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EXPANSIONS OF FILTERS IN R_0-ALGEBRAS

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Abstract. The notion of expansions of filters in R_0-algebras is introduced. Also the notion of σ-primary filters in R_0-algebras is discussed.

1. Introduction

In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Xu [11] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [12] introduced the notion of implicative filters in a lattice implication algebra, and investigated some of their properties. Turunen [8] introduced the notion of Boolean deductive system, or equivalently, Boolean filter in BL-algebras which rise as Lindenbaum algebras from many valued logic introduced by Hájek [3]. Boolean filters are important because the quotient algebras induced by Boolean filters are Boolean algebras, and a BL-algebra is bipartite if and only if it has proper Boolean filter. In [9], Wang introduced the notion of R_0-algebras in order to provide an algebraic proof of the completeness theorem of a formal deductive system. We note that R_0-algebras are different from BL-algebras because the identity $x \wedge y = x \odot (x \rightarrow y)$

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holds in BL-algebras, but does not hold in R_0-algebras. R_0-algebras are also different from lattice implication algebras because the identity $(x \to y) \to y = (y \to x) \to x$ holds in lattice implication algebras, but does not hold in R_0-algebras. Although they are different in essence, they have some similarities, that is, they all have implication operator $\to$. Therefore it is meaningful to generalize some aspects of lattice implication algebras and BL-algebras to R_0-algebras. In [2], Esteva and Godo introduced the MTL-algebra , The MTL-algebra is an extension of a BL-algebra, which is obtained by eliminating the condition $x \wedge y = x \odot (x \to y)$ in BL-algebra. In fact, MTL-algebra is an algebra induced by a left continuous t-norm and its corresponding residuum, but BL-algebra is an algebra induced by a continuous t-norm and its corresponding residuum. It is proved that an R_0-algebra is a particular type of MTL-algebra and its t-norm $\odot$ is a nilpotent minimum t-norm [2], which is obtained by taking negation operator as $1 \to x$. Hence the theory of R_0-algebras becomes one of the guides to the development of the theory of MTL-algebras. Lianzhen and Kaitai [5] extended the notions of implicative filters and Boolean filters to R_0-algebras, and considered the fuzzification of such notions and gave characterizations of fuzzy implicative filters. They also proved that fuzzy implicative filters and fuzzy Boolean filters coincide in R_0-algebras.

In this paper, we introduce the notion of expansions of filters in R_0-algebras, and discuss the notion of σ-primary filters in R_0-algebras.

2. Preliminaries

Definition 2.1. [9] Let L be a bounded distributive lattice with order-reversing involution $\neg$ and a binary operation $\to$. Then $(L, \wedge, \vee, \neg, \to)$ is called a R_0-*algebra* if it satisfies the following axioms:

(R1) $x \to y = \neg y \to \neg x$,

(R2) $1 \to x = x$,

(R3) $(y \to z) \wedge ((x \to y) \to (x \to z)) = y \to z$,

(R4) $x \to (y \to z) = y \to (x \to z)$,

(R5) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,

(R6) $(x \to y) \vee ((x \to y) \to (\neg x \vee y)) = 1$.

Let L be a R_0-algebra. For any $x, y \in L$, we define $x \odot y = \neg(x \to \neg y)$ and $x \oplus y = \neg x \to y$. It is proved that $\odot$ and $\oplus$ are commutative, associative and $x \oplus y = \neg(\neg x \odot \neg y)$, and $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a residuated lattice.

Example 2.2. [5] Let $L = [0,1]$. For any $x, y \in L$, we define $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$, $\neg x = 1 - x$, and

$$x \to y := \begin{cases} 1 & \text{if } x \leq y, \\ \neg x \vee y & \text{if } x > y. \end{cases}$$

Then $(L, \wedge, \vee, \neg, \to)$ is an R_0-algebra which is neither a BL-algebra nor a lattice implication algebra.

A R_0-algebra have the following useful properties.

Proposition 2.3. [7] *For any elements x, y and z of an R_0-algebra L, we have the following properties.*

(a1) $x \leq y$ *if and only if* $x \to y = 1$,

(a2) $x \leq y \to x$,

(a3) $\neg x = x \to 0$,

(a4) $(x \to y) \vee (y \to x) = 1$,

(a5) $x \leq y$ *implies* $y \to z \leq x \to z$,

(a6) $x \leq y$ *implies* $z \to x \leq z \to y$,

(a7) $((x \to y) \to y) \to y = x \to y$,

(a8) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,

(a9) $x \odot \neg x = 0$ *and* $x \oplus \neg x = 1$,

(a10) $x \odot y \leq x \wedge y$ *and* $x \odot (x \to y) \leq x \wedge y$,

(a11) $(x \odot y) \to z = x \to (y \to z)$,

(a12) $x \leq y \to (x \odot y)$,

(a13) $x \odot y \leq z$ *if and only if* $x \leq y \to z$,

(a14) $x \leq y$ *implies* $x \odot z \leq y \odot z$,

(a15) $x \to y \leq (y \to z) \to (x \to z)$,

(a16) $(x \to y) \odot (y \to z) \leq x \to z$.

3. Expansions of filters

In what follows let L denote an R_0-algebra unless otherwise specified.

Definition 3.1. [7] A nonempty subset F of L is called a *filter* of L if it satisfies

(i) $1 \in F$,

(ii) $(\forall x \in F)\ (\forall y \in L)\ (x \to y \in F \ \Rightarrow\ y \in F)$.

Definition 3.2. [7] A filter F of L is said to be *prime* if it satisfies

$$(\forall a, b \in L)\,(a \to b \in F \text{ or } b \to a \in F).$$

Proposition 3.3. [7] *A filter F of L is prime if and only if it satisfies:*

$$(\forall a, b \in L)\,(a \vee b \in F \implies a \in F \text{ or } b \in F).$$

Let F be a nonempty subset of L. Then the least filter containing F is called the *filter generated* by F, and denoted by $\langle F \rangle$.

The next statement gives a description of elements of $\langle F \rangle$.

Theorem 3.4. [4] *If F is a nonempty subset of L, then*

$$\langle F \rangle = \left\{ x \in L \;\middle|\; \begin{array}{l} a_1 \to (a_2 \to (\cdots \to (a_n \to x) \cdots)) = 1 \\ \text{for some } a_1, a_2, \cdots, a_n \in F \end{array} \right\}.$$

For any $n \in \mathbb{N}$, we define $n(x) \to y$ recursively as follows: $0(x) \to y = y$, $1(x) \to y = x \to y$ and $(n+1)(x) \to y = x \to (n(x) \to y)$ for all $x, y \in L$. Using (R4), we know that $y \to (n(x) \to y) = 1$, that is, $y \leq n(x) \to y$ for all $x, y \in L$.

Proposition 3.5. *Let F be a filter of L and $x \in L$. Then*

$$\langle F \cup \{x\} \rangle = \{y \in L \mid n(x) \to y \in F \text{ for some } n \in \mathbb{N}\}. \tag{1}$$

Proof. Denote by Ω_x the right hand side of (1). If $y \in \langle F \cup \{x\} \rangle$, then there exist $a_1, a_2, \cdots, a_n \in F$ and $m \in \mathbb{N}$ such that

$$m(x) \to (a_1 \to (a_2 \to (\cdots \to (a_n \to y) \cdots))) = 1. \tag{2}$$

Using (R4) repeatedly, (2) implies that

$$a_1 \to (a_2 \to (\cdots \to (a_n \to (m(x) \to y)) \cdots)) = 1 \in F. \tag{3}$$

It follows from Definition 3.1(ii) that $m(x) \to y \in F$ so that $y \in \Omega_x$. Thus $\langle F \cup \{x\} \rangle \subseteq \Omega_x$. Conversely let $y \in \Omega_x$. Then $n(x) \to y \in F$ for some $n \in \mathbb{N}$. Since $F \subseteq \langle F \cup \{x\} \rangle$, it follows that

$$x \to ((n-1)(x) \to y) = n(x) \to y \in \langle F \cup \{x\} \rangle.$$

Since $x \in \langle F \cup \{x\} \rangle$, we have $(n-1)(x) \to y \in \langle F \cup \{x\} \rangle$ by Definition 3.1(ii). Repeating this process we get $y = 0(x) \to y \in \langle F \cup \{x\} \rangle$. Hence $\Omega_x \subseteq \langle F \cup \{x\} \rangle$. This completes the proof. □

Definition 3.6. Let $\mathfrak{F}(L)$ be the set of filters in L. By an *expansion of filters* in L we shall mean a function $\sigma : \mathfrak{F}(L) \to \mathfrak{F}(L)$ such that

(o1) $(\forall G \in \mathfrak{F}(L))\,(G \subseteq \sigma(G))$.
(o2) $(\forall G, H \in \mathfrak{F}(L))\,(G \subseteq H \Rightarrow \sigma(G) \subseteq \sigma(H))$.

Example 3.7. (1) The function $\sigma_0 : \mathfrak{F}(L) \to \mathfrak{F}(L)$ defined by $\sigma_0(G) = G$ for all $G \in \mathfrak{F}(L)$ is an expansion of filters in L.

(2) The function ν that assigns the largest filter L to each filter of L is an expansion of filters in L.

(3) For each filter F of L, let

$$\mathfrak{M}(F) = \cap\{M \mid F \subseteq M,\ M \text{ is a maximal filter of } L\}.$$

Then $\mathfrak{M}$ is an expansion of filters in L.

(4) Let $F, G \in \mathfrak{F}(L)$ be such that $F \subseteq G$. Then $\langle F \cup \{x\} \rangle \subseteq \langle G \cup \{x\} \rangle$. Hence the function $\sigma_x : \mathfrak{F}(L) \to \mathfrak{F}(L)$ given by $\sigma_x(F) = \langle F \cup \{x\} \rangle$ for all $F \in \mathfrak{F}(L)$ and $x \in L$ is an expansion of filters in L.

Definition 3.8. Let σ be an expansion of filters in L. Then a filter G of L is said to be *σ-primary* if it satisfies:

$$(\forall a, b \in L)\,(a \vee b \in G,\ a \notin G \Rightarrow\ b \in \sigma(G)).$$

Note that a filter G of L is σ_0-primary if and only if it is a prime filter of L, where σ_0 is the function in Example 3.7(1).

Theorem 3.9. *If σ and δ are expansions of filters in L such that $\sigma(G) \subseteq \delta(G)$ for every $G \in \mathfrak{F}(L)$, then every σ-primary filter is also δ-primary.*

Proof. Let F be a σ-primary filter of L and let $a, b \in L$ be such that $a \vee b \in F$ and $a \notin F$. Then $b \in \sigma(F) \subseteq \delta(F)$ by assumption. Hence F is a δ-primary filter of L. □

Corollary 3.10. *Let σ be an expansion of filters in L. Then every prime filter of L is σ-primary.*

Proof. Let G be a prime filter of L. Then G is σ_0-primary, and $\sigma_0(G) = G \subseteq \sigma(G)$. It follows from Theorem 3.9 that G is a σ-primary filter of L. □

Theorem 3.11. *Let α and β be expansions of filters in L. Let $\sigma : \mathfrak{F}(L) \to \mathfrak{F}(L)$ be a function defined by $\sigma(G) = \alpha(G) \cap \beta(G)$ for all $G \in \mathfrak{F}(L)$. Then σ is an expansion of filters in L.*

Proof. For every $G \in \mathfrak{F}(L)$, we have $G \subseteq \alpha(G)$ and $G \subseteq \beta(G)$ by (o1), and so $G \subseteq \alpha(G) \cap \beta(G) = \sigma(G)$. Let $G, H \in \mathfrak{F}(L)$ be such that $G \subseteq H$. Then $\alpha(G) \subseteq \alpha(H)$ and $\beta(G) \subseteq \beta(H)$ by (o2), which imply that

$$\sigma(G) = \alpha(G) \cap \beta(G) \subseteq \alpha(H) \cap \beta(H) = \sigma(H).$$

Therefore σ is an expansion of filters in L. □

Generally, the intersection of expansions of filters is an expansion of filters.

Theorem 3.12. *Let σ be an expansion of filters in L. If $\{G_i \mid i \in D\}$ is a directed collection of σ-primary filters of L where D is an index set, then $G := \bigcup_{i\in D} G_i$ is a σ-primary filter of L.*

Proof. Clearly $G := \bigcup_{i\in D} G_i$ is a filter of L. Let $a, b \in L$ be such that $a \vee b \in G$ and $a \notin G$. Then there exists a σ-primary filter G_i such that $a \vee b \in G_i$ and $a \notin G_i$. Since G_i is σ-primary and $G_i \subseteq G$, it follows that $b \in \sigma(G_i) \subseteq \sigma(G)$ so that G is σ-primary. □

Theorem 3.13. *Let σ be an expansion of filters in L. If P is a σ-primary filter of L, then*

$$(\forall F, G \in \mathfrak{F}(L))\,(F \vee G \subseteq P,\ F \nsubseteq P \Rightarrow G \subseteq \sigma(P)),$$

where $F \vee G = \{x \vee y \mid x \in F,\ y \in G\}$.

Proof. Assume that P is a σ-primary filter of L and let $F, G \in \mathfrak{F}(L)$ be such that $F \vee G \subseteq P$ and $F \nsubseteq P$. Suppose that $G \nsubseteq \sigma(P)$. Then there exist $a \in F \setminus P$ and $b \in G \setminus \sigma(P)$, which imply that $a \vee b \in F \vee G \subseteq P$. But $a \notin P$ and $b \notin \sigma(P)$. This contradicts the assumption that P is σ-primary. Consequently, the result is valid. □

Theorem 3.14. *If σ is an expansion of filters in L, then the function $E_\sigma : \mathfrak{F}(L) \to \mathfrak{F}(L)$ defined by*

$$E_\sigma(G) := \cap\{H \in \mathfrak{F}(L) \mid G \subseteq H, \text{ and } H \text{ is } \sigma\text{-primary}\}$$

for all $G \in \mathfrak{F}(L)$ is an expansion of filters in L.

Proof. Clearly, $G \subseteq E_\sigma(G)$ for all $G \in \mathfrak{F}(L)$. Let $F, G \in \mathfrak{F}(L)$ be such that $F \subseteq G$. Then

$$\begin{aligned} E_\sigma(F) &= \cap\{H \in \mathfrak{F}(L) \mid F \subseteq H \text{ and } H \text{ is } \sigma\text{-primary}\} \\ &\subseteq \cap\{H \in \mathfrak{F}(L) \mid G \subseteq H \text{ and } H \text{ is } \sigma\text{-primary}\} \\ &= E_\sigma(G). \end{aligned}$$

Hence E_σ is an expansion of filters in L. □

Example 3.15. Let σ_0 be an expansion of filters in L given in Example 3.7(1). Then $E_{\sigma_0} : \mathfrak{F}(L) \to \mathfrak{F}(L)$ defined by

$$\begin{aligned} E_{\sigma_0}(G) &= \cap\{H \in \mathfrak{F}(L) \mid G \subseteq H \text{ and } H \text{ is } \sigma_0\text{-primary}\} \\ &= \cap\{H \in \mathfrak{F}(L) \mid G \subseteq H \text{ and } H \text{ is prime}\} \end{aligned}$$

for all $G \in \mathfrak{F}(L)$ is an expansion of filters in L.

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