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Normal Interpolation on $AX = Y$ in CSL-algebra Alg$\mathcal{L}$

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Abstract. Let $\mathcal{L}$ be a commutative subspace lattice on a Hilbert space $\mathcal{H}$ and X and Y be operators on $\mathcal{H}$. Let

$$\mathcal{M}_X = \left\{ \sum_{i=1}^{n} E_i X f_i : n \in \mathbb{N},\ f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L} \right\}$$

and

$$\mathcal{M}_Y = \left\{ \sum_{i=1}^{n} E_i Y f_i : n \in \mathbb{N},\ f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L} \right\}.$$

Then the following are equivalent.

(i) There is an operator A in Alg$\mathcal{L}$ such that $AX = Y$, $Ag = 0$ for all g in $\overline{\mathcal{M}_X}^{\perp}$, $A^*A = AA^*$ and every E in $\mathcal{L}$ reduces A.

(ii) $\sup\{K(E,f) : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\} < \infty$, $\overline{\mathcal{M}_Y} \subset \overline{\mathcal{M}_X}$ and there is an operator T acting on $\mathcal{H}$ such that $\langle EXf, Tg\rangle = \langle EYf, Xg\rangle$ and $\langle ETf, Tg\rangle = \langle EYf, Yg\rangle$ for all $f,\ g$ in $\mathcal{H}$ and E in $\mathcal{L}$, where $K(E,f) = \|\sum_{i=1}^{n} E_i Y f_i\| / \|\sum_{i=1}^{n} E_i X f_i\|$.

1. Introduction

A commutative subspace lattice or CSL$\mathcal{L}$ is a strongly closed lattice of commutative projections on a Hilbert space $\mathcal{H}$. We assume that the projections 0 and I lie in $\mathcal{L}$. We usually identify projections and their ranges, so that it makes sense to speak of an operator as leaving a projection invariant. Alg$\mathcal{L}$ is the algebra of all bounded linear operators on $\mathcal{H}$ that leave invariant all the projections in $\mathcal{L}$. If $\mathcal{L}$ is CSL, then Alg$\mathcal{L}$ is called a CSL-algebra.

Let $\mathcal{M}$ be a subset of a Hilbert space $\mathcal{H}$. Then $\overline{\mathcal{M}}$ means the closure of $\mathcal{M}$ and $\mathcal{M}^{\perp}$ the orthogonal complement of $\mathcal{M}$. Let $\mathbb{N}$ be the set of all natural numbers and let $\mathbb{C}$ be the set of all complex numbers. In this paper, we use the convention $\frac{0}{0} = 0$, when necessary.

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Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a commutative subspace lattice of orthogonal projections on $\mathcal{H}$ containing 0 and I through this paper.

Theorem A ([3]). *Let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let X and Y be operators on $\mathcal{H}$. Then the following are equivalent.*

(i) *There is an operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A.*

(ii) $\sup\{K(E, f) : n \in \mathbb{N}, f_i \in \mathcal{H}$ *and* $E_i \in \mathcal{L}\} < \infty$.

Theorem B ([4]). *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let X and Y be operators on $\mathcal{H}$. Assume that the rangeX is dense in $\mathcal{H}$. Then the following statements are equivalent.*

(i) *There exists a normal operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A.*

(ii) $\sup\{K(E, f) : n \in \mathbb{N},\ f_i \in \mathcal{H}$ *and* $E_i \in \mathcal{L}\} < \infty$ *and there is an operator.*

T acting on $\mathcal{H}$ such that $\langle Xf, Tg\rangle = \langle Yf, Xg\rangle$ and $\langle Tf, Tg\rangle = \langle Yf, Yg\rangle$ for all f and g in $\mathcal{H}$.

In Theorem B, we investigated to find a necessary and sufficient condition for normal interpolation problem in Alg$\mathcal{L}$ and we assumed the density of the range of X. In this paper, we tried to delete the range dense condition.

2. Results

Let X and Y be operators acting on $\mathcal{H}$. Let

$$\mathcal{M}_X = \left\{\sum_{i=1}^{n} E_i X f_i : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\}$$

and

$$\mathcal{M}_Y = \left\{\sum_{i=1}^{n} E_i Y f_i : n \in \mathbb{N}, f_i \in \mathcal{H} \text{ and } E_i \in \mathcal{L}\right\}.$$

Lemma 2.1. *Let A, X and Y be operators on $\mathcal{H}$. If $Y = AX$, $Af = 0$ for all f in $\overline{\mathcal{M}_X}^{\perp}$ and $AE = EA$ for all E in $\mathcal{L}$. Then the following are equivalent.*

(i) $\overline{\mathcal{M}_Y} \subset \overline{\mathcal{M}_X}$.

(ii) *For all f in $\overline{\mathcal{M}_X}^{\perp}$, A^*f is in $\overline{\mathcal{M}_X}^{\perp}$.*

Proof. (i) $\Rightarrow$ (ii). Let f be a vector in $\overline{\mathcal{M}_X}^{\perp}$. Then $\langle A^*f, EXg\rangle = \langle f, AEXg\rangle = \langle f, EYg\rangle = 0$ for all g in $\mathcal{H}$ and E in $\mathcal{L}$ because $\overline{\mathcal{M}_Y} \subset \overline{\mathcal{M}_X}$. So A^*f is a vector in $\overline{\mathcal{M}_X}^{\perp}$.

(ii) $\Rightarrow$ (i). Let f be a vector in $\overline{\mathcal{M}_X}^{\perp}$. Then $0 = \langle A^*f, EXh\rangle = \langle f, EYh\rangle$ for all E in $\mathcal{L}$ and h in $\mathcal{H}$. So f is a vector in $\overline{\mathcal{M}_Y}^{\perp}$. Hence $\overline{\mathcal{M}_Y} \subset \overline{\mathcal{M}_X}$. □

Lemma 2.2. *Let A, X and Y be operators on $\mathcal{H}$. Assume that $AX = Y$, $Af = 0$ for all f in $\overline{\mathcal{M}_X}^{\perp}$, $AE = EA$ for all E in $\mathcal{L}$ and $A^*A = AA^*$. If f is a vector in $\overline{\mathcal{M}_X}^{\perp}$, then A^*f is a vector in $\overline{\mathcal{M}_X}^{\perp}$.*

Proof. Let f be a vector in $\overline{\mathcal{M}_X}^{\perp}$ and $EXh = A^*g_1 + g_2$ for E in $\mathcal{L}$, where g_2 is a vector in $\overline{\text{range } A^*}^{\perp}$. Then

$$\begin{aligned} \langle A^*f, EXh\rangle &= \langle A^*f, A^*g_1 + g_2\rangle = \langle A^*f, A^*g_1\rangle + \langle A^*f, g_2\rangle \\ &= \langle A^*f, A^*g_1\rangle = \langle Af, Ag_1\rangle = 0. \end{aligned}$$

So A^*f is a vector in $\overline{\mathcal{M}_X}^{\perp}$. □

Theorem 2.3. *The following statements are equivalent.*

(i) *There is an operator A in Alg$\mathcal{L}$ such that $Y = AX$, $Ag = 0$ for all g in $\overline{\mathcal{M}_X}^{\perp}$, $AE = EA$ for all E in $\mathcal{L}$ and $AA^* = A^*A$.*

(ii) $\sup\{K(E,f) : n \in \mathbb{N},\ f_i \in \mathcal{H}$ *and* $E_i \in \mathcal{L}\} < \infty$, $\overline{\mathcal{M}_Y} \subset \overline{\mathcal{M}_X}$ *and there is an operator T on $\mathcal{H}$ such that $Tf \in \overline{\mathcal{M}_X}$, $\langle EXf, Tg\rangle = \langle EYf, Xg\rangle$ and $\langle ETf, Tg\rangle = \langle EYf, Yg\rangle$ for all f, g in $\mathcal{H}$ and E in $\mathcal{L}$.*

Proof. (i) ⇒ (ii). If we assume that (i) holds, then by Theorem A, $\sup\{K(E,f) : n \in \mathbb{N}$, $f_i \in \mathcal{H}$ and $E_i \in \mathcal{L}\} < \infty$. And by Lemmas 2.1 and 2.2, $\overline{\mathcal{M}_Y} \subset \overline{\mathcal{M}_X}$. Let $A^*X = T$. Then

$$\langle EXf, Tg\rangle = \langle EXf, A^*Xg\rangle = \langle AEXf, Xg\rangle = \langle EYf, Xg\rangle$$

and

$$\langle ETf, Tg\rangle = \langle EA^*Xf, A^*Xg\langle = \langle AEXf, AXg\rangle = \langle EYf, Yg\rangle$$

for all f, g in $\mathcal{H}$ and E in $\mathcal{L}$. Since

$$\langle Tf, g\rangle = \langle A^*Xf, g\rangle = \langle Xf, Ag\rangle = \langle Xf, 0\rangle = 0$$

for all f in $\mathcal{H}$ and g in $\overline{\mathcal{M}_X}^{\perp}$, $Tf \in \overline{\mathcal{M}_X}$.

Conversely, by Theorem A, there is an operator A in $\mathcal{L}$ such that $AX = Y$, $Ag = 0$ for all g in $\overline{\mathcal{M}_X}^{\perp}$ and every E in $\mathcal{L}$ reduces A. Since $\langle EXf, Tg\rangle = \langle EYf, Xg\rangle$, we have

$$\langle A(\sum_{i=1}^{n} E_iXf_i), Xg\rangle = \langle \sum_{i=1}^{n} AE_iXf_i, Xg\rangle = \langle \sum_{i=1}^{n} E_iYf_i, Xg\rangle = \langle \sum_{i=1}^{n} E_iXf_i, Tg\rangle.$$

So $\langle Ah, Xg\rangle = \langle h, Tg\rangle$ for all h in $\overline{\mathcal{M}_X}$ and g in $\mathcal{H}$. Since $\langle Ah, Xg\rangle = 0 = \langle h, Tg\rangle$ for $h \in \overline{\mathcal{M}_X}^{\perp}$ and g in $\mathcal{H}$, $A^*X = T$. Since $\langle EYf, Yg\rangle = \langle ETf, Tg\rangle$ for all E in $\mathcal{L}$ and f, g in $\mathcal{H}$,

$$\begin{aligned} \langle A(\sum_{i=1}^{n} E_iXf_i), Yg\rangle &= \langle \sum_{i=1}^{n} E_iYf_i, Yg\rangle = \langle \sum_{i=1}^{n} E_iTf_i, Tg\rangle \\ &= \langle \sum_{i=1}^{n} E_iA^*Xf_i, Tg\rangle = \langle \sum_{i=1}^{n} E_iXf_i, ATg\rangle, \end{aligned}$$

for all $n \in N$, g in $\mathcal{H}$ and $E_i \in \mathcal{L}$. So $\langle Af, Yg\rangle = \langle f, ATg\rangle$ for all f in $\overline{\mathcal{M}_X}$ and g in $\mathcal{H}$. Since $\langle Af, Yg\rangle = 0$ and $\langle f, ATg\rangle = \langle A^*f, Tg\rangle = 0$ for all f in $\overline{\mathcal{M}_X}^{\perp}$ and g in $\mathcal{H}$ ($A^*f \in \overline{\mathcal{M}_X}^{\perp}$ and $Tg \in \overline{\mathcal{M}_X}$). Hence $A^*Y = AT$. Thus $AA^*X = A^*AX$. So $AA^*f = A^*Af$ for all f in $\overline{\mathcal{M}_X}$. Since $AE = EA$, $A^*E = EA^*$ for all E in $\mathcal{L}$. Since $A^*Ag = 0 = AA^*g$ for all g in $\overline{\mathcal{M}_X}^{\perp}$ by Lemmas 2.1 and 2.2, $AA^* = A^*A$.□

Let $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ be operators acting on $\mathcal{H}$ (n is a fixed natural number). Let

$$\mathcal{N}_X = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} E_{k,i} X_i f_{k,i} : m_i \in \mathbb{N},\ l \leq n,\ f_{k,i} \in \mathcal{H} \text{ and } E_{k,i} \in \mathcal{L} \right\}$$

and

$$\mathcal{N}_Y = \left\{ \sum_{k=1}^{m_i} \sum_{i=1}^{l} E_{k,i} Y_i f_{k,i} : m_i \in \mathbb{N},\ l \leq n,\ f_{k,i} \in \mathcal{H} \text{ and } E_{k,i} \in \mathcal{L} \right\}.$$

Theorem C ([3]). *Let $\mathcal{L}$ be a commutative subspace lattice on $\mathcal{H}$. Let $X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n$ be operators on $\mathcal{H}$. Assume that the range of one of the X_p's is dense in $\mathcal{H}$ ($p = 1, 2, \cdots, n$). Let*

$$K(E, f, m) = \frac{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} E_{k,i} Y_i f_{k,i} \|}{\| \sum_{k=1}^{m_i} \sum_{i=1}^{l} E_{k,i} X_i f_{k,i} \|}.$$

Then the following statements are equivalent.

(i) *There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i (i = 1, 2, \cdots, n)$, $Ag = 0$ for all g in $\overline{\mathcal{N}_X}^{\perp}$ and every E in $\mathcal{L}$ reduces A.*

(ii) $\sup \{K(E, f, m) : m_i \in \mathbb{N},\ l \leq n, f_{k,i} \in \mathcal{H}$ *and* $E_{k,i} \in \mathcal{L}\} < \infty$.

Lemma 2.4. *Let A, X_i and Y_i be operators on $\mathcal{H}$ for $i = 1, 2, \cdots, n$. If $AX_i = Y_i (i = 1, 2, \cdots, n)$, $Ag = 0$ for all g in $\overline{\mathcal{N}_X}^{\perp}$ and $AE = EA$ for all E in $\mathcal{L}$, then the following are equivalent.*

(i) $\overline{\mathcal{N}_Y} \subset \overline{\mathcal{N}_X}$.

(ii) *For all f in $\overline{\mathcal{N}_X}^{\perp}$, A^*f is a vector in $\overline{\mathcal{N}_X}^{\perp}$.*

Proof. (i) ⇒ (ii). Let f be a vector in $\overline{\mathcal{N}_X}^{\perp}$. Then

$$\langle A^*f, EX_i f_i\rangle = \langle f, AE_i X_i f_i\rangle = \langle f, E_i Y_i f_i\rangle = 0$$

for all $i = 1, 2, \cdots, n$ and for all E_i in $\mathcal{L}$ because $\overline{\mathcal{N}_Y} \subset \overline{\mathcal{N}_X}$. So A^*f is a vector in $\overline{\mathcal{N}_X}^{\perp}$.

(ii) $\Rightarrow$ (i). Let f be a vector in $\overline{\mathcal{N}_X}^{\perp}$. Then

$$0 = \langle A^* f, E_i X_i h_i\rangle = \langle f, AE_i X_i h_i\rangle = \langle f, E_i Y_i h_i\rangle$$

for all E_i in $\mathcal{L}$, h_i in $\mathcal{H}$ and $i = 1, 2, \cdots, n$. So f is a vector in $\overline{\mathcal{N}_Y}^{\perp}$. Hence $\overline{\mathcal{N}_Y} \subset \overline{\mathcal{N}_X}$. □

Lemma 2.5. *Let A, X_i and Y_i be operators on $\mathcal{H}$ for $i = 1, 2, \cdots, n$. Assume that $AX_i = Y_i$ $(i = 1, 2, \cdots, n)$, $Ag = 0$ for all g in $\overline{\mathcal{N}_X}^{\perp}$, $AE = EA$ for all E in $\mathcal{L}$ and $A^*A = AA^*$. Then A^*f is a vector in $\overline{\mathcal{N}_X}^{\perp}$ for all f in $\overline{\mathcal{N}_X}^{\perp}$.*

Proof. Let f be a vector in $\overline{\mathcal{N}_X}^{\perp}$ and $E_i X_i f_i = A^* g_{i_1} + g_{i_2}$ for E_i in $\mathcal{L}$ and f_i in $\mathcal{H}$, where g_{i_2} is a vector in $\overline{\text{range } A^*}^{\perp}$ $(i = 1, 2, \cdots, n)$. Then

$$\begin{aligned} \langle A^* f, E_i X_i f_i\rangle &= \langle A^* f, A^* g_{i_1} + g_{i_2}\rangle = \langle A^* f, A^* g_{i_1}\rangle + \langle A^* f, g_{i_2}\rangle \\ &= \langle A^* f, A^* g_{i_1}\rangle = \langle Af, Ag_{i_1}\rangle = 0. \end{aligned}$$

So A^*f is a vector in $\overline{\mathcal{N}_X}^{\perp}$. □

Theorem 2.6. *The following are equivalent.*

(i) *There is an operator A in Alg$\mathcal{L}$ such that $Y_i = AX_i$ $(i = 1, 2, \cdots, n)$, $Ag = 0$ for all g in $\overline{\mathcal{N}_X}^{\perp}$, $AE = EA$ for all E in $\mathcal{L}$ and $AA^* = A^*A$.*

(ii) $\sup\{K(E, f, m) : m_i \in \mathbb{N},\ l \leq n,\ f_{k,i} \in \mathcal{H}$ *and* $E_{k,i} \in \mathcal{L}\} < \infty$, *$\overline{\mathcal{N}_Y} \subset \overline{\mathcal{N}_X}$ and there are operators T_p on $\mathcal{H}$ such that*

$$\langle E_q X_q f_q, T_p g_p\rangle = \langle E_q Y_q f_q, X_p g_p\rangle, \quad \langle E_q T_q f_q, T_p g_p\rangle = \langle E_q Y_q f_q, Y_p g_p\rangle$$

and $T_p f_p \in \overline{\mathcal{N}_X}$ for f_p, g_p in $\mathcal{H}$, E_q in $\mathcal{L}$ and p, $q = 1, 2, \cdots, n$.

Proof. (i) $\Rightarrow$ (ii). By Theorem C, $\sup\{K(E, f, m) : m_i \in \mathbb{N},\ l \leq n,\ f_{k,i} \in \mathcal{H}$ and $E_{k,i} \in \mathcal{L}\} < \infty$. By Lemmas 2.4 and 2.5, $\overline{\mathcal{N}_Y} \subset \overline{\mathcal{N}_X}$. Let $A^* X_p = T_p$ $(p = 1, 2, \cdots, n)$. Then

$$\begin{aligned} \langle E_q X_q f_q, T_p g_p\rangle &= \langle E_q X_q f_q, A^* X_p g_p\rangle \\ &= \langle AE_q X_q f_q, X_p g_p\rangle \\ &= \langle E_q Y_q f_q, X_p g_p\rangle \end{aligned}$$

and

$$\begin{aligned} \langle E_q T_q f_q, T_p g_p\rangle &= \langle E_q A^* X_q f_q, A^* X_p g_p\rangle \\ &= \langle AE_q X_q f_q, AX_p g_p\rangle \\ &= \langle E_q Y_q f_q, Y_p g_p\rangle. \end{aligned}$$

Since $\langle T_p f_p, g\rangle = \langle A^* X_p f_p, g\rangle = \langle X_p f_p, Ag\rangle = \langle X_p f_p, 0\rangle = 0$ for all f_p in $\mathcal{H}$ and g in $\overline{\mathcal{N}_X}^{\perp}$, $T_p f_p \in \overline{\mathcal{N}_X}$.

(ii) $\Rightarrow$ (i). By Theorem C, there is an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ $(i = 1, 2, \cdots, n)$, $Af = 0$ for all f in $\overline{\mathcal{N}_X}^{\perp}$ and every E in $\mathcal{L}$ reduces A. Since $\langle E_q X_q f_q, T_p g_p\rangle = \langle E_q Y_q f_q, X_p g_p\rangle$ for all E_q in $\mathcal{L}$ and all $p,\ q = 1, 2,\ \cdots, n$,

$$\begin{aligned}\langle A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}), X_p g_p\rangle &= \langle \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}, X_p g_p\rangle \\ &= \langle \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}, T_p g_p\rangle,\end{aligned}$$

$m_i \in \mathbb{N}$, $l \leq n$, $f_{k,i} \in \mathcal{H}$, $E_{k,i} \in \mathcal{L}$ and $p = 1, 2, \cdots, n$. So $\langle Ah, X_p g_p\rangle = \langle h, T_p g_p\rangle$ for all h in $\overline{\mathcal{N}_X}$, g_p in $\mathcal{H}$ and $p = 1, 2, \cdots, n$. Since $\langle Ah, X_p g_p\rangle = 0 = \langle h, T_p g_p\rangle$ for all h in $\overline{\mathcal{N}_X}^{\perp}$, g_p in $\mathcal{H}$ and $p = 1, 2, \cdots, n$, $A^* X_p = T_p$. Since $\langle E_q Y_q f_q, Y_p g_p\rangle = \langle E_q T_q f_q, T_p g_p\rangle$, $E_q \in \mathcal{L}$, $f_q,\ g_q \in \mathcal{H}$ and $p,\ q = 1, 2, \cdots, n$,

$$\begin{aligned}\langle A(\sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}), Y_p g_p\rangle &= \langle \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i}, Y_p g_p\rangle \\ &= \langle \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}T_i f_{k,i}, T_p g_p\rangle \\ &= \langle \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}A^* X_i f_{k,i}, T_p g_p\rangle \\ &= \langle \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i}, AT_p g_p\rangle.\end{aligned}$$

So $\langle Af, Y_p g_p\rangle = \langle f, AT_p g_p\rangle$ for all f in $\overline{\mathcal{N}_X}$ and g_p in $\mathcal{H}(p = 1, 2, \cdots, n)$. Since $\langle Af, Y_p g_p\rangle = 0$ and $\langle f, AT_p g_p\rangle = \langle A^* f, T_p g_p\rangle = 0$ for all f in $\overline{\mathcal{N}_X}^{\perp}$, g_p in $\mathcal{H}$ and $p = 1, 2, \cdots, n$ by Lemmas 2.4 and 2.5. So $A^* Y_p = AT_p(p = 1, 2, \cdots, n)$. Thus $A^* A X_p = AA^* X_p(p = 1, 2, \cdots, n)$. Hence $A^* Af = AA^* f$ for all f in $\overline{\mathcal{N}_X}$. Since $A^* Ag = 0 = AA^* g$ for all g in $\overline{\mathcal{N}_X}^{\perp}$ by Lemmas 2.4 and 2.5, $A^* A = AA^*$. □

Let $\{X_n\}$ and $\{Y_n\}$ be two infinite sequences of operators on $\mathcal{H}$. Let

$$\mathcal{K}_X = \left\{ \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}X_i f_{k,i} : m_i, l \in \mathbb{N},\ f_{k,i} \in \mathcal{H} \text{ and } E_{k,i} \in \mathcal{L} \right\}$$

and

$$\mathcal{K}_Y = \left\{ \sum_{k=1}^{m_i}\sum_{i=1}^{l} E_{k,i}Y_i f_{k,i} : m_i, l \in \mathbb{N},\ f_{k,i} \in \mathcal{H} \text{ and } E_{k,i} \in \mathcal{L} \right\}.$$

With the similar proof as Lemmas 2.4, 2.5 and Theorem 2.6, we can get the following Theorem.

Theorem 2.7. *The following statements are equivalent.*

(i) *There is an operator A in Alg$\mathcal{L}$ such that $AX_n = Y_n$ $(n = 1, 2, \cdots)$, $Ag = 0$ for all g in $\overline{\mathcal{K}_X}^{\perp}$, every E in $\mathcal{L}$ reduces A and $AA^* = A^*A$.*

(ii) $\sup\{K(E, f, m) : m_i, l \in \mathbb{N},\ f_{k,i} \in \mathcal{H}$ *and* $E_{k,i} \in \mathcal{L}\} < \infty$, $\overline{\mathcal{K}_Y} \subset \overline{\mathcal{K}_X}$ *and there are operators T_n on $\mathcal{H}$ $(n = 1, 2, \cdots)$ such that* $\langle E_q X_q f_q, T_p g_p\rangle = \langle E_q Y_q f_q, X_p g_p\rangle$, $\langle E_q T_q f_q, T_p g_p\rangle = \langle E_q Y_q f_q, Y_p g_p\rangle$ *and* $T_p f_p \in \overline{\mathcal{N}_X}$ *for* f_p, g_p *in* $\mathcal{H}$, E_q *in* $\mathcal{L}$ *and* p, $q = 1, 2, \cdots$.

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