Honam Mathematical J. **27**(2005), No. 1, pp. 101–113

GENERALIZED T-SPACES AND DUALITY

Yeon Soo Yoon

Abstract. We define and study a concept of T_A-space which is closely related to the generalized Gottlieb group. We know that X is a T_A-space if and only if there is a map $r : L(A, X) \to L_0(A, X)$ called a T_A-structure such that $ri \sim 1_{L_0(A,X)}$. The concepts of $T_{\Sigma B}$-spaces are preserved by retraction and product. We also introduce and study a dual concept of T_A-space.

1. Introduction

Let A be a compact CW complex. Let $L(A, X)$ be the space of maps from A to X with the compact open topology. Let $L_0(A, X)$ be the space of base point preserving maps in $L(A, X)$. Throughout this paper, space means a space of homotopy type of connected locally finite CW complex. According to a well known result of Milnor [8], $L(A, X)$ and $L_0(A, X)$ have the homotopy type of CW complexes. Clearly the evaluation map $p : L(A, X) \to X$ is a fibration. In 1987, Aguade introduced and studied T-spaces in [1]. A space X is called [1] a *T-space* if the fibration $L_0(S^1, X) \to L(S^1, X) \to X$ is fibre homotopically trivial. It is easy to show that any H-space is a T-space. However, there are many T-spaces which are not H-spaces in [11]. ΣX denote the reduced suspension of X and ΩX denote the based loop space of X. The adjoint functor from the group $[\Sigma X, Y]$ to the group $[X, \Omega Y]$ will be denoted by

Received January 16, 2005. Revised February 22, 2005.
2000 Mathematics Subject Classification : 55P45, 55P35.
Key words and phrases : T-space, co-T-spaces, cyclic maps, cocyclic maps.
* This work was supported by Hannam University Research Fund, 2004.

τ. The symbols e and e' denote $\tau^{-1}(1_{\Omega X})$and $\tau(1_{\Sigma X})$ respectively. On the other hand, we introduced and studied co-T-spaces in [11]. A space X is called *co-T-space* [11] if $e' : X \to \Omega\Sigma X$ is cocyclic. It is also easy to show that any co-H-space is a co-T-space.

In this paper, we introduce a T_A-space which is a generalization of T-space, and study some properties of a T_A-space. Also, we introduce and study some properties of a co-T_A-space which is a dual concept of T_A-space. In section 2, we show that X is a T_A-space if and only if there is a map $r : L(A, X) \to L_0(A, X)$ such that $ri \sim 1_{L_0(A,X)}$. We call such a retraction r a T_A-structure. The set S_{T_A} of homotopy classes of T_A-structures can be identified to the Gottlieb set $G(X, L_0(A, X))$. We show that any H-space is a T-space, and any T-space is a $T_{\Sigma B}$-space for any compact space B. We also show that if a $T_{\Sigma B}$-space X dominates Y, then Y is also a $T_{\Sigma B}$-space. Moreover, we know that $X \times Y$ is a $T_{\Sigma B}$-space if and only if X and Y are $T_{\Sigma B}$-spaces. We know that for any $[f]$, $[g] \in \pi_3(S^2)$, $L(S^3, S^2; f)$ and $L(S^3, S^2; g)$ have the same homotopy type. In section 3, we introduce and study a co-T_A-space which is a dual concept of T_A-space. We show that if a co-T_A-space X dominates Y, then Y is also a co-T_A-space. Also, we show that $X \vee Y$ is a co-T_A-space if and only if X and Y are co-T_A-spaces.

2. T_A-spaces and some properties

In this section, we introduce a T_A-space which is a generalization of T-space, and study some properties of a T_A-space. A based map $f\colon B \to X$ is called *cyclic* [10] if there exists a map $F\colon X \times B \to X$ such that $Fj \sim \nabla(1 \vee f)$,where $j\colon X \vee B \to X \times B$ is the inclusion and $\nabla\colon X \vee X \to X$ is the folding map. We call such a map F an associated map of f. The *Gottlieb set* denoted $G(B, X)$ is the set of all homotopy classes of cyclic maps from B to X. In 1982, Lim obtained a result [6] which was an H-space may be characterized by the Gottlieb sets and cyclic maps as follows; X is an H-space if and only if 1_X is cyclic if and only if $G(B,X) = [B,X]$ for any space B. It is known [10] that if $g : X \to Y$ is cyclic and $f : A \to X$ is any map, then $gf : A \to Y$ is cyclic. From the above fact, we can obtained a result [11] which characterizes a T-space by means of the Gottlieb sets and cyclic maps as follows; X is an T-space if and only if $e\colon \Sigma\Omega X \to X$ is cyclic if and only if $G(\Sigma C, X) = [\Sigma C, X]$ for any space C. For a co-H-space X, there is a map $s : X \to \Sigma\Omega X$ such that $es \sim 1$.Thus we can easily know that in the category of co-H-spaces, H-spaces and T-spaces are equivalent. It is also well known [16] that X is an H-space if and only if for any spaces M, L, $i^* : [M \times L, X] \to [M \vee L, X]$ is surjective, where $i : M \vee L \to M \times L$ is the inclusion.

For a locally compact space A, it is well known fact that there is a natural equivalence $\tau : [A \wedge B, X] \approx [B, L_0(A, X)]$ given by $(\tau(f)(b))(a) = f < a, b >$. From now on, let A be a compact CW complex.

Theorem 2.1. *The followings are equivalent;*

(1) $e_A : A \wedge L_0(A,X) \to X$ *is cyclic, where* $e_A = \tau^{-1}(1_{L_0(A,X)})$.
(2) $G(A \wedge B, X) = [A \wedge B, X]$ *for any space* B.
(3) *For any spaces* M, L, $i^* : [M \times (A \wedge L), X] \to [M \vee (A \wedge L), X]$ *is surjective.*

Proof. (1) implies (2). Let $f : A \wedge B \to X$ be a map. Then we know, from the fact that $f = e_A(1 \wedge \tau(f))$ and e_A is cyclic, that $f : A \wedge B \to X$ is cyclic. (2) implies (3). Let $f : M \vee (A \wedge X) \to X$ be a map. Let $f_1 = f_{|M} : M \to X$ and $f_2 = f_{|A\wedge X} : A \wedge X \to X$. Since f_2 is cyclic, there is a map $F : X \times (A \wedge X) \to X$ such that $Fj \sim \nabla(1 \vee f_2)$. Let $g = F(f_1 \times 1) : M \times (A \wedge L) \to X$. Then $i^*[g] = [gi] = [F(f_1 \times 1)i] = [f]$ and $i^* : [M \times A \wedge L, X] \to [M \vee A \wedge L, X]$ is surjective. (3) implies (1). Take $M = X$ and $L = L_0(A, X)$. Consider $\nabla(1 \vee e_A) : X \vee (A \wedge L_0(A, X)) \to X$. Since i^* is an epimorphism, there is a map $E : X \times (A \wedge L_0(A, X)) \to X$ such that $Ei \sim \nabla(1 \vee e_A)$. Thus e_A is cyclic. $\square$

Definition 2.2. Let A be a compact CW complex. A space X is called a T_A-space if the fibration $L_0(A, X) \to L(A, X) \to X$ is fibre homotopically trivial.

Proposition 2.3. [2] *Let $p : E \to B, p' : E' \to B$ be fibrations and $h : E \to E'$ a map with $p'h = p$. Then*

(1) *$h : E \to E'$ is a fiber homotopy equivalence if and only if the restriction of h to every fiber, $b \in B, h_{|p^{-1}(b)} : p^{-1}(b) \to p'^{-1}(b)$ is a homotopy equivalence.*

(2) *$h : E \to E'$ is a fiber homotopy equivalence if and only if $h : E \to E'$ is a homotopy equivalence.*

Remark 2.4. From the above result, it is clear that X is a T_A-space if and only if there is a homotopy equivalence $h : L(A, X) \to X \times L_0(A, X)$ such that the diagram

$$\begin{array}{ccccc} L_0(A,X) & \xrightarrow{i} & L(A,X) & \xrightarrow{p} & X \\ \| & & h\downarrow & & \| \\ L_0(A,X) & \xrightarrow{i_2} & X \times L_0(A,X) & \xrightarrow{p_1} & X \end{array}$$

is homotopy commutative.

Theorem 2.5. *X is a T_A-space if and only if there is a map $r : L(A, X) \to L_0(A, X)$ such that $ri \sim 1_{L_0(A,X)}$.*

PROOF. Suppose that X is a T_A-space. Then by the above remark, there is a homotopy equivalence $h : L(A, X) \to X \times L_0(A, X)$ such that $hi \sim i_2$. Let

$$r : L(A, X) \xrightarrow{h} X \times L_0(A, X) \xrightarrow{p_2} L_0(A, X).$$

Then $ri = p_2hi \sim p_2i_2 \sim 1_{L_0(A,X)}$. On the other hand, suppose that there is a map $r : L(A, X) \to L_0(A, X)$ such that $ri \sim 1_{L_0(A,X)}$. Define

$h : L(A,X) \to X \times L_0(A,X)$ by $h(f) = (p(f), r(f))$. Then we have the following homotopy commutative diagram;

$$\begin{array}{ccccc} L_0(A,X) & \xrightarrow{i} & L(A,X) & \xrightarrow{p} & X \\ \| & & \downarrow h & & \| \\ L_0(A,X) & \xrightarrow{i_2} & X \times L_0(A,X) & \xrightarrow{p_1} & X. \end{array}$$

Thus h induces an isomorphism of homotopy groups, and as all spaces are homotopy equivalent to CW complexes, it follows that h is a homotopy equivalence. Moreover we have $p_1 h = p$. Thus we know, from the result of Dold, that h is a fibre homotopy equivalence. □

For a T_A-space X, a retraction $r : L(A,X) \to L_0(A,X)$ is called a T_A-structure on X. It is clear that a T_A-space will in general admit many different T_A-structures. The set S_{T_A} of homotopy classes of T_A-structures can be identified to the Gottlieb set $G(X, L_0(A,X))$.

Theorem 2.6. *Let X be a T_A-space. Then there is a bijection $\phi : S_{T_A} \to G(X, L_0(A,X))$, where $S_{T_A} = \{[r] \in [L(A,X), L_0(A,X)] | r \circ i \sim 1_{L_0(A,X)}\}$.*

Proof. Since X is a T_A-space, there is a homotopy equivalence $h : L(A,X) \to X \times L_0(A,X)$ such that $h \circ i \sim i_2$. Let $k : X \times L_0(A,X) \to L(A,X)$ be a homotopy inverse of h. Define a map $\phi : S_{T_A} \to G(X, L_0(A,X))$ by $\phi([r]) = [r \circ k \circ i_1]$, where $i_1 : X \to X \times L_0(A,X)$ is the inclusion. Since $r \circ k_{|L_0(A,X)} = r \circ k \circ i_2 \sim r \circ i \sim 1_{|L_0(A,X)}$ and $r \circ k : X \times L_0(A,X) \to L_0(A,X)$, $\phi([r]) = [r \circ k \circ i_1] = [r \circ k_{|X}] \in G(X, L_0(A,X)$. On the other hand, define a map $\psi : G(X, L_0(A,X) \to S_{T_A}$ by $\psi([f]) = [F \circ h]$, where $F : X \times L_0(A,X) \to L(A,X)$ is an associated map of f. Since $F \circ h \circ i \sim F \circ i_2 \sim 1_{L_0(A,X)}$, $\psi([f]) = [F \circ h] \in S_{T_A}$. Moreover, we have that $\psi \circ \phi([r]) = \psi([r \circ k \circ i_1]) = [r \circ k \circ h] = [r] = 1([r])$ for any $[r] \in S_{T_A}$ and $\phi \circ \psi([f]) = \phi([F \circ h]) = [F \circ h \circ k \circ i_1] = [F \circ i_1] = [f] = 1([f])$ for any $[f] \in G(X, L_0(A,X)$. Thus we know that $\phi : S_{T_A} \to G(X, L_0(A,X))$ is a bijection. □

The following theorem says that if X is an H-space, then X is a T_A-space for any compact CW complex A.

Theorem 2.7. *If $e_A : A \wedge L_0(A, X) \to X$ is cyclic, then X is a T_A-space.*

PROOF. Since $e_A : A \wedge L_0(A, X) \to X$ is cyclic, there exists a map $E : (A \wedge L_0(A, X)) \times X \to X$ of e_A such that $Ej = \nabla(e_A \vee 1_X)$. Consider the map

$$E \circ (q \times 1) : A \times L_0(A, X) \times X \overset{(q\times 1)}{\to} A \wedge L_0(A, X) \times X \overset{E}{\to} X,$$

where $q : A \times L_0(A, X) \to A \wedge L_0(A, X)$ is the quotient map. Let $k : L_0(A, X) \times X \to L(A, X)$ be the adjoint of $E \circ (q \times 1) : A \times L_0(A, X) \times X \to X$. Then $pk = p_2$ and $i = ki_1$. Thus we know, by the result of Dold, that $k : L_0(A, X) \times X \to L(A, X)$ is a fiber homotopy equivalence and X is a T_A-space. □

We do not know whether the converse of the above theorem holds. However, we showed [14] that the converse of the above theorem is true for the case of a suspension $A = \Sigma B$. Thus we know that a concept of $T_{\Sigma B}$-space can be characterized by the generalized Gottlieb group.

Corollary 2.8. [14] *X is a $T_{\Sigma B}$-space if and only if $e_{\Sigma B} : \Sigma B \wedge L_0(\Sigma B, X) \to X$ is cyclic if and only if $G(\Sigma B \wedge C, X) = [\Sigma B \wedge C, X]$ for any space C.*

Remark 2.9. Any T-space X is a $T_{\Sigma B}$-space. Since X is a T-space, we have that $G(\Sigma C, X) = [\Sigma C, X]$ for any space C. Take $C = B \wedge L_0(\Sigma B, X)$. Then we know that the map $e_{\Sigma B} : \Sigma B \wedge L_0(\Sigma B, X) \to X$ is cyclic. Thus X is a $T_{\Sigma B}$-space.

From the above remark, we know that any H-space is a T-space, and any T-space is a $T_{\Sigma B}$-space for any compact space B.

Theorem 2.10. *If a $T_{\Sigma B}$-space X dominates Y, then Y is also a $T_{\Sigma B}$-space.*

PROOF. Since X dominates Y, there are maps $r : X \to Y$ and $i : Y \to X$ such that $ri \sim 1_Y$. Consider the commutative diagram

$$\begin{array}{ccc} \Sigma B \wedge L_0(\Sigma B, Y) & \xrightarrow{1 \wedge L_0(i)} & \Sigma B \wedge L_0(\Sigma B, X) \\ {\scriptstyle e^Y_{\Sigma B}}\downarrow & & \downarrow{\scriptstyle e^X_{\Sigma B}} \\ Y & \xrightarrow{\quad i \quad} & X. \end{array}$$

Since $e^X_{\Sigma B} : \Sigma B \wedge L_0(\Sigma B, X) \to X$ is cyclic, $ie^Y_{\Sigma B} = e^X_{\Sigma B}(1 \wedge L_0(i)) : \Sigma B \wedge L_0(\Sigma B, Y) \to X$ is cyclic. It is known [10] that if $g : X \to Y$ has a right homotopy inverse and $f : A \to X$ is cyclic, then $gf : A \to Y$ is cyclic. Thus we have that $e^Y_{\Sigma B} \sim r(ie^Y_{\Sigma B}) : \Sigma B \wedge L_0(\Sigma B, Y) \to Y$ is cyclic and Y is a $T_{\Sigma B}$-space. □

Theorem 2.11. *$X \times Y$ is a $T_{\Sigma B}$-space if and only if X and Y are $T_{\Sigma B}$-spaces.*

PROOF. Suppose $X \times Y$ is a $T_{\Sigma B}$-space. Then we know, from the above theorem, that X and Y are $T_{\Sigma B}$-spaces. On the other hand, let X and Y be $T_{\Sigma B}$-spaces. We show that $G(\Sigma B \wedge C, X \times Y) = [\Sigma B \wedge C, X \times Y)]$ for any space C. Let $f : \Sigma B \wedge C \to X \times Y$ be a map. Since X and Y are $T_{\Sigma B}$-spaces, $p_1 f : \Sigma B \wedge C \to X$ and $p_2 f : \Sigma B \wedge C \to Y$ are cyclic maps. It is known [6] that if $f_1 : A_1 \to X_1$ and $f_2 : A_2 \to X_2$ are cyclic, then so is $f_1 \times f_2 : A_1 \times A_2 \to X_1 \times X_2$. Thus $f = (p_1 f \times p_2 f)\Delta : \Sigma B \wedge C \to X \times Y$ is cyclic. □

Example 2.12. $S^1 \times S^1$, $S^1 \times S^3$, $S^3 \times S^7, \cdots$ are T_{S^1}-spaces (equivalently T-spaces) because S^1, S^3, S^7 are T_{S^1}-spaces.

Theorem 2.13. *If there exists a space Y such that X is a homotopy equivalent to $K \times Y$ for a $T_{\Sigma B}$-space K, then there are maps $r : X \to K$ and $i : K \to X$ such that $ri \sim 1$ and $ie^K_{\Sigma B} : \Sigma B \wedge L_0(\Sigma B, K) \to X$ is cyclic.*

PROOF. Let $f : X \to K \times Y$, $g : K \times Y \to X$ be maps such that $gf \sim 1_X$ and $fg \sim 1_{K \times Y}$. Let $r = P_1 f : X \to K$ and $i = gi_1 : K \to X$, where

$i_1 : K \to K \times Y$ is the inclusion and $p_1 : K \times Y \to K$ be the projection. Then $ri = p_1 f g i_1 \sim p_1 i_1 = 1_K$. Moreover we have, from the fact that K is a $T_{\Sigma B}$-space, that there exist a map $E : \Sigma B \wedge L_0(\Sigma B, K) \times K \to K$ such that $ej = \nabla(e^K_{\Sigma B} \vee 1)$. Consider the map $F = g \circ (E \times 1) \circ (1 \times f) : \Sigma B \wedge L_0(\Sigma B, K) \times X \to X$. Then $Fj' \sim \nabla(ie^K_{\sigma B} \vee 1)$. Thus $ie^K_{\sigma B} : \Sigma B \wedge L_0(\Sigma B, K) \to X$ is cyclic. □

For a based map $f\colon A \to X$, let $L(A, X; f)$ be the path component of $L(A, X)$ containing f. $L_0(A, X; f)$ will denote the space of base point preserving maps in $L(A, X; f)$. In general, the components of $L(\Sigma B, X)$ almost never have the same homotopy type. It is well known fact that $L(S^2, S^2; *)$ and $L(S^2, S^2; 1)$ have different homotopy type. Under what condition on X, do $L(\Sigma B, X; f)$ and $L(\Sigma B, X; g)$ have the same homotopy equivalence for any $[f]$, $[g] \in L_0(\Sigma B, X)$? For any $[f] \in L_0(\Sigma B, X)$, clearly the evaluation map $p\colon L(\Sigma B, X; f) \to X$ is a fibration with fiber $L_0(\Sigma B, X; f)$.

Proposition 2.14. [12] *The following statements are equivalent;*

(1) *$f : \Sigma B \to X$ is cyclic.*

(2) *$L(\Sigma B, X; f)$ is fiber homotopy equivalent to $L(\Sigma B, X; *)$.*

We have, from the fact that X is an H-space iff $G(C, X) = [C, X]$ for any space C, the following corollary.

Corollary 2.15. [4] *If X is an H-space,then $L(S^p, X; f)$ and $L(S^p, X; g)$ have the same homotopy type for arbitrary f and g in $\pi_p(X)$.*

Moreover, we know, from the fact that the generator $\eta_2 \in \pi_3(S^2)$ is cyclic and the sum of two cyclic maps is cyclic, that $G(S^3, S^2) = [S^3, S^2]$. Thus we have the following corollary.

Corollary 2.16. *For any $[f]$, $[g] \in \pi_3(S^2)$, $L(S^3, S^2; f)$ and $L(S^3, S^2; g)$ have the same homotopy type.*

Lemma 2.17. [5] *For any* $[f] \in L_0(\Sigma B, X)$, $L_0(\Sigma B, X; f)$ *is homotopy equivalent to* $L_0(\Sigma B, X; *)$.

Lemma 2.18. [13] *If* X *is a* $T_{\Sigma B}$*-space, then any* $[f] \in L_0(\Sigma B, X)$, *the fibration* $L_0(\Sigma B, X; f) \to L(\Sigma B, X; f) \to X$ *is fibre homotopically trivial.*

From the above two lemmas, we have an answer of the above question as follows;

Proposition 2.19. *If* X *is a* $T_{\Sigma B}$*-space, for any* $[f]$, $[g] \in L_0(\Sigma B, X)$, $L(\Sigma B, X; f)$ *and* $L(\Sigma B, X; g)$ *have the same homotopy type.*

3. co-T_A-spaces and some properties

In this section, we would like to study some properties of co-T_A-space which is a dual concept of T_A-space.

A based map $f\colon X \to A$ is called *cocyclic*[10] if there exists a map $\phi\colon X \to X \vee A$ such that $j\phi \sim (1 \times f)\Delta$, where $j\colon X \vee A \to X \times A$ is the inclusion and $\Delta\colon X \to X \times X$ is the diagonal map. We call such a map ϕ a coassociated map of f. The *dual Gottlieb set* denoted $DG(X, A)$ is the set of all homotopy classes of cocyclic maps from X to A.

A space X is called *co-T-space* [11] if $e' : X \to \Omega\Sigma X$ is cocyclic. It is also easy to show that any co-H-space is a co-T-space.

In 1987, Lim also obtained a result [7] which was a co-H-space may be characterized by the dual Gottlieb sets and cocyclic maps as follows; X is a co-H-space if and only if $1_X : X \to X$ is cocyclic if and only if $DG(X, C) = [X, C]$ for any space C. It is known [10] that if $f : X \to A$ is cocyclic and $g : A \to B$ is any map, then $gf : X \to B$ is cocyclic. From this fact, we can obtained a result [11] which characterizes a co-T-space by means of the dual Gottlieb sets as follows; X is a co-T-space if and only if $GD(X, \Omega B) = [X, \Omega B]$ for any space B.

Definition 3.1. A space X is called co-T_A-space if $e'_A : X \to L_0$ $(A, A \wedge X)$ is cocyclic, where $e'_A = \tau(1_{A \wedge X})$.

Theorem 3.2. *The followings are equivalent;*
(1) *X is a co-T_A-space.*
(2) *$DG(X, L_0(A,B)) = [X, L_0(A,B)]$ for any space B.*
(3) *For any spaces M, L, i_* : $[X, M \vee L_0(A,L)] \to [X, M \times L_0(A,L)]$ is surjective, where $i : M \vee L_0(A,L) \to M \times L_0(A,L)$ is the inclusion.*

PROOF. (1) implies (2). Let $f : X \to L_0(A,B)$ be a map. Then there is a map $\tau^{-1}(f) : A \wedge X \to B$. Then we know, from the fact that $f = L_0(\tau^{-1}(f)) \circ e'_A$ and e'_A is cocyclic, that $f : X \to L_0(A,B)$ is cocyclic. (2) implies (3). Let $f : X \to M \times L_0(A,L)$ be a map. Let $f_1 = p_1 f : X \to M$ and $f_2 = p_2 f : X \to L_0(A,L)$. Since f_2 is cocyclic, there is a map $\theta : X \to X \vee L_0(A,L)$ such that $j\theta \sim (1 \times f_2)\Delta$. Let $g = (f_1 \vee 1)\theta : X \to M \vee L_0(A,L)$. Then $i_*([g]) = [ig] = [i(f_1 \vee 1)\theta] = [f]$ and $i_* : [X, M \vee L_0(A,L)] \to [X, M \times L_0(A,L)]$ is surjective. (3) implies (1). Take $M = X$ and $L = A \wedge X$. Consider $(1 \times e'_A)\Delta : X \to X \times L_0(A, A \wedge X)$. Since i_* is an epimorphism, there is a map $\theta : X \to X \vee L_0(A, A \wedge X)$ such that $j\theta \sim (1 \times e'_A)\Delta$. Thus $e'_A : X \to L_0(A, A \wedge X)$ is cocyclic. □

Corollary 3.3. *If X is a co-H-space, then for any space A, X is a co-T_A-space.*

Corollary 3.4. *If X is a co-T-space, then for any space B, X is a co-$T_{\Sigma B}$-space.*

PROOF. Let C be any space. Then there is a homoeomorphism $h : L_0(\Sigma B, C) \to \Omega L_0(B, C)$ given by $h(f)(s)(b) = f(< s, b >)$. Let $f : X \to L_0(\Sigma B, C)$ be a map. Since X is co-T-space, $[X, \Omega L_0(B,C)] = DG(X, \Omega L_0(B,C))$. Thus we know that $hf : X \to \Omega L_0(B,C)$ is cocyclic and $f = h^{-1}(hf) : X \to L_0(\Sigma B, C)$ is cocyclic. Therefore X is a co-$T_{\Sigma B}$-space. □

Thus we know that any co-H-space is a co-T-space, and any co-T-space is a co-$T_{\Sigma B}$-space for any space B.

Theorem 3.5. *If a co-T_A-space X dominates Y, then Y is also a co-T_A-space.*

PROOF. Since X dominates Y, there are maps $r : X \to Y$ and $i : Y \to X$ such that $ri \sim 1_Y$. Consider the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{r} & Y \\ {\scriptstyle e'^X_A}\downarrow & & \downarrow{\scriptstyle e'^Y_A} \\ L_0(A, A \wedge X) & \xrightarrow{L_0(1\wedge r)} & L_0(A, A \wedge Y). \end{array}$$

Since $e'^X_A : X \to L_0(A, A \wedge X)$ is cocyclic, $e'^Y_A r = L_0(1 \wedge r)e'^X_A : X \to L_0(A, A \wedge Y)$ is cocyclic. It is known [10] that if $f : X \to A$ is cocyclic and $i : Y \to X$ has a left homotopy inverse, then $fi : Y \to A$ is cocyclic. Thus we have that $e'^Y_A \sim (e'^Y_A r)i : Y \to L_0(A, A \wedge Y)$ is cocyclic and Y is a co-T_A-space. □

Theorem 3.6. *$X \vee Y$ is a co-T_A-space if and only if X and Y are co-T_A-spaces.*

PROOF. Suppose $X \vee Y$ is a co-T_A-space. Let $r_1 = p_1 j : X \vee Y \to X$ and $r_2 = p_2 j : X \vee Y \to Y$, where $j : X \vee Y \to X \times Y$ is the inclusion and $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are projections. Then $r_1 i_1 = 1_X$ and $r_2 i_2 = 1_Y$. Then we know, from the above theorem, that X and Y are co-T_A-spaces. On the other hand, let X and Y be co-T_A-spaces. Then e'^X_A and e'^Y_A are cocyclic maps. It is known [L2] that if $f : X \to A$ and $g : Y \to B$ are cocyclic, then so is $f \vee g : X \vee Y \to A \vee B$. Thus we know that $e'^X_A \vee e'^Y_A : X \vee Y \to L_0(A, A \wedge X) \vee L_0(A, A \wedge Y)$ is cocyclic. Let $h : A \wedge (X \vee Y) \to (A \wedge X) \vee (A \wedge Y)$ be the natural homeomorphism. Define a map $k : L_0(A, A \wedge X) \vee L_0(A, A \wedge Y) \to L_0(A, (A \wedge X) \vee (A \wedge Y))$ by $k(f, *) = i_1 \circ f$, $k(*, g) = i_2 \circ g$, where $i_1 : A \wedge X \to (A \wedge X) \vee (A \wedge Y)$, $i_2 : A \wedge Y \to (A \wedge X) \vee (A \wedge Y)$ are natural inclusions. Then we have the following commutative diagram;

$$\begin{array}{ccc} X \vee Y & \xrightarrow{e'^{X\vee Y}_A} & L_0(A, A\wedge(X\vee Y)) \\ {\scriptstyle e'^X_A \vee e'^Y_A}\downarrow & & \downarrow{\scriptstyle L_0(h)} \\ L_0(A, A\wedge X)\vee L_0(A, A\wedge Y) & \xrightarrow{k} & L_0(A, A\wedge X\vee A\wedge Y). \end{array}$$

Since $e'^X_A \vee e'^Y_A$ is cocyclic and $L_0(h)\circ e'^{X\vee Y}_A = k\circ e'^X_A \vee e'^Y_A$, we know that $L_0(h)\circ e'^{X\vee Y}_A$ is cocyclic. Since $L_0(h)$ is a homotopy equivalence, we have that $e'^{X\vee Y}_A \sim L_0(h)^{-1}\circ L_0(h)\circ e'^{X\vee Y}_A$ is cocyclic and $X\vee Y$ is a co-T_A-space. □

Theorem 3.7. *If there exists a space Y such that X is a homotopy equivalent to $Y\vee K$ for a co-T_A-space K, then there are maps $r : X \to K$ and $i : K \to X$ such that $ri \sim 1$ and $e'^K_A r : X \to L_0(A, A\wedge K)$ is cocyclic.*

PROOF. Let $f : X \to Y\vee K$, $g : Y\vee K \to X$ be maps such that $gf \sim 1_X$ and $fg \sim 1_{Y\vee K}$. Let $r = P_2 f : X \to K$ and $i = gi_2 : K \to X$, where $i_2 : K \to Y\vee K$ is the inclusion and $p_2 : Y\vee K \to K$ be the projection. Then $ri = p_2 fgi_2 \sim p_2 i_2 = 1_K$. Moreover we have, from the fact that K is a co-T_A-space, that there exist a map $\mu : K \to K\vee L_0(A, A\wedge K)$ such that $j\mu \sim (1\times e'^K_A)\Delta$. Consider the map $\rho = (g\vee 1)\circ(1\vee\mu)\circ f : X \to X\vee L_0(A, A\wedge K)$. Then $j'\rho \sim (1\times e'^K_A r)\Delta$. Thus $e'^K_A r : X \to L_0(A, A\wedge K)$ is cocyclic. □

On the other hand, on the category of spaces of homotopy type of 1-connected locally finite CW complexes, we studied in [14] that some properties of T'-space which is another dual concept of T-space, and we also studied in [15] that some properties of T'_A which is a generalized concept of T'-space.

References

[1] J. Aguade, *Decomposable free loop spaces*, Can. J. Math. 39(1987), 938-955.

[2] A. Dold, *Partitions of unity in the theory of fibrations*, Ann. Math. 78(2)(1963), 223-255.

[3] H. B. Haslam, *G-spaces and H-spaces*, Ph. D. Thesis, Univ. of California, Irvine, 1969.

[4] S. S. Koh, *Note on the properties of the components of the mapping space* X^{S^q}, Proc. Amer. Math. Soc., 11(1960), 896-904.

[5] G. E. Lang, *The Evaluation map and EHP sequences*, Pacific J. Math., 44(1)(1973), 201-210.

[6] K. L. Lim, *On cyclic maps*, J. Austral. Math. Soc.,(Series A)32(1982), 349-357.

[7] K. L. Lim, *Cocyclic maps and coevaluation subgroups*, Canad. Math. Bull. 30(1987), 63-71.

[8] J. W. Milnor, *On spaces having the homotopy type of a CW complex*, Trans. Amer. Math. Soc. 90(1959), 272-280.

[9] J. Stasheff, *On homotopy abelian H-spaces*, Proc. Camb. Phil. Soc. 57(1961), 734.

[10] K. Varadarajan, *Genralized Gottlieb groups*, J. Indian Math. Soc. 33(1969), 141-164.

[11] M. H. Woo and Y. S. Yoon, *T-spaces by the Gottlieb groups and duality*, J. Austral. Math. Soc., (Series A)59(1995), 193-203.

[12] Y. S. Yoon, *On n-cyclic maps*, J. Korean. Math. Soc., 26(1)(1989), 17-25.

[13] Y. S. Yoon, *Decomposable reduced tori*, Math. Japonica, 39(3)(1994), 481-486.

[14] Y. S. Yoon, *Decomposability of Evaluation Fibrations*, Kyungpook Math. J., 35(1995), 361-369.

[15] Y. S. Yoon and J. O. Yu, *Decomposable right half smash product spaces*, Comm. Korean Math. Soc. 11(1996), 225-233.

[16] A. Zabrodsky, *Hopf Spaces*, North-Holland Pub. Com. 1976.

Yeon Soo Yoon
epartment of Mathematics
Hannam University
Daejeon 306-791, Korea
E-mail address : yoon@hannam.ac.kr