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On the Rate of Convergence of Modified Baskakov Type Operators on Functions of Bounded Variation

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Abstract. The aim of this paper is to establish the rate of convergence of Baskakov-Durrmeyer operators for bounded variation function. We have given the better estimate over the results due to Guo ([4]), Anial and Teberska ([1]) and Gupta and Srivastava ([8]).

1. Introduction

We first recall the construction of Baskakov-Durrmeyer operators. The Baskakov-Durrmeyer operators L_n is the linear positive operator defined on $[0,\infty)$ by

$$L_n(f,x)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)dt,\quad x\in[0,\infty)$$

where

$$p_{n,k}(x)=\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}},$$

see e.g. ([9]). In [3] Bojnic estimated the rate of convergence of Fourier series of functions of bounded variation. Aniol and Taberska ([1]) and Guo ([4]) obtained analogous results for Durrmeyer type operators. A lot of work has been done in this direction by Vijay Gupta and collaborators (see e.g. [5], [6], [7], [8]). We remark here that the rate of convergence for the modified Baskakov operators was obtained by Gupta and Srivastava ([8]), but there is some misprint in the estimate of R_{33} in their main result, also very recently Bastien and Ragalski ([2]) gave the optimum bound for Baskakov basis function. This along with the improvement in the result of Gupta and Srivastava ([8]) motivated us to study further on such type of operator. In the present paper, we study the rate of convergence of Baskakov-Durrmeyer

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operators for function of bounded variation.

2. Auxiliary results

We need some auxiliary results for proving the main theorem.

Lemma 2.1 ([2]). *For all $x \in (0,\infty)$ and $n,k \in \boldsymbol{N}$ there holds*

$$p_{n,k}(x) \leq \frac{C}{\sqrt{nx(1+x)}}$$

where

$$C = \begin{cases} 1, & \text{if } n = 1 \\ \dfrac{2\sqrt{2}}{3\sqrt{3}}, & \text{if } n \geq 2,\ k = 0 \\ \left(\dfrac{3}{2}\right)^{3/2} \dfrac{n^{3/2}(n-1)^{n-1}}{(n+\frac{1}{2})^{n+1/2}}, & \text{if } n \geq 2,\ k \geq 1. \end{cases}$$

Lemma 2.2 ([9]). *Let the m^{th} order moment for the operator $L_n(f,x)$ be defined by*

$$T_{n,m}(x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty}(t-x)^m p_{n,k}(t)dt$$

then

$$\begin{aligned} T_{n,1}(x) &= \frac{1+2x}{n-2}, \quad n > 2 \\ T_{n,2}(x) &= \frac{2(n-1)x(1+x)+2(1+2x)^2}{(n-2)(n-3)}, \quad n > 3. \end{aligned}$$

If particular, given any $\lambda > 2$ and any $x > 0$ there is an integer $\boldsymbol{N}(\lambda,x) > 2$ such that

$$T_{n,m}(x) \leq \frac{\lambda x(1+x)}{n} \text{ for all } n \geq \boldsymbol{N}(\lambda,x).$$

Next let

$$K_n(x,t) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,t}(t),\ \lambda > 2 \text{ and } n > \boldsymbol{N}(\lambda,x),$$

then

(i) *For $0 \leq y < x$, we get*

$$\int_0^y K_n(x,t)dt \leq \frac{\lambda x(1+x)}{n(x-y)^2}. \tag{2.1}$$

(ii) *For $x < z < \infty$, we get*

$$\int_z^\infty K_n(x,t)dt \le \frac{\lambda x(1+x)}{n(z-x)^2}. \tag{2.2}$$

The proof of (2.1) *and* (2.2) *are simple and are left for the readers.*

Lemma 2.3. *For every $k \in \boldsymbol{N}$, $x \in (0,\infty)$, we have*

$$\left| \sum_{j=0}^{k} p_{n,j}(x) - \sum_{j=0}^{k} p_{n-1,j}(x) \right| \le \frac{C}{2\sqrt{nx(1+x)}}.$$

The proof of the above lemma is simple just we have to apply Lemma 2.1.

3. Main results

In this section, we shall give our main results.

Theorem 3.1. *Let f be a function of bounded variation an every finite subinterval of $[0,\infty)$ and let*

$$g_x(t) = \begin{cases} f(t) - f(x+), & \text{if } x < t < \infty \\ 0, & \text{if } t = x \\ f(t) - f(x-), & \text{if } 0 \le t < x. \end{cases}$$

$V_a^b(g_x)$ be the total variation of g_x on $[a,b]$. If $\left|f(t)\right| < M(1+t)^\alpha$ for $t \in [0,\infty)$, where $M > 0$, $\alpha \in \boldsymbol{N}_0$ and choose a number $\lambda > 2$. Then for $n > max\{1+\alpha, \boldsymbol{N}(\lambda,x)\}$, we get

$$\begin{aligned} &\left| L_n(f,x) - \frac{1}{2}\{f(x+)+f(x-)\} \right| \\ \le\ &\left| \{f(x+)-f(x-)\} \right| \frac{C}{\sqrt{nx(1+x)}} + \frac{3\lambda+(3\lambda+1)x}{nx} \sum_{k=1}^{n} V_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) \\ &+ \frac{\lambda M K_\alpha (1+x)^{\alpha+1}}{nx}, \end{aligned} \tag{3.1}$$

where C is a constant defined in Lemma 2.1.

Proof. First, we have

$$\begin{aligned} &\left| L_n(f,x) - \frac{1}{2}\{f(x+)+f(x-)\} \right| \\ \le\ &\left| L_n(g_x,x) \right| + \frac{1}{2}\left| \{f(x+)-f(x-)\} \right| \left| L_n\left(sign(t-x),x\right) \right|. \end{aligned} \tag{3.2}$$

Thus to estimate (3.1), we need the estimates for $L_n(g_x,x)$ and $L_n\left(sign(t-x),x\right)$. Now using Lemma 2.1, Lemma 2.3 and using the similar methods as given in [8], we have

$$\left| L_n\left(sign(t-x),x\right)\right| \le \frac{2C}{\sqrt{nx(1+x)}}. \tag{3.3}$$

Now to estimate $L_n(g_x,x)$,

$$\begin{aligned} L_n(g_x,x) &= \int_0^\infty K_n(x,t)g_x(t)dt \\ &= \Big(\int_{I_1}+\int_{I_2}+\int_{I_3}\Big)K_n(x,t)g_x(t)dt \\ &= R_1+R_2+R_3, \ \text{ say.} \end{aligned}$$

Where $I_1=[0,x-x/\sqrt{n}]$, $I_2=[x-x/\sqrt{n},x+x/\sqrt{n}]$ and $I_3=[x+x/\sqrt{n},\infty)$. Suppose $\lambda_n(x,t)=\int_0^t K_n(x,u)du$. First, we estimate R_1. Writing $y=x-x/\sqrt{n}$ and using partial integration, we get

$$\begin{aligned} R_1 &= \int_0^y g_x(t)K_n(x,t)dt \\ &= \int_0^y g_x(t)d_t\big(\lambda_n(x,t)\big) \\ &= g_x(y+)\lambda_n(x,y)-\int_0^y \lambda_n(x,t)d_t\big(g_x(t)\big). \end{aligned}$$

Since

$$\left| g_x(y+)\right| = \left| g_x(y+)-g_x(x)\right| \le V_{y+}^x(g_x),$$

then by (2.1), we get

$$\begin{aligned} \left| R_1\right| &\le V_{y+}^x(g_x)\lambda_n(x,y)+\int_0^y \lambda_n(x,t)d_t(-V_t^x(g_x)) \\ &\le V_{y+}^x(g_x)\frac{\lambda x(1+x)}{n(x-y)^2}+\frac{\lambda x(1+x)}{n}\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)). \end{aligned}$$

Integrating by parts, we have

$$\int_0^y \frac{1}{(x-t)^2}d_t(-V_t^x(g_x)) = \frac{-V_{y+}^x(g_x)}{(x-y)^2}+\frac{V_0^x(g_x)}{x^2}+2\int_0^y \frac{\big(\widehat{V}_t^x(g_x)\big)}{(x-t)^3}dt,$$

where $\widehat{V}_t^x(g_x)$ is the normalized from of $V_t^x(g_x)$ and $\widehat{V}_t^x(g_x)=V_t^x(g_x)$. Consequently, we get

$$\begin{aligned} &\left| R_1\right| \\ \le\ & V_{y+}^x(g_x)\frac{\lambda x(1+x)}{n(x-y)^2}+\frac{\lambda x(1+x)}{n}\left[\frac{-V_{y+}^x(g_x)}{(x-y)^2}+\frac{V_0^x(g_x)}{x^2}+2\int_0^y \frac{\big(V_t^x(g_x)\big)}{(x-t)^3}dt\right] \\ =\ & \frac{\lambda x(1+x)}{n}\left[\frac{V_0^x(g_x)}{x^2}+2\int_0^y \frac{\big(V_t^x(g_x)\big)}{(x-t)^3}dt\right]. \end{aligned}$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\int_0^{x-x/\sqrt{n}} \frac{V_t^x(g_x)}{(x-t)^3}dt = \frac{1}{2x^2}\int_1^n V_{x-x/\sqrt{n}}^x(g_x)dt \le \frac{1}{2x^2}\sum_{k=1}^n V_{x-x/\sqrt{n}}^x(g_x).$$

Hence

$$\left| R_1 \right| \le \frac{2\lambda(1+x)}{nx}\sum_{k=1}^n V_{x-x/\sqrt{n}}^x(g_x). \tag{3.4}$$

Since $\int_a^b d_t\lambda_n(x,t) \le 1,\ for\ (a,b)\subset[0,\infty)$, therefore

$$\left| R_2 \right| \le \frac{1}{n}\sum_{k=0}^n V_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x). \tag{3.5}$$

Finally, we estimate R_3, writing $z = x - x/\sqrt{n}$, we have

$$R_3 = \int_x^\infty g_x(t)K_n(x,t)dt = \int_z^\infty g_x(t)d_t\big(\lambda_n(x,t)\big).$$

We define $Q_n(x,t)$ on $[0,2x]$ as

$$Q_n(x,t) = \begin{cases} 1-\lambda_n(x,t), & \text{if } 0 \le t \le 2x \\ 0, & \text{if } t = 2x. \end{cases}$$

Therefore

$$\begin{aligned} R_3 &= -\int_2^{2x} g_x(t)d_t\big(Q_n(x,t)\big) \\ &\quad - g_x(2x)\int_{2x}^\infty K_n(x,t)dt + \int_{2x}^\infty g_x(t)d_t\big(\lambda_n(x,t)\big) \\ &= R_{31} + R_{32} + R_{33}, \quad \text{say.} \end{aligned} \tag{3.6}$$

Using (2.2) and integrating partially the first term, we get

$$\begin{aligned} \left| R_{31} \right| &\le V_x^{z-}(g_x)\frac{\lambda x(1+x)}{n(z-x)^2} + \frac{\lambda x(1+x)}{n}\int_z^{2x-}\frac{1}{(x-t)^2}d_t\big(V_x^t(g_x)\big) \\ &\quad + \frac{1}{2}V_x^{2x-}(g_x)\int_{2x}^\infty K_n(x,u)du \\ &\le V_x^{z-}(g_x)\frac{\lambda x(1+x)}{n(z-x)^2} + \frac{\lambda x(1+x)}{n}\int_z^{2x-}\frac{1}{(x-t)^2}d_t\big(V_x^t(g_x)\big) \\ &\quad + \frac{1}{2}V_x^{2x-}(g_x)\frac{\lambda x(1+x)}{nx^2} \\ &\le V_x^{z-}(g_x)\frac{\lambda x(1+x)}{n(z-x)^2} + \frac{\lambda x(1+x)}{n} \\ &\quad \Big[\frac{V_x^{2x}(g_x)}{x^2} - \frac{V_x^{z-}(g_x)}{(z-x)^2} + 2\int_z^{2x}\frac{V_x^t(g_x)}{(x-t)^3}dt\Big]. \end{aligned}$$

Thus, by replacing the variable in the last integral by $x+x/\sqrt{n}$, we get

$$(3.7)\qquad \left|R_{31}\right| \leq \frac{\lambda x(1+x)}{nx^2}\left[V_x^{2x}(g_x)+\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x)\right] \leq \frac{2\lambda(1+x)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x).$$

From (2.1), we get

$$(3.8)\qquad \left|R_{32}\right| \leq g_x(2x)\frac{\lambda x(1+x)}{nx^2}+\frac{\lambda x(1+x)}{nx}\sum_{k=1}^{n} V_x^{x+x/\sqrt{k}}(g_x).$$

Finally for $n>\alpha$, we obtain

$$\left|R_{33}\right| \leq M(n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_{2x}^{\infty}\left[(1+t)^{\alpha}+(1+x)^{\alpha}\right]p_{n,k}(t)dt,$$

(i) If $\alpha=0$, then applying (2.1), we obtain

$$\left|R_{33}\right| \leq 2M(n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_{2x}^{\infty} p_{n,k}(t)dt \leq \frac{2\lambda M(1+x)}{nx},\ for\ n>\boldsymbol{N}(\lambda,x).$$

(ii) If $\alpha=1$, then by (2.1) and Lemma 2.2, for $n>\mathbf{N}(\lambda,x)$, we get

$$\left|R_{33}\right| \leq M(n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_{2x}^{\infty}(2+2x+t-x)\,p_{n,k}(t)dt \leq \frac{\lambda M(1+x)(2+3x)}{nx} \leq \frac{3\lambda M(1+x)^2}{nx}.$$

(iii) If $\alpha=2$, then for $n>\alpha$

$$\left|R_{33}\right| \leq M(n-1)(1+2^{\alpha-1})(1+x)^{\alpha}\sum_{k=0}^{\infty} p_{n,k}(x)\int_{2x}^{\infty} p_{n,k}(t)dt + M2^{\alpha-1}(n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_{2x}^{\infty} p_{n,k}(t)(t-x)^{\alpha}dt.$$

Consequently in case $\alpha=2$

$$\left|R_{33}\right| \leq \frac{3\lambda M(1+x)^3}{nx}+\frac{2\lambda Mx(1+x)}{n} \leq \frac{5\lambda M(1+x)}{nx},\ \text{for } n\geq \boldsymbol{N}(\lambda,x).$$

In general case, if α is even or odd then by Lemma 2.2, we may easily verify that there exist a constant K_α depending only on α, such that

$$\left| R_{33} \right| \leq \frac{\lambda M K_\alpha (1+x)^{\alpha+1}}{nx}, \text{ for all } n \geq max\{(1+\alpha), \boldsymbol{N}(\lambda, x)\}. \tag{3.9}$$

Collecting the estimates of (3.2) to (3.9), we get the required result. □

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