic fuzzy normal subgroup. Finally, we show that there is a one-to-one correspondence the intuitionistic fuzzy cosets of an intuitionistic fuzzy subgroup A of a group G and the cosets of a certain subgroup H of G.

0. Introduction

The concept of a fuzzy set was introduced by Zadeh[19], and it is now a rigorous area of research with manifold applications raging from engineering and computer science to medical diagnosis and social behavior studies. In particular, several researchers [6, 15-18] applied the notion of an fuzzy set to group theory.

Received March 24, 2005. Revised July 28, 2005.

2000 Mathematics Subject Classification : 03F55, 20A05, 20K27, 20N25.

Key words and phrases : intuitionistic fuzzy normal subgroup, intuitionistic fuzzy coset, intuitionistic fuzzy characteristic fuzzy subgroup, normalizer, Abelian group.

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov[1] in 1986. After that time, Çoker and his colleagues [4,5,7], Lee and Lee[14], and Hur and his colleagues [12] applied the concept of intuitionistic fuzzy sets to topology. In particular, Hur and his colleagues [11] applied the notion of intuitionistic fuzzy sets to topological group. Also, several researchers [2,3,8-10,13] applied one to group theory.

The present paper is a sequel to [13]. We obtain a number of further analogs of the properties of groups, thereby enriching the theory of intuitionistic fuzzy groups and, in particular, corroborating the concept of intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets introduced in [9,13]. Moreover, we obtain an analog of the following standard result from group theory that if θ is an automorphism of a group G which leaves invariant some normal subgroup N, then θ induces an automorphism of the quotient group G/N.

Some variations of this result are also considered, for which we obtain analogs for intuitionistic fuzzy groups. Also we show that there is a natural one-to-one correspondence between the intuitionistic fuzzy cosets of an intuitionistic fuzzy subgroup A of a group G and the cosets of a subgroup G_A of G defined by

$$G_A = \{g \in G : A(g) = A(e)\},$$

where e denotes, as usual, the identity element of the group G. Our analysis illustrates that the subgroup G_A defined above plays a significant role in investigating the structure of the corresponding intuitionistic fuzzy subgroup.

1. Preliminaries

In this section, we list some basic concepts and well-known results which are needed in the later sections.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as I.

Definition 1.1[1,4]. Let X be a nonempty set. A complex mapping $A = (\mu_A,\ \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if $\mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$, where the mapping $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definition 1.2[9]. Let G be a group with the identity e and let $A \in$ IFS(G). Then A is called an *intuitionistic fuzzy subgroup*(in short, *IFG*) of G if

(i) $\mu_A(xy) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(x) \vee \nu_A(y)$ for any $x, y \in G$.

(ii) $\mu_A(x^{-1}) \geq \mu_A(x)$ and $\nu_A(x^{-1}) \leq \nu_A(x)$ for each $x \in G$.

We will denote the set of all IFGs of G as IFG(G).

Result 1.A[9, Proposition 2.6]. Let G be a group with the identity e and let $A \in$ IFG(G). Then $A(x^{-1}) = A(x)$ and $\mu_A(x) \leq \mu_A(e)$, $\nu_A(x) \geq \nu_A(e)$ for each $x \in G$.

Result 1.B[9, Proposition 2.7]. If $A \in \text{IFG}(G)$, then $G_A = \{x \in G : A(x) = A(e)\}$ is a subgrpup of G.

Result 1.C[9, Proposition 2.3]. Let $\{A_\alpha\}_{\alpha\in\Gamma} \subset \text{IFG}(G)$. Then $\bigcap_{\alpha\in\Gamma} A_\alpha \in \text{IFG}(G)$.

Definition 1.3[9]. Let G be a group with the identity e and let $A \in \text{IFG}(G)$. Then A is called an *intuitionistic fuzzy normal subgroup*(in short, *IFNG*) of G if $A(xy) = A(yx)$ for any $x, y \in G$.

We will denote the set of all IFNGs of G as IFNG(G).

2. Intuitionistic fuzzy subgroups

Definition 2.1. Let A be an IFG of a group G and let $\theta : G \to G$ be a mapping. We define a complex mapping $A^\theta = (\mu_{A^\theta}, \nu_{A^\theta}) : G \to I \times I$ as follows : for each $g \in G$, $A^\theta(g) = A(\theta(g))$.

For a group G, a subgroup K is called a *characteristic subgroup* if $\theta(K) = K$ for every automorphism θ of G.

We now define an analog.

Definition 2.2. Let A be an IFG of a group G. Then A is called an *intuitionistic fuzzy characteristic subgroup* of G if $A^\theta = A$ for every automorphism θ of G.

Theorem 2.3. Let G be a group, let $A \in \text{IFS}(G)$ and let $\theta : G \to G$ be a mapping,

(1) If $A \in \text{IFG}(G)$ and θ is a homomorphism, then $A^\theta \in \text{IFG}(G)$.

(2) If A is an intuitionistic fuzzy characteristic subgroup of G, then $A \in \text{IFNG}(G)$.

Proof. (1) Let $x, y \in G$. Then

$A^\theta(xy) = A(\theta(xy))$

$= A(\theta(x)\theta(y))$. (Since θ is a homomorphism)

Since $A \in \text{IFG}(G)$,

$\mu_A(\theta(x)\theta(y)) \geq \mu_A(\theta(x)) \wedge \mu_A(\theta(y)) = \mu_{A^\theta}(x) \wedge \mu_{A^\theta}(y)$

and

$\nu_A(\theta(x)\theta(y)) \leq \nu_A(\theta(x)) \vee \nu_A(\theta(y)) = \nu_{A^\theta}(x) \vee \nu_{A^\theta}(y)$.

Thus $\mu_{A^\theta}(xy) \geq \mu_{A^\theta}(x) \wedge \mu_{A^\theta}(y)$ and $\nu_{A^\theta}(xy) \leq \nu_{A^\theta}(x) \vee \nu_{A^\theta}(y)$.

On the other hand,

$A^\theta(x^{-1}) = A(\theta(x^{-1}))$

$= A(\theta(x)^{-1})$ (Since θ is a homomorphism)

$= A(\theta(x))$ (By Result 1.A)

$= A^\theta(x)$.

Hence $A^\theta \in \text{IFG}(G)$.

(2) Let $\theta : G \to G$ be the automorphism of G defined by

$\theta(g) = x^{-1}gx$ for each $g \in G$.

Then clearly it is standard result that θ is an automorphism of G, called the *inner automorphism* induced by x. Let $x, y \in G$. Since A is intuitionistic fuzzy characteristic, $A^\theta = A$. Thus

$A(xy) = A^\theta(xy) = A(\theta(xy)$

$= A(x^{-1}(xy)x)$ (By the definition of θ)

$= A(yx)$.

Hence $A \in \text{IFNG}(G)$. This completes the proof. ■

Remark 2.4. Theorem 2.3(2) is an analog of the result that a characteristic subgroup of a group is normal.

Now we obtain analogs of the concepts of *conjugacy*, *normalizer* regarding a group, and their properties.

Definition 2.5. Let G be a group and let $A_1, A_2 \in \text{IFG}(G)$. Then we say that A_1 is *conjugate* to A_2 if there exists an $x \in G$ such that $A_1(g) = A_2(x^{-1}gx)$ for each $g \in G$.

It is easy to show that the relation of conjugacy is an equivalence relation on IFG(G). Hence IFG(G) is a union of pairwise disjoint classes of intuitionistic fuzzy subgroups each consisting of intuitionistic fuzzy subgroups which are equivalent to one another. Now we shall obtain an expression giving the number of distinct conjugates of an intuitionistic fuzzy subgroups.

Notation. Let G be a group, let $A \in \text{IFG}(G)$ and let $g \in G$. We define a complex mapping $A^g = (\mu_{A^g}, \nu_{A^g}) : G \to I \times I$ as follows : for each $u \in G$,

$A^g(u) = A(g^{-1}ug)$, i.e., $\mu_{A^g}(u) = \mu_A(g^{-1}ug)$ and $\nu_{A^g}(u) = \nu_A(g^{-1}ug)$.

From Theorem 2.3(1), it is clear that $A^g \in \text{IFG}(G)$.

Definition 2.6. Let A be an IFG of a group G. Then the set

$$N(A) = \{g \in G : A^g = A\}$$

is called the *normalizer* of A.

Theorem 2.7. Let A be an IFG of a group G. Then

(1) $N(A)$ is a subgroup of G.

(2) $A \in \text{IFNG}(G)$ id and only if $N(A) = G$.

(3) If G is a finite group, then the number of distinct conjugates of A is equal to the index of $N(A)$ in G.

Proof. (1) Let $g, h \in N(A)$ and let $u \in G$. Then
$A^{gh}(u) = A((gh)^{-1}u(gh)) = A(h^{-1}(g^{-1}ug)h) = A^h(g^{-1}ug) = (A^h)^g(u)$.
Thus $A^{gh} = (A^g)^h = A^h = A$. So $gh \in N(A)$. Let $x \in N(A)$ and let $y = x^{-1}$. Let $u \in G$. Then

$A^y(u) = A(y^{-1}uy) = A(xux^{-1}) = A((x^{-1}u^{-1}x)^{-1})$
$= A(x^{-1}u^{-1}x)$ (By Result 1.A)
$= A^x(u^{-1})$ (By the definition of A^x)
$= A(u^{-1})$ (Since $A^x = A$)
$= A(u)$. (By Result 1.A)

Thus $A^y = A$. So $y = x^{-1} \in N(A)$. Hence $N(A)$ is a subgroup of G.

(2)($\Rightarrow$): Suppose $A \in \mathrm{IFNG}(G)$ and let $g \in G$. Let $u \in G$. Then

$A^g(u) = A(g^{-1}ug) = A((g^{-1}u)g)$
$= A(g(g^{-1}u))$ (Since $A \in \mathrm{IFNG}(G)$)
$= A(u)$.

Thus $A^g = A$. So $g \in N(A)$, i.e., $G \subset N(A)$. Hence $N(A) = G$.

($\Leftarrow$): Suppose $N(A) = G$ and let $x, y \in G$. Then

$A(xy) = A(xyxx^{-1}) = A(x(yx)x^{-1})$
$= A^{x^{-1}}(yx)$ (By the definition of $A^{x^{-1}}$)
$= A(yx)$. (By the hypothesis)

Hence $A \in \mathrm{IFNG}(G)$.

(3) Consider the decomposition of G as a union of cosets of $N(A)$;

$G = x_1N(A) \cup x_2N(A) \cup \cdots \cup x_kN(A)$ (1)

, where k is the number of distinct cosets, i.e., the index of $N(A)$ in G. Let $x \in N(A)$ and choose i such that $1 \le i \le k$. Let $g \in G$. Then

$A^{x_ix}(g) = A((x_ix)^{-1}g(x_ix)) = A(x^{-1}(x_i^{-1}gx_i)x) = A^x(x_i^{-1}gx_i)$
$= A(x_i^{-1}gx_i)$ (Since $x \in N(A)$)
$= A^{x_i}(g)$.

Thus $A^{x_ix} = A^{x_i}$ for each $x \in N(A)$ and $1 \le i \le k$. So any two elements of G which lie in the same coset $x_iN(A)$ give rise to the same conjugate A^{x_i} of A. Now we show that two distinct cosets give two distinct conjugates of A. Assume that $A^{x_i} = A^{x_j}$, where $i \neq j$ and $1 \leq i \leq k$, $1 \leq j \leq k$. Let $g \in G$. Then

$$A^{x_i}(g) = A^{x_j}(g), \text{ i.e., } A(x_i^{-1}gx_i) = A(x_j^{-1}gx_j). \quad (2)$$

Let $h \in G$ such that $g = x_j h x_j^{-1}$. Then, by (2),

$$A(x_i^{-1}x_j h x_j^{-1}x_i) = A(x_j^{-1}x_j h x_j^{-1}x_j)$$
$$\Rightarrow A((x_i^{-1}x_j)h(x_j^{-1}x_i)) = A(h), \text{ i.e., } A((x_j^{-1}x_i)^{-1}h(x_j^{-1}x_i)) = A(h)$$
$$\Rightarrow A^{x_j^{-1}x_i}(h) = A(h), \text{ i.e., } A^{x_j^{-1}x_i} = A.$$

Thus $x_j^{-1}x_i \in N(A)$. So $x_iN(A) = x_jN(A)$. Since (1) represent a partition of G into pairwise disjoint cosets and $i \neq j$, this is not possible. Hence the number of distinct conjugates of A is equal to the index of $N(A)$ in G. This completes the proof. ■

Remark 2.8. Theorem 2.7(2) illustrates the motiviation behind the term "normalizer" and it shows the analogy with the fact that a subgroup H of a group G is normal in G if and only if the normalizer of H in G is equal to G itself. And Theorem 2.7(3) is an analog of a basic result in group theory.

Definition 2.9[13]. Let A be an IFG of a group G and let $x \in G$. We define two complex mappings

$$Ax = (\mu_{Ax}, \nu_{Ax}) : G \to I \times I$$

and

$$xA = (\mu_{xA}, \nu_{xA}) : G \to I \times I$$

as follows, respectively : for each $g \in G$,

$$Ax(g) = A(gx^{-1}) \text{ and } xA(g) = A(x^{-1}g).$$

Then Ax[resp.xA] is called the *intuitionistic fuzzy right* [resp.*left*] *coset* of G determined by x and A.

Lemma 2.10. Let A be an IFG of a group G and let

$K = \{x \in G : Ax = Ae\}$,

where e denotes the identity element of G. Then K is a subgroup of G. Furthermore, $G_A = K$.

Proof. Let $k \in K$ and let $g \in G$. Then $Ak(g) = Ae(g)$. Thus $A(gk^{-1}) = A(g)$. In particular, $A(ek^{-1}) = A(e)$, i.e., $A(k^{-1}) = A(e)$. Thus $k^{-1} \in G_A$. By Result 1.B, G_A is a subgroup of G. Thus $k \in G_A$. So $K \subset G_A$. Now let $h \in G_A$. Then

$A(h) = A(e)$. (3)

Let $g \in G$. Then $Ah(g) = A(gh^{-1})$ and $Ae(g) = A(g)$.

Thus

$$\begin{aligned}&\mu_A(gh^{-1}) \geq \mu_A(g) \wedge \mu_A(h^{-1})\\ &= \mu_A(g) \wedge \mu_A(h) \quad \text{(By Result 1.A)}\\ &= \mu_A(g) \wedge \mu_A(e) \quad \text{(By (3))}\\ &= \mu_A(g) \quad \text{(By Result 1.A)}\end{aligned}$$

and

$$\begin{aligned}&\nu_A(gh^{-1}) \leq \nu_A(g) \vee \nu_A(h^{-1}) = \nu_A(g) \vee \nu_A(h)\\ &= \nu_A(g) \vee \nu_A(e) = \nu_A(g)\end{aligned}$$

Also,

$$\begin{aligned}&\mu_A(g) = \mu_A(gh^{-1}h) \geq \mu_A(gh^{-1}) \wedge \mu_A(h)\\ &= \mu_A(gh^{-1}) \wedge \mu_A(e) \quad \text{(By (3))}\\ &= \mu_A(gh^{-1}) \quad \text{(By Result 1.A)}\end{aligned}$$

and

$$\begin{aligned}&\nu_A(g) = \nu_A(gh^{-1}h) \leq \nu_A(gh^{-1}) \vee \nu_A(h)\\ &= \nu_A(gh^{-1}) \vee = \nu_A(gh^{-1}) \ .\end{aligned}$$

So $A(gh^{-1}) = A(g)$, i.e., $Ah = Ae$, i.e., $h \in K$. Hence $G_A \subset K$. Therefore $G_A = K$. This completes the proof. ■

Corollary 2.10[9, Proposition 3.5]. Let G be a group. If $A \in \mathrm{IFNG}(G)$, then $G_A \lhd G$.

Proof. Let $g \in G$ and let $x \in G_A$. Then

$$\begin{aligned} A(g^{-1}xg) &= A(gg^{-1}x) \text{ (Since } A \in \text{IFNG}(G)) \\ &= A(x) \\ &= A(e). \text{ (Since } x \in G_A) \end{aligned}$$

Thus $g^{-1}xg \in G_A$. Hence $G_A \lhd G$. ■

For a group G, the commutator $[x, y]$ of two elements x, y in G is defined as $[x, y] = x^{-1}y^{-1}xy$. If $xy = yx$, then obviously $[x, y] = e$. Thus G is abelian if $[x, y] = e$ for all $x, y \in G$. This motivates the following definition.

Remark 2.11. A special case of Lemma 2.10 is implicit in our previous paper[13, Theorem 3.12], where it was tacitly assumed that A is intuitionistic fuzzy normal. But, as we see now, it is not necessarily to assume that A is intuitionistic fuzzy normal, and this fact straightens the proof of the intuitionistic fuzzy Lagrange's theorem [13, Theorem 3.12].

Definition 2.12. Let A be an IFG of a group G. Then A is said to be *intuitionistic fuzzy abelian* if

$A([x, y]) = A(e)$ for any $x, y \in G$.

Result 2.A[13]. Let $A \in \text{IFG}(G)$. Then $A \in \text{IFNG}(G)$ if and only if $\mu_A([x, y]) \geq \mu_A(x)$ and $\nu_A([x, y]) \leq \nu_A(x)$ for any $x, y \in G$.

Analogous to some well-known properties of abelian group, we prove.

Theorem 2.13. (1) An intuitionistic fuzzy abelian subgroup of a group is intuitionistic fuzzy normal.

(2) Given an intuitionistic fuzzy abelian subgroup of G, there is a normal subgroup N of G such that G/N is abelian.

Proof. (1) Let A be an intuitionistic fuzzy abelian subgroup of G. Let $x, y \in G$. Then, by Result 1.A, $\mu_A([x,y]) = \mu_A(e) \geq \mu_A(x)$ and $\nu_A([x,y]) = \nu_A(e) \leq \nu_A(x)$. Hence, by Result 2.A, $A \in \text{IFNG}(G)$.

(2) Let A be an intuitionistic fuzzy abelian subgroup of G. Then, by (1), $A \in \text{IFNG}(G)$. Thus, by Corollary 2.10, $G_A \lhd G$. Also, it is easy to see that $G' \subset G_A$, where G' denotes the commutator subgroup of G (i.e., the subgroup generated by all elements $[x,y]$, $x, y \in G$). Hence G/G_A is abelian. ■

The following is the immediate result of Definition 1.3 and Result 1.C.

Proposition 2.14. If $\{A_\alpha\}_{\alpha\in\Gamma}$ is a family of IFNGs of a group G, then $\bigcap_{\alpha\in\Gamma} A_\alpha \in \text{IFNG}(G)$. Furthermore, if $A, B \in \text{IFNG}(G)$, then $A \cap B \in \text{IFNG}(G)$.

It is a standard result in group theory that if G is a group, $H \leq G, K \leq G$ and $H \lhd G$, then $H \cap K \lhd K$ is normal in K. Now we derive an analog for intuitionistic fuzzy subgroups.

Proposition 2.15. Let G be a group and let $A \in \text{IFG}(G)$, $B \in \text{IFNG}(G)$. Then $A \cap B$ is an intuitionistic fuzzy normal subgroup of the group G_A.

Proof. It is clear that G_A is a subgroup of G by Result 1.B. By Proposition 2.14, $A \cap B \in \text{IFG}(G)$. Thus $A \cap B \in \text{IFG}(G_A)$. Let $x, y \in G_A$. Since G_A is a subgroup of G, $xy \in G_A$ and $yx \in G_A$. Thus

$$A(xy) = A(yx) = A(e).$$

Since $B \in \text{IFNG}(G)$, $B(xy) = B(yx)$. So

$(A \cap B)(xy) = (\mu_A(xy) \wedge \mu_B(xy), \nu_A(xy) \vee \nu_B(xy))$
$= (\mu_A(yx) \wedge \mu_B(yx), \nu_A(yx) \vee \nu_B(yx))$
$= (A \cap B)(yx).$

Hence $A \cap B \in \text{IFNG}(G_A)$. ■

3. Intuitionistic fuzzy cosets

Result 3.A[13, Proposition 2.6]. Let A be an IFG of a group G. Then the followings are equivalent :

(1) $\mu_A(xyx^{-1}) \geq \mu_A(y)$ and $\nu_A(xyx^{-1}) \leq \nu_A(y)$ for any $x, y \in G$.

(2) $A(xyx^{-1}) = A(y)$ for any $x, y \in G$.

(3) $A \in \text{IFNG}(G)$.

(4) $xA = Ax$ for each $x \in G$.

(5) $xAx^{-1} = A$ for each $x \in G$.

Remark 3.1. We shall restric ourselves in the subsequent discussion, without any loss of generality, with intuitionistic fuzzy right cosets only(corresponding results for intuitionistic fuzzy left cosets could be obtained without any difficulty). Consequently from now on we call an intuitionistic fuzzy right coset an *intuitionistic fuzzy coset* and denote it as Ax for each $x \in G$.

Definition 3.2[13]. Let A be an IFG of a finite group G. Then the cardinality $|G/A|$ of G/A is called an *index* of A, where G/A denotes the set of all intuitionistic fuzzy cosets of A.

Result 3.B[13, Theorem 3.7]. Let A be an IFNG of a group G. We define an operation $*$ on G/A as follows : for any $x, y \in G$,

$$Ax * Ay = Axy.$$

Then $(G/A, *)$ is a group.

Result 3.C[13, Theorem 3.12]. Let A be an IFG of a finite group G. Then the index of A divides the order of G.

It is a well-known result in group theory that subgroup of index 2 is a normal subgroup. We now obtain an analog of a generalization of this result.

Theorem 3.3. Let A be an IFG of a finite group G such that the index of A is p, where p is the smallest prime dividing the order of G. Then $A \in$ IFNG(G).

Proof. By Result 1.B, G_A is a subgroup of G. Then, by Lemma 2.10 and Result 3.C, G_A has index p in G, i.e., G_A has p distinct (right) cosets, say, $\{G_A x_i : 1 \leq i \leq p\}$. Now consider the permutation representation of G on the cosets of G_A given by the map

$$\pi : x \longrightarrow \pi_{x^{-1}},$$

where $\pi_{x^{-1}} : G_A x_i \rightarrow G_A x_i x^{-1}$, $1 \leq i \leq p$. Since the index of G_A in G is p, π is an isomorphism of G into the symmetric group S_p. Futhermore, $Ker\pi = Core(G_A)$, where $Core(G_A)$ denotes the intersection of all the conjugates $g^{-1}G_A g$, $g \in G$. By the fundamental theorem of homomorphism of groups and using Lagrange's theorem, the order of $G/Core(G_A)$ divides $p!$ which is the order of S_p. Since

$$G/Core(G_A) \cong (G/G_A)\ (G_A/Core(G_A))$$

and the order of G/G_A is p, it follows that the order of $G_A/Core(G_A)$ divides $(p-1)!$. Since the order of G_A divides the order of G, $G_A = Core(G_A)$; otherwise we get a contradiction to the fact that p is the smallest prime dividing the order of G. Since $Core(G_A)$ is a normal subgroup of G, G_A is a normal subgroup of G. Now consider the quotient group G/H. Since the order of G/G_A is p, G/G_A is abelian. Let $x, y \in$

G. Then $(G_Ax)(G_Ay) = (G_Ay)(G_Ax)$. Thus $G_Axy = G_Ayx$. So there exists an $h \in G_A$ such that $xy = hyx$. Then

$$\mu_A(xy) = \mu_A(hyx) \geq \mu_A(h) \wedge \mu_A(yx) = \mu_A(e) \wedge \mu_A(yx) = \mu_A(yx)$$

and

$$\nu_A(xy) = \nu_A(hyx) \leq \nu_A(h) \vee \nu_A(yx) = \nu_A(e) \vee \nu_A(yx) = \nu_A(yx).$$

Similarly, we have

$\mu_A(yx) \geq \mu_A(xy)$ and $\nu_A(yx) = \nu_A(xy)$.

So $A(xy) = A(yx)$ for any $x, y \in G$. Hence $A \in \text{IFNG}(G)$. This completes the proof. ■

The following is the immediate result of Theorem 3.3.

Corollary 3.3. Let A be an IFG of a group G such that the index of A is 2, then $A \in \text{IFNG}(G)$.

It is well-known in group theory that of θ is a homomorphism of a group G into itself whose kernal is N, then θ induces a homomorphism from G/N into itself. Now we derive an analog of the following result.

Theorem 3.4. Let A be an IFNG of a group G and let θ be an homomorphism of G into itself such that $\theta(G_A) = G_A$. Then θ induces a homomorphism $\bar{\theta}$ of the intuitionistic fuzzy cosets of A defined as follows :

$\bar{\theta}(Ax) = A\theta(x)$ for each $x \in G$.

Proof. Suppose $x, y \in G$ such that $Ax = Ay$. Then $Ax(x) = Ay(x)$ and $Ax(y) = Ay(y)$. Thus $A(e) = A(xy^{-1}) = A(yx^{-1})$. So $xy^{-1}, yx^{-1} \in G_A$. Since $\theta(G_A) = G_A$, $\theta(xy^{-1})$, $\theta(yx^{-1}) \in G_A$. Then

$A(\theta(xy^{-1})) = A(\theta(yx^{-1})) = A(e)$. (4)

Let $g \in G$. Then

$\mu_{A\theta(x)}(g) = \mu_A(g\theta(x)^{-1})$
$= \mu_A(g\theta(x^{-1}))$ (Since θ is a homomorphism)
$= \mu_A(g\theta(y^{-1})\theta(y)\theta(x^{-1}))$
$\geq \mu_A(g\theta(y^{-1})) \wedge \mu_A(\theta(y)\theta(x^{-1}))$ (Since $A \in \text{IFG}(G)$)
$= \mu_A(g\theta(y^{-1})) \wedge \mu_A(\theta(yx^{-1}))$ (Since θ is a homomorphism)
$= \mu_{A\theta(y)}(g) \wedge \mu_A(e)$ (By (4))
$= \mu_{A\theta(y)}(g)$ (By Result 1.A)

and

$\nu_{A\theta(x)}(g) = \nu_A(g\theta(x)^{-1}) = \nu_A(g\theta(x^{-1})) = \nu_A(g\theta(y^{-1})\theta(y)\theta(x^{-1}))$
$\leq \nu_A(g\theta(y^{-1})) \vee \nu_A(\theta(y)\theta(x^{-1})) = \nu_A(g\theta(y^{-1})) \vee \nu_A(\theta(yx^{-1}))$
$= \nu_{A\theta(y)}(g) \vee \nu_A(e) = \nu_{A\theta(y)}(g)$.

By the similar arguments, we have

$\mu_{A\theta(y)}(g) \geq \mu_{A\theta(x)}(g)$ and $\nu_{A\theta(y)}(g) \leq \nu_{A\theta(x)}(g)$.

Thus $A\theta(x) = A\theta(y)$. So $\bar{\theta}$ is well-defined.

Now let $x, y \in G$. Then

$\bar{\theta}(Ax * Ay) = \bar{\theta}(Axy)$ (By Result 3.B)
$= A\theta(xy)$ (By the definition of $\bar{\theta}$)
$= A\theta(x)\theta(y)$ (Since θ is a homomorphism)
$= A\theta(x) * A\theta(y)$ (By Result 3.B)
$= \bar{\theta}(Ax) * \bar{\theta}(Ay)$. (By the definition of $\bar{\theta}$)

Hence $\bar{\theta}$ is a homomorphism. This completes the proof. ■

Corollary 3.4-1. In the same hypothesis as in Theorem 3.4, if θ is an automorphism and G is finite, then $\bar{\theta}$ is an automorphism.

Proof. Since G has finite order, it is easy to see that θ has finite order. Suppose that θ has order k. Then $\theta^k = id_G$, where id_G denotes the identity mapping. Let $x, y \in G$ such that $\bar{\theta}(Ax) = \bar{\theta}(Ay)$. Then, by the definition of $\bar{\theta}$,

$A\theta(x) = A\theta(y)$.

Since $\theta^k = id_G$, $\theta^k(x) = x$ and $\theta^k(y) = y$. Thus $Ax = A\theta^k(x) = A\theta^k(y) = Ay$.

So $\bar{\theta}$ is injective. Hence $\bar{\theta}$ is an automorphism. ■

Corollary 3.4-2. In the same hypothesis as in Theorem 3.4, if $\bar{\theta}$ is an automorphism and $G_A = (e)$, then θ is an automorphism.

Proof. Let $x, y \in G$ such that $\theta(x) = \theta(y)$. Then $A\theta(x) = A\theta(y)$, i.e., $\bar{\theta}(Ax) = \bar{\theta}(Ay)$. Since $\bar{\theta}$ is injective, $Ax = Ay$. Then $Ax(y) = Ay(y)$. Thus $A(yx^{-1}) = A(e)$. So $yx^{-1} \in G_A$. Since $G_A = (e)$, $yx^{-1} = e$. Thus $x = y$. So θ is injective. Hence θ is an automorphism. ■

The motivation of the following result stems from the standard theorem in group theory that if θ is an automorphism of G and N is a normal subgroup of G such that $N^\theta \subset N$, then θ induces an automorphism of the quotient group G/N into itself.

Remark 3.5. In Theorem 3.4, we have assumed A to be intuitionistic fuzzy normal instead of assuming only that A is an intuitionistic fuzzy subgroup. This has been done to ensure that the law of composition of intuitionistic fuzzy cosets is well-defined, and this fact is used in the proof of Theorem 3.4 to show that $\bar{\theta}$ is a homomorphism(refer to Result 3.B). However, it is clear from the proof that to show $\bar{\theta}$ is well-defined it is not necessary to assume A to be intuitionistic fuzzy normal.

Theorem 3.6. Let A be an IFNG of a group G and let θ be an automorphism of G such that $A^\theta = A$(recall the definition of A^θ given by Definition 2.1). Then θ induces an automorphism $\bar{\theta}$ of G/A defined as follows : for each $x \in G$,

$$\bar{\theta}(Ax) = A\theta(x).$$

Proof. Let $x, y \in G$ such that $Ax = Ay$. We show that $\bar{\theta}(Ax) = \bar{\theta}(Ay)$, i.e., $A\theta(x)(g) = A\theta(y)(g)$ for each $g \in G$. Let $g \in G$. Since θ is an automorphism of G, there exists a $g^* \in G$ such that $\theta(g^*) = g$. Since $Ax = Ay$, $Ax(g^*) = Ay(g^*)$, i.e., $A(g^*x^{-1}) = A(g^*y^{-1})$. Since $A^\theta = A$, $A^\theta(g^*x^{-1}) = A^\theta(g^*y^{-1})$. By Definition 2.1, $A(\theta(g^*x^{-1})) = A(\theta(g^*y^{-1}))$. Since θ is an automorphism of G, $A(\theta(g^*)\theta(x^{-1})) = A(\theta(g^*)\theta(y^{-1}))$. Thus $A(g\theta(x^{-1})) = A(g\theta(y^{-1}))$, i.e., $A\theta(x)(g) = A\theta(y)(g)$. So $\bar{\theta}(Ax) = \bar{\theta}(Ay)$. Hence $\bar{\theta}$ is well-defined. The proof of the fact that $\bar{\theta}$ is a homomorphism is analogous to the corresponding part of the proof of Theorem 3.4, and thus we omit the details. Now suppose $Ax \in Ker\bar{\theta}$ for each $x \in G$. Then $\bar{\theta}(Ax) = A\theta(x) = Ae$. Let $g \in G$. Then $A\theta(x)(\theta(g)) = Ae\theta(g)$, i.e., $A(\theta(g)\theta(x^{-1})) = A\theta(g)$. Thus $A\theta(gx^{-1}) = A\theta(g)$, i.e., $A^\theta(gx^{-1}) = A^\theta(g)$. Since $A^\theta = A$, $A(gx^{-1}) = A(g)$. Then $Ax(g) = Ae(g)$. Thus $Ax = Ae$, i.e., $Ker\bar{\theta} = \{Ae\}$. So $\bar{\theta}$ is injective. Hence $\bar{\theta}$ is an automorphism of G/A. This completes the proof. ■

Theorem 3.7. Let A be an IFG of a finite group G and let $x, y \in G$. Then $G_{A,x} = G_{A,y}$ if and only if $Ax = Ay$.

Proof. By Result 1.B and Lemma 2.10, G_A is a subgroup of G and $G_A = \{x \in G : Ax = Ae\}$.

($\Rightarrow$): Suppose $G_{A,x} = G_{A,y}$ for any $x, y \in G$. Then $xy^{-1} \in G_A$. Thus $Axy^{-1} = Ae$. Let $g \in G$. Then $Axy^{-1}(g) = Ae(g)$, i.e., $A(gyx^{-1}) = A(g)$. Repacing g by gy^{-1}, which is also an arbitrary element of G, we get that

$$A(gx^{-1}) = A(gy^{-1}) \text{ for each } y \in G.$$

Thus $Ax(g) = Ay(g)$ for each $y \in G$. So $Ax = Ay$.

($\Leftarrow$): Suppose $Ax = Ay$ for any $x, y \in G$ and let $g \in G$. Then

$$Ax(g) = Ay(g), \text{ i.e., } A(gx^{-1}) = A(gy^{-1}).$$

In particular, $A(yx^{-1}) = A(yy^{-1}) = A(e)$. Thus $yx^{-1} \in G_A$. So $G_{A,x} = G_{A,y}$. This completes the proof. ■

Remark 3.8. Theorem 3.6 shows that there is a one-to-one correspondence between the (right) cosets of G_A in G and the intuitionistic fuzzy cosets of A, given by the mapping

$x \leftrightarrow Ax$ for each $x \in G$.

Hence we see that the subgroup G_A plays a key role in the analysis of intuitionistic fuzzy cosets.

4. Acknowledgements

The authors are highly grateful to referees for their valuable comments and suggestions for improving the paper.

References

[1] K.Atanassov, *Intuitionistic fuzzy sets,* Fuzzy Sets and Systems 20(1986), 87-96.

[2] Baldev Banerjee and Dhiren Kr. Basnet, *Intuitionistic fuzzy subrings and ideals,* J.Fuzzy Mathematics 11(1)(2003), 139-155.

[3] R.Biswas, *Intuitionistic fuzzy subgroups,* Mathematical Forum x(1989), 37-46.

[4] D. Çoker, *An introduction to intuitionistic fuzzy topological spaces,* Fuzzy Sets and Systems 88(1997), 81-89.

[5] D. Çoker and A.Haydar Es, *On fuzzy compactness in intuitionistic fuzzy topological spaces,* J. Fuzzy Math. 3(1995), 899-909.

[6] P.S.Das, *Fuzzy groups and level subgroups,* J. Math. Anal. Appl. 84(1981), 264-269.

[7] H.Gürçay, D. Çoker and A.Haydar Es, *On fuzzy continuity in intuitionistic fuzzy topological spaces,* J. Fuzzy Math. 5(1997), 365-378.

[8] K.Hur, S.Y.Jang and H.W.Kang, *Intuitionistic fuzzy subgroupoids,* International Journal of Fuzzy Logic and Intelligent Systems 3 (1) (2003), 72-77.

[9] K.Hur, H.W.Kang and H.K.Song, *Intuitionistic fuzzy subgroups and subrings,* Honam Mathematical J. 25 (1) (2003), 19-41.

[10] K.Hur, S.Y.Jang and H.W.Kang, *Intuitionistic fuzzy subgroups and cosets,* Honam Mathematical J. 26(1) (2004), 17-41.

[11] K.Hur, Y.B.Jun and J.H.Ryou, *Intuitionistic fuzzy topological groups,* Honam Math.J.26(2)(2004),163-192.

[12] K.Hur, J.H.Kim and J.H.Ryou, *Intuitionistic fuzzy topological spaces,* J.Korea Soc. Math.Edu.Ser.B : Pure Appl. Math. 11(3)(2004),243-265.

[13] K.Hur, S.Y.Jang and H.W.Kang, *Intuitionistic fuzzy normal subgroups and intuitionistic fuzzy cosets,* Honam Math.J.26(4)(2004),559-587.

[14] S.J.Lee and E.P.Lee, *The category of intuitionistic fuzzy topological spaces,* Bull. Korean Math. Soc. 37(1)(2000), 63-76.

[15] Wang-jin Liu, *Fuzzy invariant subgroups and fuzzy ideals,* Fuzzy Sets and Systems 8(1982), 133-139.

[16] N.P.Mukherjee and P.Bhattacharya, *Fuzzy normal subgroups and fuzzy cosets,* Inform. Sci. 34(1984), 225-239.

[17] ________________________________, *Fuzzy groups : Some group-theoretic analogs,* Inform.Sci. 39(1986), 249-268.

[18] A.Rosenfeld, *Fuzzy groups,* J. Math. Anal. Appl. 35(1971), 512-517.

[19] L.A.Zadeh, *Fuzzy sets,* Inform. and Control 8(1965), 338-353.

Kul Hur
Division of Mathematics and Informational Statistics
Wonkwang University,
Iksan, Chonbuk, Korea 570-749
Email : kulhur@wonkwang.ac.kr

Hee Won Kang
Dept. of Mathematics Education
Woosuk University
Hujong-Ri Samrae-Eup, Wanju-kun Chonbuk, Korea 565-701
Email : khwon@woosuk.ac.kr

Dae Sig Kim
Dept. of Digital Contents
Dongshin University
252 Daehodong Naju Chonnam, Republic of Korea
Email : dskim@dsu.ac.kr