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Strongly Solid Varieties and Free Generalized Clones

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Abstract. Clones are sets of operations which are closed under composition and contain all projections. Identities of clones of term operations of a given algebra correspond to hyperidentities of this algebra, i.e., to identities which are satisfied after any replacements of fundamental operations by derived operations ([7]). If any identity of an algebra is satisfied as a hyperidentity, the algebra is called solid ([3]). Solid algebras correspond to free clones. These connections will be extended to so-called *generalized clones*, to *strong hyperidentities* and to *strongly solid* varieties. On the basis of a *generalized superposition operation* for terms we generalize the concept of a unitary Menger algebra of finite rank ([6]) to *unitary Menger algebras with infinitely many nullary operations* and prove that strong hyperidentities correspond to identities in free unitary Menger algebras with infinitely many nullary operations.

1. Menger algebras with infinitely many nullary operations

Generalized hypersubstitutions were introduced in [4] on the basis of a generalized superposition operation which is defined as an $(m+1)$-ary operation $S^m, m \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$, on the set of all terms of type τ. Here we consider terms of type τ_n defined by using of operation symbols $f_i, i \in I$, where f_i is n-ary for every $i \in I$ and by elements of the alphabet $X = \{x_1, \cdots, x_n, \cdots\}$. Let $W_{\tau_n}(X)$ be the set of all terms of type τ_n. Then for any $n \geq 1, n \in \mathbb{N}^+$, we define $S^n : W_{\tau_n}(X)^{n+1} \to W_{\tau_n}(X)$ inductively by the following steps:

Definition 1.1.

(i) If $t = x_i, 1 \leq i \leq n,$ then $S^n(x_i, t_1, \cdots, t_n) := t_i$ for $t_1, \cdots, t_n \in W_{\tau_n}(X)$.

(ii) If $t = x_i,\ n < i$, then $S^n(x_i, t_1, \cdots, t_n) := x_i$.

(iii) If $t = f_i(s_1, \cdots, s_n)$, then

$$S^n(t, t_1, \cdots, t_n) := f_i(S^n(s_1, t_1, \cdots, t_n), \cdots, S^n(s_n, t_1, \cdots, t_n)).$$

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Then we may consider the algebraic structure

$$clone_g\tau_n := (W_{\tau_n}(X); S^n, (x_i)_{i\in\mathbb{N}^+})$$

with the universe $W_{\tau_n}(X)$, with one $(n+1)$-ary operation and infinitely many nullary operations. This algebra is called a *generalized clone.* Now we prove

Theorem 1.2. *The algebra $clone_g\tau_n$ satisfies the following identities:*

(Cg1) $\tilde{S}^n(T, \tilde{S}^n(F_1, T_1, \cdots, T_n), \cdots, \tilde{S}^n(F_n, T_1, \cdots, T_n))$
$\approx \tilde{S}^n(\tilde{S}^n(T, F_1, \cdots, F_n), T_1, \cdots, T_n).$

(Cg2) $\tilde{S}^n(T, \lambda_1, \cdots, \lambda_n) = T.$

(Cg3) $\tilde{S}^n(\lambda_i, T_1, \cdots, T_n) = T_i$ *for* $1 \leq i \leq n.$

(Cg4) $\tilde{S}^n(\lambda_j, T_1, \cdots, T_n) = \lambda_j$ *for* $j > n.$

(*Here $\tilde{S}^n, \lambda_i$ are operation symbols corresponding to the operations S^n and x_i, $i \in \mathbb{N}^+$, respectively and T, T_j, F_i are new variables.*)

Proof. (Cg1) We give a proof by induction on the complexity of the term t.

(i) If $t = x_j, 1 \leq j \leq n$, then

$$\begin{array}{rl} & S^n(x_j, S^n(s_1, t_1, \cdots, t_n), \cdots S^n(s_n, t_1, \cdots, t_n)) \\ = & S^n(s_j, t_1, \cdots, t_n) \\ = & S^n(S^n(x_j, s_1, \cdots, s_n), t_1, \cdots, t_n). \end{array}$$

(ii) If $t = x_j, j > n$, then

$$\begin{array}{rl} & S^n(x_j, S^n(s_1, t_1, \cdots, t_n), \cdots S^n(s_n, t_1, \cdots, t_n)) \\ = & x_j \\ = & S^n(x_j, t_1, \cdots, t_n) \\ = & S^n(S^n(x_j, s_1, \cdots, s_n), t_1, \cdots, t_n). \end{array}$$

(iii) If $t = f_i(t'_1, \cdots, t'_n)$ and if we assume that our proposition is satisfied for $t'_1, \cdots, t'_n$, then

$$\begin{array}{rl} & S^n(f_i(t'_1, \cdots, t'_n), S^n(s_1, t_1, \cdots, t_n), \cdots, S^n(s_n, t_1, \cdots, t_n)) \\ = & f_i(S^n(t'_1, S^n(s_1, t_1 \cdots, t_n), \cdots, S^n(s_n, t_1, \cdots, t_n)), \cdots, \\ & S^n(t'_n, S^n(s_1, t_1, \cdots, t_n), \cdots, S^n(s_n, t_1, \cdots, t_n))) \\ = & f_i(S^n(S^n(t'_1, s_1, \cdots, s_n), t_1, \cdots, t_n), \cdots, S^n(S^n(t'_n, s_1, \cdots, s_n), t_1, \cdots, t_n)) \\ = & S^n(S^n(f_i(t'_1, \cdots, t'_n), s_1, \cdots, s_n), t_1, \cdots, t_n) \\ = & S^n(S^n(t, s_1, \cdots, s_n), t_1, \cdots, t_n). \end{array}$$

(Cg2) If t contains variables from the set $\{x_1, \cdots, x_n\}$, then we substitute in t for these variables the same variables and obtain t. If t contains a variable which does not belong to the set $\{x_1, \cdots, x_n\}$, then this variable will not be replaced by another term. Therefore the result is t.
(Cg3) and (Cg4) correspond to (i) and (ii), respectively, in the definition of S^n. $\square$

Algebras $(M; S^n)$ of type $\tau_n = (n+1)$, $n \geq 1$, which satisfy the so-called *superassociative law*

(C1) $S^n(x, S^n(z_1, y_1, \cdots, y_n), \cdots, S^n(z_n, y_1, \cdots, y_n)) \approx S^n(S^n(x, z_1, \cdots, z_n), y_1, \cdots, y_n)$ are called *Menger algebras of rank* n (see e.g. [6]). An algebra $(M; S^n, \lambda_1, \cdots, \lambda_n)$ of type $\tau_n = (n+1, 0, \cdots, 0)$ is called a *unitary Menger algebra of rank* n (see [6]) if $(M; S^n)$ is a Menger algebra of rank n and if the nullary fundamental operations $\lambda_1, \cdots, \lambda_n$ satisfy the identities

(C2) $S^n(\lambda_i, x_1, \cdots, x_n) = x_i$.

(C3) $S^n(x, \lambda_1, \cdots, \lambda_n) = x$.

The set of all terms of a given type built up by variables from an n-element alphabet $X_n = \{x_1, \cdots, x_n\}$ together with an $(n+1)$-ary superposition operation which is defined by (i) and (iii) from Definition 1.1 forms a Menger algebra of rank n. If we add the variables $x_1, \cdots, x_n$ as nullary operations, we have a unitary Menger algebra of rank n. Another example is the Menger algebra of all n-ary term operations of a given algebra $\mathcal{A}$ together with the superposition. Generalizing the concept of a Menger algebra we define

Definition 1.3. An algebra $(M; S^n, (e_i)_{i \in \mathbb{N}^+})$ where S^n is $(n+1)$-ary and $e_i, i \in \mathbb{N}^+$ are nullary, is called unitary Menger algebra with infinitely many nullary operations if (Cg1), (Cg2), (Cg3), (Cg4) are satisfied.

The class of all unitary Menger algebras with infinitely many nullary operations forms a variety V_M and the algebra $clone_g \tau_n$ belongs to this variety. In [6] Menger algebras were characterized by semigroups. If $(M; S^n)$ is an algebra of type $\tau_n = (n+1)$, then on the cartesian power M^n one can define a binary operation $* : (M^n)^2 \rightarrow M^n$ by $(x_1, \cdots, x_n) * (y_1, \cdots, y_n) := (S^n(x_1, y_1, \cdots, y_n), \cdots, S^n(x_n, y_1, \cdots, y_n))$. Then an algebra $(M; S^n)$ of type $\tau_n = (n+1)$ is a Menger algebra if and only if $(M^n; *)$ is a semigroup. This can be generalized to unitary Menger algebras with infinitely many nullary operations.

Let V_M be the variety of all unitary Menger algebras with infinitely many nullary operations. Let $\{X_i \mid i \in I\}$ be a new set of variables. This set is indexed by the index set I of the set of operation symbols of type τ_n. By $\tilde{S}^n$ we denote an $(n+1)$-ary operation symbol and let $(\lambda_i)_{i \in \mathbb{N}^+}$ be an indexed set of nullary operation symbols. Let $\mathcal{F}_{V_M}(\{X_i \mid i \in I\})$ be the free algebra with respect to the variety V_M, freely generated by $\{X_i \mid i \in I\}$. Then we have:

Theorem 1.4. *The algebra $clone_g \tau_n$ is isomorphic with the free algebra $\mathcal{F}_{V_M}(\{X_i \mid$*

$i \in I\}$) and therefore free with respect to the variety of unitary Menger algebras with infinitely many nullary operations.

Proof. We define a map $\varphi : W_{\tau_n}(X) \to F_{V_M}(\{X_i \mid i \in I\})$ inductively as follows:

(i) $\varphi(x_j) := \lambda_j, j \in \mathbb{N}^+$,

(ii) $\varphi(f_i(t_1, \cdots, t_n)) := \tilde{S}^n(X_i, \varphi(t_1), \cdots, \varphi(t_n))$.

We prove the homomorphism property

$$\varphi(S^n(t_0, t_1, \cdots, t_n)) = \tilde{S}^n(\varphi(t_0), \varphi(t_1), \cdots, \varphi(t_n))$$

by induction on the complexity of the term t_0.

$$\begin{array}{rcl} t_0 = x_j, 1 \leq j \leq n: & & \varphi(S^n(x_j, t_1, \cdots, t_n)) \\ & = & \varphi(t_j) \\ & = & \tilde{S}^n(\lambda_j, \varphi(t_1), \cdots, \varphi(t_n)) \\ & = & \tilde{S}^n(\varphi(t_0), \varphi(t_1), \cdots, \varphi(t_n)) \quad \text{by} \quad \text{(Cg3)}. \end{array}$$

$$\begin{array}{rcl} t_0 = x_k, k > n: & & \varphi(S^n(x_k, t_1, \cdots, t_n)) \\ & = & \varphi(x_k) \\ & = & \tilde{S}^n(\lambda_k, \varphi(t_1), \cdots \varphi(t_n)) \\ & = & \tilde{S}^n(\varphi(x_k), \varphi(t_1), \cdots, \varphi(t_n)) \quad \text{by} \quad \text{(Cg4)}. \end{array}$$

Inductively, assume that $t_0 = f_i(s_1, \cdots, s_n)$ and that $\varphi(S^n(s_j, t_1, \cdots, t_n)) = \tilde{S}^n(\varphi(s_j), \varphi(t_1), \cdots, \varphi(t_n))$ for all $1 \leq j \leq n$. Then

$$\begin{array}{rl} & \varphi(S^n(f_i(s_1, \cdots, s_n), t_1, \cdots, t_n)) \\ = & \varphi(f_i(S^n(s_1, t_1, \cdots, t_n), \cdots, S^n(s_n, t_1, \cdots, t_n))) \\ = & \tilde{S}^n(X_i, \varphi(S^n(s_1, t_1, \cdots, t_n)), \cdots, \varphi(S^n(s_n, t_1, \cdots, t_n))) \\ = & \tilde{S}^n(X_i, \tilde{S}^n(\varphi(s_1), \varphi(t_1), \cdots, \varphi(t_n)), \cdots, \tilde{S}^n(\varphi(s_n, \varphi(t_1), \cdots, \varphi(t_n)) \\ = & \tilde{S}^n(\tilde{S}^n(X_i, \varphi(s_1), \cdots, \varphi(s_n)), \varphi(t_1), \cdots, \varphi(t_n)) \\ = & \tilde{S}^n(\varphi(f_i(s_1, \cdots, s_n)), \varphi(t_1), \cdots, \varphi(t_n)). \end{array}$$

Thus φ is a homomorphism. It maps the generating set $F_{\tau_n} := \{f_i(x_1, \cdots, x_n) \mid i \in I\}$ of the algebra $clone_g\tau_n$ onto the set $\{X_i \mid i \in I\}$ since

$$\begin{array}{rl} & \varphi(f_i(x_1, \cdots, x_n) \\ = & \tilde{S}^n(X_i, \varphi(x_1), \cdots, \varphi(x_n)) \\ = & \tilde{S}^n(X_i, \lambda_1, \cdots, \lambda_n) \\ = & X_i \quad \text{by} \quad \text{(Cg2)}. \end{array}$$

Furthermore, since $\{X_i \mid i \in I\}$ is a free independent set we have

$$X_i = X_j \;\Rightarrow\; i = j \;\Rightarrow\; f_i(x_1, \cdots, x_n) = f_j(x_1, \cdots, x_n).$$

Thus φ is a bijection between the generating sets of $clone_g\tau_n$ and $\mathcal{F}_{V_M}(\{X_i \mid i \in I\})$. Altogether, φ is an isomorphism. □

2. Generalized hypersubstitutions and endomorphisms

In [4] the concept of a generalized hypersubstitution was defined with the aim to consider strong hyperidentities and strongly solid varieties.

Definition 2.1. A mapping $\sigma : \{f_i \mid i \in I\} \to W_{\tau_n}(X)$ is called a *generalized hypersubstitution* of type τ_n. If σ maps n-ary operation symbols to n-ary terms of type τ_n, it is called a *hypersubstitution* of type τ_n. Generalized hypersubstitutions can be inductively extended to mappings $\hat{\sigma}$ defined on $W_{\tau_n}(X)$ by

(i) $\hat{\sigma}[x_i] \;:=\; x_i \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \cdots, t_n)] \;:=\; S^n(\sigma(f_i), \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_n])$

Let $Hyp_G(\tau_n)$ be the set of all generalized hypersubstitutions of type τ_n and let $Hyp(\tau_n)$ be the set of all (arity-preserving) hypersubstitutions. We denote the hypersubstitution which maps the operation symbol f_i to the term $f_i(x_1, \cdots, x_n)$ for every $i \in I$ by σ_{id}.

(For arity-preserving hypersubstitutions see e.g. [3])

Proposition 2.2. *The extension of a generalized hypersubstitution is an endomorphism of the algebra $clone_g\tau_n$.*

Proof. Let S^n be the $(n+1)$-ary fundamental operation of the algebra $clone_g\tau_n$. Then we show by induction on the complexity of the term t that

$$\hat{\sigma}[S^n(t, t_1, \cdots, t_n)] \;=\; S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_n]). \qquad (*)$$

If $t = x_i,\ 1 \le i \le n$, then $\hat{\sigma}[S^n(x_i, t_1, \cdots, t_n)] = \hat{\sigma}[t_i] = S^n(\hat{\sigma}[x_i], \hat{\sigma}[t_1]), \cdots, \hat{\sigma}[t_n])$.

If $t = x_j,\ j > n$, then $\hat{\sigma}[S^n(x_j, t_1, \cdots, t_n)] = x_j = S^n(\hat{\sigma}[x_j], \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_n])$.

If $t = f_i(s_1, \cdots, s_n)$ and if we assume that equation $(*)$ is satisfied for $s_1, \cdots, s_n$, then

$$\begin{aligned}
& \hat{\sigma}[S^n(f_i(s_1, \cdots, s_n), t_1, \cdots, t_n))] \\
= \;& \hat{\sigma}[f_i(S^n(s_1, t_1, \cdots, t_n), \cdots, S^n(s_n, t_1, \cdots, t_n))] \\
= \;& S^n(\sigma(f_i), \hat{\sigma}[S^n(s_1, t_1, \cdots, t_n)], \cdots, \hat{\sigma}[S^n(s_n, t_1, \cdots, t_n)]) \\
= \;& S^n(\sigma(f_i), S^n(\hat{\sigma}[s_1], \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_n]), \cdots, S^n(\hat{\sigma}[s_n], \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_n])) \\
= \;& S^n(S^n(\sigma(f_i), \hat{\sigma}[s_1], \cdots, \hat{\sigma}[s_n]), \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_n]) \\
= \;& S^n(\hat{\sigma}[f_i(s_1, \cdots, s_n)], \hat{\sigma}[t_1], \cdots, \hat{\sigma}[t_n]).
\end{aligned}$$

For the nullary operations we have $\hat{\sigma}[x_j] = x_j$. □

The converse is also satisfied, i.e.,

Proposition 2.3. *Every endomorphism of $clone_g\tau_n$ is the extension of a generalized hypersubstitution.*

Proof. If $\varphi : clone_g\tau_n \to clone_g\tau_n$ is an endomorphism, then we consider the restriction of φ' of φ to the set $F_{\tau_n} := \{f_i(x_1, \cdots, x_n) \mid i \in I\}$ and form the mapping $\varphi' \circ \sigma_{id} : \{f_i \mid i \in I\} \to W_{\tau_n}(X)$. This mapping is a generalized hypersubstitution. We prove that φ is equal to the extension $(\varphi' \circ \sigma_{id})\,\hat{}$ of this generalized hypersubstitution. Indeed, $\varphi(x_i) = x_i = (\varphi' \circ \sigma_{id})\,\hat{}[x_i]$ for every variable x_i and for composite terms $f_i(t_1, \cdots, t_n)$ we have:

$$\begin{aligned} & \varphi(f_i(t_1, \cdots, t_n)) \\ = \;& \varphi(S^n(f_i(x_1, \cdots, x_n), t_1, \cdots, t_n)) \\ = \;& S^n(\varphi(f_i(x_1, \cdots, x_n), \varphi(t_1), \cdots, \varphi(t_n)) \\ = \;& S^n((\varphi' \circ \sigma_{id})(f_i)(\varphi' \circ \sigma_{id})\,\hat{}[t_1], \cdots, (\varphi' \circ \sigma_{id})\,\hat{}[t_n]) \\ = \;& (\varphi' \circ \sigma_{id})\,\hat{}[f_i(t_1, \cdots, t_n)]. \end{aligned}$$

□

It is easy to see that the set $Hyp_G(\tau_n)$ together with the binary operation $\circ_G$ defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and σ_{id} forms a monoid $(Hyp_G(\tau_n); \circ_G, \sigma_{id})$. Let $End(clone_g\tau_n)$ be the endomorphism monoid of $clone_g\tau_n$. Then we have:

Proposition 2.4. *The monoid $(Hyp_G(\tau_n); \circ_G, \sigma_{id})$ of all generalized hypersubstitutions is isomorphic with the endomorphism monoid $End(clone_g\tau_n)$.*

Proof. We consider the mapping $\psi : End(clone_g\tau_n) \to Hyp_G\tau_n$ defined by $\varphi \mapsto \varphi \circ \sigma_{id}$ which maps each endomorphism of $clone_g\tau_n$ to a generalized hypersubstitution. Clearly, ψ is well-defined and injective since from $\varphi \circ \sigma_{id} = \varphi' \circ \sigma_{id}$ by multiplication with σ_{id}^{-1} from the right hand side there follows $\varphi = \varphi'$. The mapping ψ is surjective since for $\sigma \in Hyp_G(\tau_n)$ the extension $\hat{\sigma}$ is an endomorphism and $\psi(\hat{\sigma}) = \hat{\sigma} \circ \sigma_{id} = \sigma$. Therefore, ψ is a bijection. Let $id_{W_{\tau_n}(X)}$ be the identity mapping on $W_{\tau_n}(X)$, then $\psi(id_{W_{\tau_n}(X)}) = id_{W_{\tau_n}(X)} \circ \sigma_{id} = \sigma_{id}$ and $\psi(\varphi_1 \circ \varphi_2) = (\varphi_1 \circ \varphi_2) \circ \sigma_{id} = \varphi_1 \circ (\varphi_2 \circ \sigma_{id}) = (\varphi_1 \circ \sigma_{id})\,\hat{}\circ(\varphi_2 \circ \sigma_{id}) = \psi(\varphi_1)\,\hat{}\circ\psi(\varphi_2) = \psi(\varphi_1) \circ_G \psi(\varphi_2)$. Here we used that φ_1 is equal to the extension of the hypersubstitution $\varphi_1 \circ \sigma_{id}$ (see the proof of Proposition 2.3). □

The algebra $clone_g\tau_n = (W_{\tau_n}(X); S^n$, The algebra $clone_g\tau_n = (W_{\tau_n}(X); S^n$, $(x_i)_{i \in \mathbb{N}^+})$ is generated by the set $F_{\tau_n} = \{f_i(x_1, \cdots, x_n) \mid i \in I\}$. Any mapping from F_{τ_n} to $W_{\tau_n}(X)$ is called a *generalized clone substitution.* Since $clone_g\tau_n$ is free, every generalized clone substitution can be uniquely extended to an endomorphism of the algebra $clone_g\tau_n$. Let $Subst_G$ be the set of all generalized clone substitutions. We introduce a binary composition operation $\otimes$ on this set, by setting $\eta_1 \otimes \eta_2 := \overline{\eta_1} \circ \eta_2$ where $\circ$ denotes the usual composition f functions. Denoting by $id_{F_{\tau_n}}$ the identity mapping on F_{τ_n} we see that $(Subst_G; \otimes, id_{F_{\tau_n}})$ is a monoid. Further we have:

Proposition 2.5. *The monoids* $(Subst_G; \otimes, id_{F_{\tau_n}})$ *and* $(Hyp_G(\tau_n); \circ_G, \sigma_{id})$ *are isomorphic.*

3. Strong hyperidentities and identities in generalized clones

Generalized hypersubstitutions can be used to define strong hyperidentities in algebras or in varieties ([4]).

Definition 3.1. Let V be a variety of type τ_n and let $Id\ V$ be the set of all identities satisfied in V. An identity $s \approx t \in Id\ V$ is called a *strong hyperidentity* in V ([4]) if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\ V$ for all generalized hypersubstitutions $\sigma \in Hyp_G(\tau_n)$.

We consider the following example: Let *Rec* be the variety of semigroups which is defined by the following identity: $x_1x_2x_3 \approx x_1x_3$, i.e., $Rec := Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3\}$. We want to prove that the associative law is a strong hyperidentity in *Rec*. We introduce the binary operation symbol F and write the associative identity in the form $F(x_1, F(x_2, x_3)) \approx F(F(x_1, x_2), x_3)$. Arbitrary n-ary terms over the variety *Rec* have the form x_i or $x_ix_j, i, j \in \mathbb{N}^+$. Then we have

$$\hat{\sigma}_{x_i}[F(x_1, F(x_2, x_3))] = x_i = \hat{\sigma}_{x_i}[F(F(x_1, x_2), x_3)] = \hat{\sigma}_{x_i}[F(x_1, x_3)] \quad \text{if} \quad i \neq 2$$

and

$$\hat{\sigma}_{x_2}[F(x_1, F(x_2, x_3))] = x_3 = \hat{\sigma}_{x_2}[F(F(x_1, x_2), x_3)] = \hat{\sigma}_{x_2}[F(x_1, x_3)].$$

Actually, if $i, j \in \{1, 2\}$, we can use that the variety *Rec* is solid and therefore its identities are closed under any application of arity-preserving hypersubstitutions. Then the following cases are left:

$$\begin{aligned} \hat{\sigma}_{x_ix_j}[F(x_1, F(x_2, x_3))] &= S^2(x_ix_j, x_1, S^2(x_ix_j, x_2, x_3)) \\ &= \begin{cases} x_1x_j & \text{if } i = 1, j > 2, \\ x_3x_jx_j & \text{if } i = 2, j > 2, \\ x_ix_1 & \text{if } j = 1, i > 2, \\ x_ix_ix_3 & \text{if } j = 2, i > 2, \\ x_ix_j & \text{if } i, j > 2 \end{cases} \end{aligned}$$

and further we have

$$\begin{aligned} \hat{\sigma}_{x_ix_j}[F(F(x_1, x_2), x_3)] &= S^2(x_ix_j, S^2(x_ix_j, x_1, x_2), x_3) \\ &= \begin{cases} x_1x_jx_j & \text{if } i = 1, j > 2, \\ x_3x_j & \text{if } i = 2, j > 2, \\ x_ix_ix_1 & \text{if } j = 1, i > 2, \\ x_ix_3 & \text{if } j = 2, i > 2, \\ x_ix_j & \text{if } i, j > 2. \end{cases} \end{aligned}$$

Because of $x_1x_j \approx x_1x_jx_j, x_3x_jx_j \approx x_3x_j, x_ix_1 \approx x_ix_ix_1, x_ix_ix_3 \approx x_ix_3$, this shows that the associative law is a strong hyperidentity in *Rec*.

If every identity of a variety V is satisfied as a strong hyperidentity, the variety is called *strongly solid*. It is not difficult to check that the variety Rec is a strongly solid variety of semigroups. Indeed, we have

$$\begin{array}{rcl} \hat{\sigma}_{x_i x_j}[F(x_1,x_3)] & = & S^2(x_i x_j, x_1, x_3) \\ & = & \begin{cases} x_1 x_j & if\ i=1, j>2, \\ x_3 x_j & if\ i=2, j>2, \\ x_i x_1 & if\ j=1, i>2, \\ x_i x_3 & if\ j=2, i>2, \\ x_i x_j & if\ i,j>2. \end{cases} \end{array}$$

Because of $x_1 x_j x_j \approx x_1 x_j, x_i x_i x_1 \approx x_i x_1$ the equation $F(F(x_1,x_2),x_3) \approx F(x_1,x_3)$ is a strong hyperidentity. This proves that the variety Rec is strongly solid since it is enough to consider the identities from a basis.

On the set $W_{\tau_n}(X)$ of all terms of type τ_n the operation symbols $f_i, i \in I$, we define operations of type τ_n by

$$\overline{f_i} : W_{\tau_n}(X)^n \to W_{\tau_n}(X) \quad \text{with} \quad \overline{f_i}(t_1,\cdots,t_n) := f_i(t_1,\cdots,t_n).$$

Together with these operations one obtains the *absolutely free algebra* $\mathcal{F}_{\tau_n}(X) := (W_{\tau_n}(X); (\overline{f_i})_{i\in I})$ of type τ_n. If V is a variety of type τ_n, then IdV, the set of all identities satisfied in V, forms a fully invariant congruence on $\mathcal{F}_{\tau_n}(X)$, i.e., a congruence which is closed under all endomorphisms of $\mathcal{F}_{\tau_n}(X)$. Now we prove that $Id\ V$ is also a congruence relation on the unitary Menger algebra $clone_g \tau_n$ with infinitely many nullary operations.

Theorem 3.2. *Let V be a variety of type τ_n and let $Id\ V$ be the set of all identities satisfied in V. Then $Id\ V$ is a congruence relation on $clone_g \tau_n$.*

Proof. At first we prove by induction on the complexity of the term t that for every $n \in \mathbb{N}^+$ from $t_1 \approx s_1, \cdots, t_n \approx s_n \in Id\ V$ follows $S^n(t,t_1,\cdots,t_n) \approx S^n(s,s_1,\cdots,s_n) \in Id\ V$.

(a) $t = x_i, 1 \leq i \leq n$: Then $S^n(t,t_1,\cdots,t_n) = t_i \approx s_i = S^n(t,s_1,\cdots,s_n) \in Id\ V$.

(b) $t = x_j, j > n$: Then $S^n(t,t_1,\cdots,t_n) = x_j \approx x_j = S^n(t,s_1,\cdots,s_n) \in Id\ V$.

(c) Assume now that $t = f_i(l_1,\cdots,l_n)$ and that for $l_j,\ 1 \leq j \leq n$ we have already $S^n(l_j,t_1,\cdots,t_n) \approx S^n(l_j,s_1,\cdots,s_n) \in Id\ V,\ 1 \leq j \leq n$. Then

$$\begin{array}{rl} & S^n(t,t_1,\cdots,t_n) \\ = & f_i(S^n(l_1,t_1,\cdots,t_n),\cdots,S^n(l_n,t_1,\cdots,t_n)) \\ \approx & f_i(S^n(l_1,s_1\cdots,\cdots,s_n),\cdots,S^n(l_n,s_1,\cdots,s_n)) \in Id\ V \end{array}$$

since $Id\ V$ is compatible with the operations of $\mathcal{F}_{\tau_n}(X)$.

The next step consists in showing

$$t \approx s \quad \Rightarrow \quad S^n(t, s_1, \cdots, s_n) \approx S^n(s, s_1, \cdots, s_n) \in Id\ V.$$

Since $Id\ V$ is a fully invariant congruence on $\mathcal{F}_{\tau_n}(X)$ from $t \approx s \in Id\ V$ we obtain $S^n(t, s_1, \cdots, s_n) \approx S^n(s, s_1, \cdots, s_n) \in Id\ V$ by substitution if t and s contain only variables from $X = \{x_1, \cdots, x_n\}$. Variables $x_j, j < n$, will not be changed. Therefore in all cases we get $S^n(t, s_1, \cdots, s_n) \approx S^n(s, s_1, \cdots, s_n)$.
Assume now that $t \approx s, t_1 \approx s_1, \cdots, t_n \approx s_n \in Id\ V$. Then

$$S^n(t, t_1, \cdots, t_n) \approx S^n(t, s_1, \cdots, s_n) \approx S^n(s, s_1, \cdots, s_n) \approx S^n(s, t_1, \cdots, t_n)) \in Id\ V.$$

The compatibility with the nullary operations of $clone_g\tau_n$ is also clear. □

Further we have:

Theorem 3.3. *Let V be a variety of type τ_n. Then V is strongly solid if and only if $Id\ V$ is a fully invariant congruence relation on $clone_g\tau_n$.*

Proof. Let V be strongly solid, let $s \approx t \in Id\ V$ and let $\varphi : clone_g\tau_n \to clone_g\tau_n$ be an endomorphism of $clone_g\tau_n(\varphi \in End(clone_g\tau_n))$. Then we have

$$\varphi(s) \ = \ (\varphi \circ \sigma_{id})\hat{}\,[s] \approx (\varphi \circ \sigma_{id})\hat{}\,[t] \ = \ \varphi(t) \in Id\ V$$

since $\varphi \circ \sigma_{id}$ is a generalized hypersubstitution with $\varphi = (\varphi \circ \sigma_{id})\hat{}$ (see Proposition 2.4 and the proof of Proposition 2.3). Therefore $Id\ V$ is fully invariant. If conversely $Id\ V$ is fully invariant, $s \approx t \in Id\ V$ and let $\sigma \in Hyp_G(\tau_n)$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\ V$ since by Proposition 2.2 the extension of a generalized hypersubstitution is a clone endomorpism. This shows that every identity $s \approx t \in Id\ V$ is satisfied as a strong hyperidentity and then V is strongly solid. □

Since $Id\ V$ is a congruence relation on $clone_g\tau_n$, we may form the quotient algebra $clone_gV := clone_g\tau_n/Id\ V$. The operations $\tilde{S}^n$ of this algebra are defined as usual by

$$\tilde{S}^n([t]_{Id\ V}, [t_1]_{Id\ V}, \cdots, [t_n]_{Id\ V}) := [S^n(t, t_1, \cdots, t_n)]_{Id\ V}.$$

The nullary operations are $[x_i]_{Id\ V}, i \in \mathbb{N}^+$. Since for a strongly solid variety V the relation $Id\ V$ is fully invariant on $clone_g\tau_n$, it corresponds to a fully invariant congruence on the absolutely free algebra of the type of unitary Menger algebras with infinitely many nullary operations (see [1]). Fully invariant congruences on absolutely free algebras of a given type correspond to equational theories, i.e., to sets of identities of certain varieties. Therefore we have:

Theorem 3.4. *Let V be a variety of type τ_n and let $s \approx t \in Id\ V$. Then $s \approx t$ is a strong hyperidentity in V iff $s \approx t$ is an identity in $clone_gV$.*

Proof. Assume at first that $s \approx t$ is a strong hyperidentity of V. This means that for each $\sigma \in Hyp_G(\tau_n)$ we have $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\ V$. Let $v : \{f_i(x_1, \cdots, x_n) \mid$

$i \in I\} \to clone_g\tau_n$ be a valuation mapping and let Φ be a choice function which chooses from each block with respect to $Id\ V$ exactly one element and from the class $[x_j]_{Id\ V}$ the variable x_j. Then the mapping $\eta := \Phi \circ v$ is a clone substitution of $clone_g\tau_n$. Let $nat(Id\ V)$ be the natural mapping which maps each term t to the class $[t]_{Id\ V}$. Under the isomorphism between clone substitutions and generalized hypersubstitutions from Proposition 2.5 the generalized hypersubstitution which corresponds to η is $\eta \circ \sigma_{id}$. Since $clone_g\tau_n$ is free, freely generated by $F_{\tau_n} := \{f_i(x_1, \cdots, x_n) \mid i \in I\}$, the extension $\overline{\eta}$ is an endomorphism of $clone_g\tau_n$ and is uniquely determined. Therefore, we have $\overline{\eta} = (\eta \circ \sigma_{id})\hat{}$. A consequence of the freeness of $clone_g\tau_n$ is that from $v = nat(IdV) \circ \eta$ there follows $\overline{v} = natId\ V \circ \overline{\eta}$. Then we obtain from $s \approx t \in Id\ V$

$$\begin{aligned} \overline{v}(s) &= (nat(Id\ V) \circ \overline{\eta})(s) \\ &= nat(Id\ V)((\eta \circ \sigma_{id})\hat{}\,[s]) = nat(Id\ V)((\eta \circ \sigma_{id})\hat{}\,[t]) = \overline{v}(t). \end{aligned}$$

This shows that $s \approx t$ is an identity in $clone_gV$.
Conversely, assume that $s \approx t$ is an identity in $clone_gV$ and let σ be a generalized hypersubstitution. Then $nat(Id\ V) \circ \sigma \circ \sigma_{id}^{-1}$ is a valuation mapping and $nat(Id\ V)(\hat{\sigma}[s]) = (nat(Id\ V) \circ \overline{(\sigma \circ \sigma_{id}^{-1})})(s) = \overline{nat(Id\ V) \circ \sigma \circ \sigma_{id}^{-1}}(s) = \overline{natId\ V \circ \sigma \circ \sigma_{id}^{-1}}(t) = nat(Id\ V)(\hat{\sigma}[t])$ using $\overline{\sigma \circ \sigma_{id}^{-1}}$. This means $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in Id\ V$ and $s \approx t$ is a strong hyperidentity. □

Corollary 3.5. *Let V be a variety of type τ_n. Then V is strongly solid iff $clone_gV$ is free with respect to itself, freely generated by the set $\{[f_i(x_1, \cdots, x_n)]_{Id\ V} \mid i \in I\}$, meaning that every mapping from $\{[f_i(x_1, \cdots, x_n)]_{Id\ V} \mid i \in I\}$ to the universe of $clone_gV$ can be extended to an endomorphism of $clone_gV$.*

Proof. If V is strongly solid, then by Theorem 3.3, $Id\ V$ is a fully invariant congruence relation on $clone_g\tau_n$. The algebra $clone_g\tau_n$ is the quotient algebra of the absolutely free algebra $\mathcal{F}(\{X_i \mid i \in I\})$ of the type of unitary Menger algebras with infinitely many nullary operations . Using the "Correspondence Teorem" (Theorem 6.20 in [1]) the fully invariant congruence relation $Id\ V$ on $clone_g\tau_n$ corresponds to a fully invariant congruence θ on $\mathcal{F}(\{X_i \mid i \in I\})$ and then the quotient algebra $\mathcal{F}(\{X_i \mid i \in I\})/\theta$ is free with respect to itself ([1], Lemma 14.7) and is by the isomorphism theorem isomorphic to $clone_gV$.

For the converse direction we use Theorem 3.4, and we will show that V is strongly solid if every identity $s \approx t \in Id\ V$ is also an identity in $clone_gV$. Suppose that $clone_gV$ is free with respect to itself, freely generated by the set $\{[f_i(x_1, \cdots, x_n)]_{Id\ V} \mid i \in I\}$. Let $s \approx t$ be any identity in $Id\ V$. To show that $s \approx t$ is an identity in $clone_gV$, we will show that $\overline{v}(s) = \overline{v}(t)$ for any valuation mapping $v : F_{\tau_n} \longrightarrow clone_gV$. Given v, we define a mapping $\alpha_v : \{[f_i(x_1, \cdots, x_n)]_{IdV} \mid i \in I\} \longrightarrow clone_gV$ by $\alpha_v([f_i(x_1, \cdots, x_n)]_{Id\ V}) = v(f_i(x_1, \cdots, x_n))$. Since $clone_gV$ is free with respect to itself and is freely generated by the independent set $\{[f_i(x_1, \cdots, x_n)]_{Id\ V} \mid i \in I\}$, we get

$$\begin{array}{rrcl} & [f_i(x_1,\cdots,x_n)]_{Id\ V} & = & [f_j(x_1,\cdots,x_n)]_{Id\ V} \\ \Longrightarrow & i & = & j \\ \Longrightarrow & f_i(x_1,\cdots,x_n) & = & f_j(x_1,\cdots,x_n) \\ \Longrightarrow & v(f_i(x_1,\cdots,x_n)) & = & v(f_j(x_1,\cdots,x_n)) \\ \Longrightarrow & \alpha_v([f_i(x_1,\cdots,x_n)]_{Id\ V}) & = & \alpha_v([f_j(x_1,\cdots,x_n)]_{Id\ V}), \end{array}$$

and the mapping α_v is well-defined. Since the set F_{τ_n} generates the free algebra $clone_g\tau_n$, the mapping v can be uniquely extended to the homomorphism $\overline{v} : clone_g\tau_n \to clone_gV$. Then we have

$$s \approx t \implies [s]_{Id\ V} = [t]_{Id\ V} \implies \overline{\alpha}_v([s]_{Id\ V}) = \overline{\alpha}_v([t]_{Id\ V}) \implies \overline{v}(s) = \overline{v}(t),$$

showing that $s \approx t \in Id(clone_gV)$. □

(For algebras which are free with respect to itself and for free independent sets see e.g. [2] and [5])

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