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Some Theorems Connecting the Unified Fractional Integral Operators and the Laplace Transform

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Abstract. In the present paper, we obtain two Theorems connecting the unified fractional integral operators and the Laplace transform. Due to the presence of a general class of polynomials, the multivariable H-function and general functions θ and ϕ in the kernels of our operators, a large number of (new and known) interesting results involving simpler polynomials (which are special cases of a general class of polynomials) and special functions involving one or more variables (which are particular cases of the multivariable H-function) obtained by several authors and hitherto lying scattered in the literature follow as special cases of our findings. Thus the Theorems obtained by Srivastava et al. [9] follow as simple special cases of our findings.

1. Introduction and definitions

The field of fractional calculus is presently receiving keen attention by many researchers. The monograph on the subject [7] gives fairly good account of the developments in fractional calculus.

In this paper we study the following fractional integral operators which are generalizations of the operators studied recently [2], [3].

$$
\begin{aligned}
(1.1)\qquad & R_x^{\eta,\alpha}\left[f(x)\right] \\
= \; & R^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\cdots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\cdots,u_r,v_r}_{x;e;z_1,\cdots,z_r,a_j,\alpha'_j,\cdots,\alpha_j^{(r)},b_j,\beta'_j,\cdots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\cdots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}\left[f\left(x\right)\right] \\
= \; & x^{-\eta-\alpha-1}\int_0^x t^{\eta}\left(x-t\right)^{\alpha} S_n^m\left[e\left(\frac{t}{x}\right)^{\mu}\left(1-\frac{t}{x}\right)^{\nu}\right] \\
& H\left[z_1\left(\frac{t}{x}\right)^{u_1}\left(1-\frac{t}{x}\right)^{v_1},\cdots,z_r\left(\frac{t}{x}\right)^{u_r}\left(1-\frac{t}{x}\right)^{v_r}\right]\theta\left(\frac{t}{x}\right)f\left(t\right)\,dt
\end{aligned}
$$

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and

$$(1.2)\qquad \begin{aligned} & W_x^{\eta,\alpha}\left[f(x)\right] \\ = \ & W^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\cdots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\cdots,u_r,v_r}_{x;e;z_1,\cdots,z_r,a_j,\alpha'_j,\cdots,\alpha_j^{(r)},b_j,\beta'_j,\cdots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\cdots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}\left[f\left(x\right)\right] \\ = \ & x^{\eta}\int_x^{\infty} t^{-\eta-\alpha-1}\left(t-x\right)^{\alpha} S_n^m\left[e\left(\frac{x}{t}\right)^{\mu}\left(1-\frac{x}{t}\right)^{\nu}\right] \\ & H\ \left[z_1\left(\frac{x}{t}\right)^{u_1}\left(1-\frac{x}{t}\right)^{v_1},\cdots,z_r\left(\frac{x}{t}\right)^{u_r}\left(1-\frac{x}{t}\right)^{v_r}\right]\phi\left(\frac{x}{t}\right)f\left(t\right)\ dt \end{aligned}$$

provided that the general θ and ϕ functions and f are so specified that the integrals (1.1) and (1.2) exist. The detailed conditions of existence of the fractional integral operators defined by (1.1) and (1.2) can be worked out for specific θ and ϕ functions by following the procedure given earlier [3].

Here, $S_n^m[x]$ denotes the general class of polynomials introduced by Srivastava [5, p.1,eq.(1)]

$$(1.3)\qquad S_n^m[x]=\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}}{k!}A_{n,k}x^k,\quad n=0,\ 1,\ 2,\ \cdots$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}$ $(n,k\geq 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould-Hopper polynomials, the Brafman polynomials and several others [10, pp. 158-161].

The H-function of r complex variables $z_1,\cdots,z_r$ was introduced by Srivastava and Panda [8]. In this paper we shall define and represent it in the following from [6, p.251, eq. (C.1)]

$$(1.4)\qquad \begin{aligned} & H\left[z_1,\cdots,z_r\right] \\ = \ & H^{0,N:\ M',N'\ ;\cdots;\ M^{(r)},N^{(r)}}_{P,Q:\ P',Q'\ ;\cdots;\ P^{(r)},Q^{(r)}} \\ & \left[\begin{array}{c|c} z_1 & \left(a_j;\ \alpha'_j,\cdots,\alpha_j^{(r)}\right)_{1,P}:\left(c'_j,\ \gamma'_j\right)_{1,P'};\cdots;\left(c_j^{(r)},\ \gamma_j^{(r)}\right)_{1,P^{(r)}} \\ \vdots & \\ z_r & \left(b_j;\beta'_j,\cdots,\beta_j^{(r)}\right)_{1,\ Q}:\left(d'_j,\delta'_j\right)_{1,Q'};\cdots;\left(d_j^{(r)},\ \delta_j^{(r)}\right)_{1,Q^{(r)}} \end{array}\right] \\ = \ & \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi_1(\xi_1)\cdots\phi_r(\xi_r)\psi(\xi_1,\cdots,\xi_r)\ z_1^{\xi_1}\cdots z_r^{\xi_r}d\xi_1\cdots d\xi_r \end{aligned}$$

where

(1.5)

$$\phi_i(\xi_i)=\frac{\prod\limits_{j=1}^{M^{(i)}}\Gamma\big(d_j^{(i)}-\delta_j^{(i)}\xi_i\big)\prod\limits_{j=1}^{N^{(i)}}\Gamma\big(1-c_j^{(i)}+\gamma_j^{(i)}\xi_i\big)}{\prod\limits_{j=M^{(i)}+1}^{Q^{(i)}}\Gamma\big(1-d_j^{(i)}+\delta_j^{(i)}\xi_i\big)\prod\limits_{j=N(i)+1}^{P^{(i)}}\Gamma\big(c_j^{(i)}-\gamma_j^{(i)}\xi_i\big)},\quad\forall i\in\{1,\cdots,r\}$$

$$\psi(\xi_1,\cdots,\xi_r)=\frac{\prod_{j=1}^{N}\Gamma\Big(1-a_j+\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\Big)}{\prod_{j=N+1}^{P}\Gamma\Big(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}\xi_i\Big)\prod_{j=1}^{Q}\Gamma\Big(1-b_j+\sum_{i=1}^{r}\beta_j^{(i)}\xi_i\Big)}. \tag{1.6}$$

The nature of contours $L_1,\cdots,L_r$ in (1.4), the various special cases and other details of the above function can be found in the book referred to above. It may be remarked here that all the Greek letters occurring in the left-hand side of (1.4) are assumed to be positive real numbers for standardization purposes; the definition of this function will, however, be meaningful even if some of these quantities are zero. Again, it is assumed throughout the present work that this function always satisfies the appropriate existence and convergence conditions of its defining integral [6, pp. 252-253, eqs. (C.4-C.6)].

The main object of this paper is to establish two Theorems connecting the fractional integral operators defined by (1.1) and (1.2) and the Laplace transform defined by

$$L\big\{f(x);\ s\big\}\ =\ \int_0^\infty e^{-sx}f(x)dx\ =\ F(s). \tag{1.7}$$

2. Main theorems

Theorem 1. *If*

$$L\big\{f(x);s\big\}\ =\ \int_0^\infty e^{-sx}f(x)dx \tag{2.1}$$

then

$$\begin{aligned} &L\big\{R[f(x)];s\big\} \\ =\ &L\big\{R^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\cdots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\cdots,u_r,v_r}_{x;e;z_1,\cdots,z_r,a_j,\alpha'_j,\cdots,\alpha_j^{(r)},b_j,\beta'_j,\cdots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\cdots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}\ f(x);s\big\} \\ =\ &\int_0^\infty t^{\eta}f(t)\ A_1(s,t)\ dt \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} &L\big\{W[f(x)];s\big\} \\ =\ &L\big\{W^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\cdots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\cdots,u_r,v_r}_{x;e;z_1,\cdots,z_r,a_j,\alpha'_j,\cdots,\alpha_j^{(r)},b_j,\beta'_j,\cdots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\cdots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}}\ f(x);s\big\} \\ =\ &\int_0^\infty t^{-\eta-\alpha-1}f(t)\ B_1(s,t)dt \end{aligned} \tag{2.3}$$

where

$$(2.4)\qquad A_1(s,t) = L\Big\{x^{-\eta-\alpha-1}(x-t)^{\alpha}\, S_n^m\left[e\Big(\frac{t}{x}\Big)^{\mu}\Big(1-\frac{t}{x}\Big)^{\nu}\right] H\left[z_1\Big(\frac{t}{x}\Big)^{u_1}\Big(1-\frac{t}{x}\Big)^{v_1},\cdots,z_r\Big(\frac{t}{x}\Big)^{u_r}\Big(1-\frac{t}{x}\Big)^{v_r}\right]\theta\Big(\frac{t}{x}\Big)U(x-t);s\Big\},$$

$$(2.5)\qquad B_1(s,t) = L\Big\{x^{\eta}(t-x)^{\alpha} S_n^m\left[e\Big(\frac{x}{t}\Big)^{\mu}\Big(1-\frac{x}{t}\Big)^{\nu}\right] H\left[z_1\Big(\frac{x}{t}\Big)^{u_1}\Big(1-\frac{x}{t}\Big)^{v_1},\cdots,z_r\Big(\frac{x}{t}\Big)^{u_r}\Big(1-\frac{x}{t}\Big)^{v_r}\right]\phi\Big(\frac{x}{t}\Big)U(t-x);s\Big\}$$

and U *denotes the well known unit step function. The functions* θ, ϕ *and* f *are so specified that the integrals* (2.2), (2.3), (2.4) *and* (2.5) *exist and* $\mathrm{Re}(s) > 0$.

Proof. From (2.1) and (1.1), we get

$$(2.6)\qquad L\{R[f(x)];\ s\} = \int_0^{\infty} e^{-sx}\Big\{x^{-\eta-\alpha-1}\int_0^{x} t^{\eta}(x-t)^{\alpha} S_n^m\Big[e\Big(\frac{t}{x}\Big)^{\mu}\Big(1-\frac{t}{x}\Big)^{\nu}\Big] H\Big[z_1\Big(\frac{t}{x}\Big)^{u_1}\Big(1-\frac{t}{x}\Big)^{v_1},\cdots,z_r\Big(\frac{t}{x}\Big)^{u_r}\Big(1-\frac{t}{x}\Big)^{v_r}\Big]\theta\Big(\frac{t}{x}\Big)f(t)\,dt\Big\}\,dx.$$

On changing the order of integration in the above equation (which is permissible under the conditions stated) we find that

$$(2.7)\qquad L\{R[f(x)];\ s\} = \int_0^{\infty} t^{\eta} f(t)\Big\{\int_t^{\infty} e^{-sx}x^{-\eta-\alpha-1}(x-t)^{\alpha} S_n^m\Big[e\Big(\frac{t}{x}\Big)^{\mu}\Big(1-\frac{t}{x}\Big)^{\nu}\Big] H\Big[z_1\Big(\frac{t}{x}\Big)^{u_1}\Big(1-\frac{t}{x}\Big)^{v_1},\cdots,z_r\Big(\frac{t}{x}\Big)^{u_r}\Big(1-\frac{t}{x}\Big)^{v_r}\Big]\theta\Big(\frac{t}{x}\Big)\,dx\Big\}\,dt.$$

Now making use of definition of the unit step function, (2.7) easily yields formula (2.2). The second assertion (2.3) of Theorem 1 can be established by proceeding in a similar manner. □

Theorem 2. *Let $F(s)$, the Laplace transform of $f(x)$ as defined by* (1.7) *exist, then*

(2.8)
$$
\begin{aligned}
&R[F(s)]\\
&= R^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\cdots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\cdots,u_r,v_r}_{x;e;z_1,\cdots,z_r,a_j,\alpha'_j,\cdots,\alpha_j^{(r)},b_j,\beta'_j,\cdots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\cdots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}} F(s)\\
&= s^{-\eta-\alpha-1}\int_0^\infty f(x)\, B_2(x,s)\, dx
\end{aligned}
$$

and

(2.9)
$$
\begin{aligned}
&W[F(s)]\\
&= W^{\eta,\alpha;m,n,\mu,\nu;N,P,Q,M',N',P',Q',\cdots,M^{(r)},N^{(r)},P^{(r)},Q^{(r)},u_1,v_1,\cdots,u_r,v_r}_{x;e;z_1,\cdots,z_r,a_j,\alpha'_j,\ldots,\alpha_j^{(r)},b_j,\beta'_j,\cdots,\beta_j^{(r)},c'_j,\gamma'_j,d'_j,\delta'_j,\cdots,c_j^{(r)},\gamma_j^{(r)},d_j^{(r)},\delta_j^{(r)}} F(s)\\
&= s^{\eta}\int_0^\infty f(x) A_2(x,s) dx
\end{aligned}
$$

where

(2.10)
$$
\begin{aligned}
&B_2(x,s)\\
&= L\Bigg\{ t^{\eta}(s-t)^{\alpha} S_n^m\Big[e\Big(\frac{t}{s}\Big)^{\mu}\Big(1-\frac{t}{s}\Big)^{\nu}\Big]\ H\Big[z_1\Big(\frac{t}{s}\Big)^{u_1}\Big(1-\frac{t}{s}\Big)^{v_1},\cdots,\\
&\quad z_r\Big(\frac{t}{s}\Big)^{u_r}\Big(1-\frac{t}{s}\Big)^{v_r}\Big]\theta\Big(\frac{t}{}\\
&\quad \frac{}{t}\Big)^{\mu}\Big(1-\frac{s}{t}\Big)^{\nu}\Big]\ H\Big[z_1\Big(\frac{s}{t}\Big)^{u_1}\Big(1-\frac{s}{t}\Big)^{v_1},\cdots,\\
&\quad z_r\Big(\frac{s}{t}\Big)^{u_r}\Big(1-\frac{s}{t}\Big)^{v_r}\Big]\phi\Big(\frac{s}{t}\Big)U(t-s)\ ;x\Bigg\}
\end{aligned}
$$

and the functions θ, ϕ and f are so specified that the integrals (2.8), (2.9), (2.10) *and* (2.11) *exist and* $\mathrm{Re}(s) > 0$.

Proof. The proof of Theorem 2 is much akin to that of Theorem 1, and we omit the details. □

3. Special cases and applications

The Theorems 1 and 2 established here are unified in nature and act as key formulae, due to the presence of the product of a general class of polynomials, the multivariable H-function and the general θ and ϕ functions in the kernels of the fractional integral operators given by (1.1) and (1.2). Thus the general class of polynomials involved in the Theorems 1 and 2 reduce to a large spectrum of polynomials listed by Srivastava and Singh [10, pp. 158-161], and so from the Theorems 1 and 2 we can further obtain various Theorems involving a number of simpler polynomials. Again, the multivariable H-function occurring in these formulae can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of E, F, G and H-functions of one, two or more variables. For example, if $N = P = Q = 0$, the multivariable H-function occurring in the left-hand side of these formulae would reduce immediately to the product of r different H-functions of Fox [1], thus the table listing various special cases of the H-function [4, pp. 145-159] can be used to derive from these Theorems a number of other Theorems involving any of these simpler special functions. Thus if in the Theorem 1 and Theorem 2, we take n=o (the polynomial S_0^m will reduce to $A_{0,0}$ which can be taken to be unity without loss of generality), reduce the multivariable H-function occurring therein to the Gaussian hypergeometric function and also reduce the general θ and ϕ functions to unity, we get in essence the Theorems established by Srivastava, Saigo and Raina [9, p. 6, Th.3; p.8, Th. 4].

On suitably reducing some of the parameters occurring in the formula (2.2), the general class of polynomials S_n^m into the Hermite polynomials [10, p. 158, eq. (1.4)] and the multivariable H-function to the product of r different Whittaker functions [6, p. 18, eq. (2.6.7)], we arrive at the following new and interesting special case of the formula (2.2) after a little simplification

$$L\Big\{R_1[f(x)];s\Big\} = \int_0^\infty t^\eta f(t)\, A_3(s\ ,t)\, dt \tag{3.1}$$

where

$$R_1[f(x)] = \prod_{i=1}^{r} z_i^{a_i} \sum_{k=0}^{[n/2]} \frac{(-n)_{2k}}{k!}\,(-1)^k \int_0^\infty t^{\eta+\frac{n}{2}+\sum_{i=1}^{r} a_i} g(x,t)\,\theta\,(\frac{t}{x}) f(t)\,dt \tag{3.2}$$

$$A_3(s,\ t) = L\Big\{g(x,t)\,\theta\left(\frac{t}{x}\right) U(x-t)\ ;s\Big\} \tag{3.3}$$

and

$$g(x,t) = x^{-\eta-\alpha-1-\frac{n}{2}-\sum_{i=1}^{r} a_i}(x-t)^\alpha H_n\Big(\frac{1}{2}\sqrt{\frac{x}{t}}\Big)\prod_{i=1}^{r}(\exp)^{-z_i t/2x} W_{\lambda_i,\nu_i}\Big(\frac{z_i t}{x}\Big) \tag{3.4}$$

and the integrals (3.1), (3.2) and (3.3) exist and $\mathrm{Re}(s) > 0$.

Several other interesting and useful special cases of our main formula (2.2), (2.3), (2.8) and (2.9) involving the product of a large variety of polynomials (which are special cases of S_n^m) and numerous simple special functions involving one or more variables (which are particular cases of the multivariable H-function) can also be obtained. Again due to the presence of the general functions θ and ϕ in the kernels of the fractional integral operators (1.1) and (1.2), many further results can also be established by replacing θ and ϕ by a wide range of functions of one or more variable, which are not the special cases of a general class of polynomials and the multivariable H-function, but we do not record them here for the lack of space.

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